Evolutionary Game Theory and Population Dynamics

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1 Short overview

We begin these lecture notes by a crash course in game theory. In particular, we introduce a fundamental notion of a Nash equilibrium. To address the problem of the equilibrium selection in games with multiple equilibria, we review basic properties of the deterministic replicator dynamics and stochastic dynamics of finite populations.

We show the almost global asymptotic stability of an efficient equilibrium in the replicator dynamics with a migration between subpopulations. We also show that the stability of a mixed equilibrium depends on the time delay introduced in replicator equations. For large time delays, a population oscillates around its equilibrium.

We analyze the long-run behaviour of stochastic dynamics in well-mixed populations and in spatial games with local interactions. We review results concerning the effect of the number of players and the noise level on the stochastic stability of Nash equilibria. In particular, we present examples of games in which when the number of players increases or the noise level decreases, a population undergoes a transition between its equilibria. We discuss similarities and differences between systems of interacting players in spatial games maximizing their individual payoffs and particles in lattice-gas models minimizing their interaction energy.

In short, there are two main themes of our lecture notes: the selection of efficient equilibria (providing the highest payoffs to all players) in population dynamics and the dependence of the long-run behaviour of a population on various parameters such as the time delay, the noise level, and the size of the population.

2 Introduction

Many socio-economic and biological processes can be modeled as systems of interacting individuals; see for example econophysics bulletin [16] and statistical mechanics and quantitative biology archives [13]. One may then try to derive their global behaviour from individual interactions between their basic entities such as animals in ecological and evolutionary models, genes in population genetics and people in social processes. Such approach is fundamental in statistical physics which deals with systems of interacting particles. One can therefore try to apply methods of statistical physics to investigate the population dynamics of interacting individuals. There are however profound differences between these two systems. Physical systems tend in time to states which are characterized by the minimum of some global quantity, the total energy or free energy of the system. Population dynamics lacks such general principle. Agents in social models maximize their own payoffs, animals and genes maximize their individual darwinian fitness. The long-run behavior of such populations cannot in general be characterized by the global or even local maximum of the payoff or fitness of the whole population. We will explore similarities and differences between these systems.

The behaviour of systems of interacting individuals can be often described within game-theoretic models [48, 24, 25, 103, 100, 79, 36, 106, 27, 14, 37, 66, 67, 68]. In such models, players have at their disposal certain strategies and their payoffs in a game depend on strategies chosen both by them and by their opponents. The central concept in game theory is that of a **Nash equilibrium**. It is an assignment of strategies to players such that no player, for fixed strategies of his opponents, has an incentive to deviate from his current strategy; no change can increase his payoff.

In Chapter 3, we present a crash course in game theory. One of the fundamental problems in game theory is the equilibrium selection in games with multiple Nash equilibria. Some two-player symmetric games with two strategies, have two Nash equilibria and it may happen that one of them is payoff dominant (also called efficient) and the other one is risk-dominant. In the efficient equilibrium, players receive highest possible payoffs. The strategy is risk-dominant if it has a higher expected payoff against a player playing both strategies with equal probabilities. It is played by individuals averse to risk. One of the selection methods is to construct a dynamical system where in the long run only one equilibrium is played with a high frequency.

John Maynard Smith [46, 47, 48] has refined the concept of the Nash equilibrium to include the stability of equilibria against mutants. He introduced the fundamental notion of an **evolutionarily stable strategy**. If everybody plays such a strategy, then the small number of mutants playing a different strategy is eliminated from the population. The dynamical interpretation of the evolutionarily stable strategy was later provided by several authors [94, 35, 109]. They proposed a system of difference or differential replicator equations which describe the time-evolution of frequencies of strategies. Nash

equilibria are stationary points of this dynamics. It appears that in games with a payoff dominant equilibrium and a risk-dominant one, both are asymptotically stable but the second one has a larger basin of attraction in the replicator dynamics.

In Chapter 4, we introduce **replicator dynamics** and review theorems concerning asymptotic stability of Nash equilibria [103, 36, 37]. Then in Chapter 5, we present our own model of the replicator dynamics [55] with a migration between two subpopulations for which an efficient equilibrium is almost globally asymptotically stable.

It is very natural, and in fact important, to introduce a **time delay** in the population dynamics; a time delay between acquiring information and acting upon this knowledge or a time delay between playing games and receiving payoffs. Recently Tao and Wang [92] investigated the effect of a time delay on the stability of interior stationary points of the replicator dynamics. They considered two-player games with two strategies and a unique asymptotically stable interior stationary point. They proposed a certain form of a time-delay differential replicator equation. They showed that the mixed equilibrium is asymtotically stable if a time delay is small. For sufficiently large delays it becomes unstable.

In Chapter 6, we construct two models of discrete-time replicator dynamics with a time delay [2]. In the social-type model, players imitate opponents taking into account average payoffs of games played some units of time ago. In the biological-type model, new players are born from parents who played in the past. We consider two-player games with two strategies and a unique mixed Nash equilibrium. We show that in the first type of dynamics, it is asymptotically stable for small time delays and becomes unstable for large ones when the population oscillates around its stationary state. In the second type of dynamics, however, the Nash equilibrium is asymptotically stable for any time delay. Our proofs are elementary, they do not rely on the general theory of delay differential and difference equations.

Replicator dynamics models population behaviour in the limit of the infinite number of individuals. However, real populations are finite. Stochastic effects connected with random matchings of players, mistakes of players and biological mutations can play a significant role in such systems. We will discuss various stochastic adaptation dynamics of populations with a fixed number of players interacting in discrete moments of time. In well-mixed populations, individuals are randomly matched to play a game [40, 76, 54]. The deterministic selection part of the dynamics ensures that if the mean payoff of a given strategy is bigger than the mean payoff of the other one, then the number of individuals playing the given strategy increases. However, players may mutate hence the population may move against a selection pressure. In spatial games, individuals are located on vertices of certain graphs and they interact only with their neighbours; see for example [62, 63, 64, 5, 17, 106, 18, 44, 41, 7, 86, 87, 89, 30, 31, 32, 33] and a recent review [90] and references therein. In discrete moments of times, players adapt

to their opponents by choosing with a high probability the strategy which is the best response, i.e. the one which maximizes the sum of the payoffs obtained from individual games. With a small probability, representing the noise of the system, they make mistakes. The above described stochastic dynamics constitute ergodic Markov chains with states describing the number of individuals playing respective strategies or corresponding to complete profiles of strategies in the case of spatial games. Because of the presence of random mutations, our Markov chains are ergodic (irreducible and periodic) and therefore they possess unique stationary measures. To describe the long-run behavior of such stochastic dynamics, Foster and Young [22] introduced a concept of stochastic stability. A configuration of the system is **stochastically stable** if it has a positive probability in the stationary measure of the corresponding Markov chain in the zero-noise limit, that is the zero probability of mistakes. It means that in the long run we observe it with a positive frequency along almost any time trajectory.

In Chapter 7, we introduce the concept of stochastic stability and present a useful representation of stationary measures of ergodic Markov chains [107, 23, 83].

In Chapter 8, we discuss populations with random matching of players in **well-mixed populations**. We review recent results concerning the dependence of the long-run behavior of such systems on the number of players and the noise level. In the case of two-player games with two symmetric Nash equilibria, an efficient one and a risk-dominant one, when the number of players increases, the population undergoes twice a transition between its equilibria. In addition, for a sufficiently large number of individuals, the population undergoes another **equilibrium transition** when the noise decreases.

In Chapter 9, we discuss **spatial games**. We will see that in such models, the notion of a Nash equilibrium (called there a Nash configuration) is similar to the notion of a ground-state configuration in classical lattice-gas models of interacting particles. We discuss similarities and differences between systems of interacting players in spatial games maximizing their individual payoffs and particles in lattice-gas models minimizing their interaction energy.

The concept of stochastic stability is based on the zero-noise limit for a fixed number of players. However, for any arbitrarily low but fixed noise, if the number of players is large enough, the probability of any individual configuration is practically zero. It means that for a large number of players, to observe a stochastically stable configurations we must assume that players make mistakes with extremely small probabilities. On the other hand, it may happen that in the long run, for a low but fixed noise and sufficiently large number of players, the stationary configuration is highly concentrated on an ensemble consisting of one Nash configuration and its small perturbations, i.e. configurations where most players play the same strategy. We will call such configurations **ensemble stable.** It will be shown that these two stability concepts do not necessarily coincide. We will present examples of spatial games with three strategies where concepts of stochastic stability and ensem-

ble stability do not coincide [51, 53]. In particular, we may have the situation, where a stochastically stable strategy is played in the long run with an arbitrarily low frequency. In fact, when the noise level decreases, the population undergoes a sharp transition with the coexistence of two equilibria for some noise level. Finally, we discuss the influence of **dominated strategies** on the long-run behaviour of population dynamics.

In Chapter 10, we shortly review other results concerning stochastic dynamics of finite populations.

3 A crash course in game theory

To characterize a game-theoretic model one has to specify players, strategies they have at their disposal and payoffs they receive. Let us denote by $I = \{1, ..., n\}$ the set of players. Every player has at his disposal m different strategies. Let $S = \{1, ..., m\}$ be the set of strategies, then $\Omega = S^I$ is the set of strategy profiles, that is functions assigning strategies to players. The payoff of any player depends not only on his strategy but also on strategies of all other players. If $X \in \Omega$, then we write $X = (X_i, X_{-i})$, where $X_i \in S$ is a strategy of the i-th player and $X_{-i} \in S^{I - \{i\}}$ is a strategy profile of remaining players. The payoff of the i-th player is a function defined on the set of profiles,

$$U_i : \Omega \rightarrow R, \quad i, ..., n$$

The central concept in game theory is that of a Nash equilibrium. An assignment of strategies to players is a **Nash equilibrium**, if for each player, for fixed strategies of his opponents, changing his current strategy cannot increase his payoff. The formal definition will be given later on when we enlarge the set of strategies by mixed ones.

Although in many models the number of players is very large (or even infinite as we will see later on in replicator dynamics models), their strategic interactions are usually decomposed into a sum of two-player games. Only recently, there have appeared some systematic studies of truly multi-player games [42, 10, 11, 75]. Here we will discuss only two-player games with two or three strategies. We begin with games with two strategies, A and B. Payoffs functions can be then represented by 2×2 payoff matrices. A general payoff matrix is given by

$$U = \begin{array}{c c c} & A & B \\ A & a & b \\ B & c & d, \end{array}$$

where U_{kl}, $k, l = A, B$, is a payoff of the first (row) player when he plays the strategy k and the second (column) player plays the strategy l.

We assume that both players are the same and hence payoffs of the column player are given by the matrix transposed to U; such games are called symmetric. In this classic set-up of static games (called matrix games or games in the normal form), players know payoff matrices, simultaneously announce (use) their strategies and receive payoffs according to their payoff matrices.

We will present now three main examples of symmetric two-player games with two strategies. We begin with an anecdote, then an appropriate game-theoretic model is build and its Nash equilibria are found.

Example 1 (Stag-hunt game)

Jean-Jacques Rousseau wrote, in his Discourse on the Origin and Basis of Equality among Men, about two hunters going either after a stag or a hare [72, 24]. In order to get a stag, both hunters must be loyal one to another and stay at their positions. A single hunter, deserting his companion, can get his own hare. In the game-theory language, we have two players and each of them has at his disposal two strategies: Stag (St) and Hare (H). In order to present this example as a matrix game we have to assign some values to animals. Let a stag (which is shared by two hunters) be worth 10 units and a hare 3 units. Then the payoff matrix of this symmetric game is as follows:

$$
U = \quad
\begin{array}{c c c}
 & \text{St} & \text{H} \\
\text{St} & 5 & 0 \\
\text{H} & 3 & 3
\end{array}
$$

It is easy to see that there are two Nash equilibria: (St, St) and (H, H).

In a general payoff matrix, if $a > c$ and $d > b$, then both (A, A) and (B, B) are Nash equilibria. If $a + b < c + d$, then the strategy B has a higher expected payoff against a player playing both strategies with the probability $1/2$. We say that B risk dominates the strategy A (the notion of the risk-dominance was introduced and thoroughly studied by Harsányi and Selten [29]). If at the same time $a > d$, then we have a selection problem of choosing between the payoff-dominant (Pareto-efficient) equilibrium (A, A) and the risk-dominant (B, B).

Example 2 (Hawk-Dove game)

Two animals are fighting for a certain territory of a value V. They can be either aggressive (hawk strategy - H) or peaceful (dove strategy - D). When two hawks meet, they accure the cost of fighting $C > V$ and then they split the territory. When two dove meets, they split the territory without a fight. A dove gives up the territory to a hawk. We obtain the following payoff matrix:

$$U = \begin{array}{c|cc} & \text{H} & \text{D} \\ \hline \text{H} & \text{(V-C)/2} & \text{V} \\ \text{D} & 0 & \text{V}/2, \end{array}$$

The Hawk-Dove game was analyzed by John Maynard Smith [48]. It is also known as the Chicken game [77] or the Snowdrift game [31]. It has two non-symmetric Nash equilibria: (H, D) and (D, H).

Example 3 (Prisoner's Dilemma)

The following story was discussed by Melvin Dresher, Merill Flood, and Albert Tucker [4, 73, 84]. Two suspects of a bank robbery are caught and interrogated by the police. The police offers them separately the following deal. If a suspect testifies against his colleague (a strategy of defection - D), and the other does not (cooperation - C), his sentence will be reduced by five years. If both suspects testify, that is defect, they will get the reduction of only one year. However, if they both cooperate and do not testify, their sentence, because of the lack of a hard evidence, will be reduced by three years. We obtain the following payoff matrix:

$$U = \begin{array}{c|cc} & \text{C} & \text{D} \\ \hline \text{C} & 3 & 0 \\ \text{D} & 5 & 1 \end{array}$$

The strategy C is a **dominated strategy** - it results in a lower payoff than the strategy D, regardless of a strategy used by the other player. Therefore, (D, D) is the unique Nash equilibrium but both players are much better off when they play C - this is the classic Prisoner's Dilemma.

A novel behaviour can appear in games with three strategies.

Example 4 (Rock-Scissors-Paper game)

In this game, each of two players simultaneously exhibits a sign of either a scissors (S), a rock (R), or a paper (P). The game has a cyclic behaviour: rock crashes scissors, scissors cut paper, and finally paper wraps rock. The payoffs can be given by the following matrix:

$$U = \begin{array}{c|ccc} & \text{R} & \text{S} & \text{P} \\ \hline \text{R} & 1 & 2 & 0 \\ \text{S} & 0 & 1 & 2 \\ \text{P} & 2 & 0 & 1 \end{array}$$

It is easy to verify that this game, because of its cyclic behavior, does not have any Nash equilibria as defined so far. However, we intuitively feel that when we repeat it many times, the only way not to be exploited is to mix randomly strategies, i.e. to choose each strategy with the probability $1/3$.

This brings us to a concept of a mixed stategy, a probability mass function on the set of pure strategies S. Formally, a **mixed strategy** x is an element of a simplex

$$\Delta = \{x \in R^m, 0 \le x_k \le 1, \sum_{k=1}^{m} x_k = 1\}.$$

By the support of a mixed strategy x we mean the set of pure strategies with positive probabilities in x. Payoffs of mixed strategies are defined as appropriate expected values. In two-player games, a player who uses a mixed strategy x against a player with a mixed strategy y receives a payoff given by

$$\sum_{k,l \in S} U_{kl} x_k y_l.$$

In general n-player games, profiles of strategies are now elements of $\Theta = \Delta^I$. We are now ready to define formally a Nash equilibrium.

Definition 1 $X \in \Theta$ is a **Nash equilibrium** *if for every* $i \in I$ *and every* $y \in \Delta$,

$$U_i(X_i, X_{-i}) \ge U_i(y, X_{-i})$$

In the mixed Nash equilibrium, expected payoffs of all strategies in its support should be equal. Otherwise a player could increase his payoff by increasing the probability of playing a strategy with the higher expected payoff. In two-player games with two strategies, we identify a mixed strategy with its first component, $x = x_1$. Then the expected payoff of A is given by $ax + b(1-x)$ and that of B by $cx + d(1-x)$. $x^* = (d-b)/(d-b+a-c)$ for which the above two expected values are equal is a mixed Nash equilibrium or more formally, a profile (x, x) is a Nash equilibrium.

In Examples 1 and 2, in addition to Nash equilibria in pure strategies, we have mixed equilibria, $x^* = 3/5$ and $x^* = V/C$ respectively. It is obvious that the Prisoner's Dilemma game does not have any mixed Nash equilibria . On the other hand, the only Nash equilibrium of the Rock-Scissors-Paper game is a mixed one assigning the probability $1/3$ to each strategy.

We end this chapter by a fundamental theorem due to John Nash [59, 60].

Theorem 1 *Every game with a finite number of players and a finite number of strategies has at least one Nash equilibrium.*

In any Nash equilibrium, every player uses a strategy which is a best reply to the profile of strategies of remaining players. Therefore a Nash equilibrium can be seen as a best reply to itself - a fixed point of a certain best-reply correspondence. Then one can use the Kakutani fixed point theorem to prove the above theorem.

4 Replicator dynamics

The concept of a Nash equilibrium is a static one. Here we will introduce the classical replicator dynamics and review its main properties [103, 36, 37]. Replicator dynamics provides a dynamical way of achieving Nash equilibria in populations. We will see that Nash equilibria are stationary points of such dynamics and some of them are asymptotically stable.

Imagine a finite but a very large population of individuals. Assume that they are paired randomly to play a symmetric two-player game with two strategies and the payoff matrix given in the beginning of the previous chapter. The complete information about such population is provided by its strategy profile, that is an assignment of pure strategies to players. Here we will be interested only in the proportion of individuals playing respective strategies. We assume that individuals receive average payoffs with respect to all possible opponents - they play against the average strategy.

Let $r_i(t)$, $i = A, B$, be the number of individuals playing the strategy A and B respectively at the time t. Then $r(t) = r_A(t) + r_B(t)$ is the total number of players and $x(t) = \frac{r_1(t)}{r(t)}$ is a fraction of the population playing A.

We assume that during the small time interval ϵ, only an ϵ fraction of the population takes part in pairwise competitions, that is plays games. We write

$$r_i(t + \epsilon) = (1 - \epsilon)r_i(t) + \epsilon r_i(t)U_i(t); \quad i = A, B, \tag{1}$$

where $U_A(t) = ax(t) + b(1 - x(t))$ and $U_B(t) = cx(t) + d(1 - x(t))$ are average payoffs of individuals playing A and B respectively. We assume that all payoffs are not smaller than 0 hence r_A and r_B are always non-negative and therefore $0 \leq x \leq 1$.

The equation for the total number of players reads

$$r(t + \epsilon) = (1 - \epsilon)r(t) + \epsilon r(t)\bar{U}(t), \tag{2}$$

where $\bar{U}(t) = x(t)U_A(t) + (1 - x(t))U_B(t)$ is the average payoff in the population at the time t. When we divide (1) by (2) we obtain an equation for the frequency of the strategy A,

$$x(t + \epsilon) - x(t) = \epsilon \frac{x(t)[U_A(t) - \bar{U}(t)]}{1 - \epsilon + \epsilon\bar{U}(t)}. \tag{3}$$

Now we divide both sides of (3) by ϵ, perform the limit $\epsilon \to 0$, and obtain the well known differential replicator equation:

$$\frac{dx(t)}{dt} = x(t)[U_A(t) - \bar{U}(t)]. \tag{4}$$

The above equation can also be written as

$$\frac{dx(t)}{dt} = x(t)(1 - x(t))[U_A(t) - U_B(t)]$$

$$= (a - c + d - b)x(t)(1 - x(t))(x(t) - x^*) \tag{5}$$

For games with m strategies we obtain a system of m differential equations for $x_k(t)$, fractions of the population playing the k-th strategy at the time t, $k = 1, ..., m$,

$$\frac{dx_k(t)}{dt} = x_k(t)[\sum_{l=1}^{m} U_{kl}x_l(t) - \sum_{k,l=1}^{m} U_{kl}x_k(t)x_l(t)], \tag{6}$$

where on the right hand-size of (6) there is a difference of the average payoff of the k-th strategy and the average payoff of the population. The above system of differential equations or analogous difference equations, called replicator dynamics was proposed in [94, 35, 109]. For any initial condition $x^0 \in \Delta$, it has the unique global solution, $\xi(x^0, t)$, which stays in the simplex Δ.

Now we review some theorems relating replicator dynamics and Nash equilibria [103, 36, 37]. We consider symmetric two-player games. We denote the set of strategies corresponding to symmetric Nash equilibria by

$$\Delta^{NE} = \{x \in \Delta : (x, x) \text{ is a Nash equilibrium}\}.$$

It follows from the definition of the Nash equilibrium (see also discussion in the previous chapter concerning mixed strategies) that

$$\Delta^{NE} = \{x \in \Delta : u(i, x) = \max_{z \in \Delta} u(z, x) \text{ for every } i \text{ in the support of } x\}.$$

It is easy to see that

$$\Delta^0 = \{x \in \Delta : u(i, x) = u(x, x) \text{ for every } i \text{ in the support of } x\}$$

is the set of stationary points of the replicator dynamics.

It follows that symmetric Nash equilibria are stationary points of the replicator dynamics.

Theorem 2 $S \cup \Delta^{NE} \subset \Delta^0$

The following two theorems relate stability of stationary points to Nash equilibria [103, 36].

Theorem 3 *If $x \in \Delta$ is Lyapunov stable, then $x \in \Delta^{NE}$.*

Theorem 4 *If $x^0 \in interior(\Delta)$ and $\xi(x^0, t) \to_{t \to \infty} x$, then $x \in \Delta^{NE}$.*

Below we present the replicator dynamics in the examples of two-player games discussed in the previous chapter. We write replicator equations and show their phase diagrams.
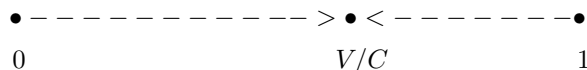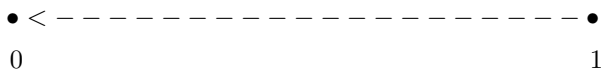
Stag-hunt game

$$\frac{dx}{dt} = x(1-x)(5x-3)$$

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•<– – – – – – – – – – – –•– – – – – – –>•
0                             3/5            1
```
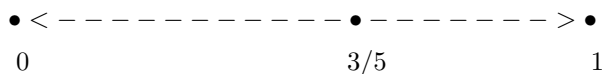
Hawk-Dove game

$$\frac{dx}{dt} = -x(1-x)(x - C/V)$$

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•– – – – – – – – – – –>•<– – – – – – –•
0                         V/C            1
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Prisoner's Dilemma

$$\frac{dx}{dt} = -x(1-x)(x+1)$$

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•<– – – – – – – – – – – – – – – – – – – –•
0                                         1
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We see that in the Stag-hunt game, both pure Nash equilibria are asymptotically stable. The risk-dominant one has the larger basin of attraction which is true in general because $x^* = (d-b)/(d-b+a-c) > 1/2$ for games with an efficient equilibrium and a risk dominant one.

In the Hawk-Dove game, the unique symmetric mixed Nash equilibrium is asymptotically stable.

In the Prisoner's Dilemma, the strategy of defection is globally asymptotically stable.

In the Rock-Scissors-Paper game, a more detailed analysis has to be done. One can show, by straithforward computations, that the time derivative of $lnx_1x_2x_3$ is equal to zero. Therefore $lnx_1x_2x_3 = c$ is an equation of a closed orbit for any constant c. The stationary point $(1/3, 1/3, 1/3)$ of the replicator dynamics is Lyapunov stable and the population cycles on a closed trajectory (which depends on the initial condition) around its Nash equilibrium.

5 Replicator dynamics with migration

We discuss here a game-theoretic dynamics of a population of replicating who can migrate between two subpopulations or habitats [55]. We consider symmetric two-player games with two strategies: A and B. We assume that $a > d > c$, $d > b$, and $a + b < c + d$ in a general payoff matrix given in the beginning of Chapter 3. Such games have two Nash equilibria: the efficient one (A, A) in which the population is in a state with a maximal fitness (payoff) and the risk-dominant (B, B) where players are averse to risk. We show that for a large range of parameters of our dynamics, even if the initial conditions in both habitats are in the basin of attraction of the risk-dominant equilibrium (with respect to the standard replication dynamics without migration), in the long run most individuals play the efficient strategy.

We consider a large population of identical individuals who at each time step can belong to one of two different non-overlapping subpopulations or habitats which differ only by their replication rates. In both habitats, they take part in the same two-player symmetric game. Our population dynamics consists of two parts: the standard replicator one and a migration between subpopulations. Individuals are allowed to change their habitats. They move to a habitat in which the average payoff of their strategy is higher; they do not change their strategies.

Migration helps the population to evolve towards an efficient equilibrium. Below we briefly describe the mechanism responsible for it. If in a subpopulation, the fraction of individuals playing the efficient strategy A is above its unique mixed Nash equilibrium fraction, then the expected payoff of A is bigger than that of B in this subpopulation, and therefore the subpopulation evolves to the efficient equilibrium by the replicator dynamics without any migration. Let us assume therefore that such fraction is below the Nash equilibrium in both subpopulations. Without loss of generality we assume that initial conditions are such that the fraction of individuals playing A is bigger in the first subpopulation than in the second one. Hence the expected payoff of A is bigger in the first subpopulation than in the second one, and the expected payoff of B is bigger in the second subpopulation than in the first one. This implies that a fraction of A-players in the second population will switch to the first one and at the same time a fraction of B-players from the first population will switch to the second one - migration causes the increase of the fraction of individual of the first population playing A. However, any B-player will have more offspring than any A-player (we are below a mixed Nash equilibrium) and this has the opposite effect on relative number of A-players in the first population than the migration. The asymptotic composition of the whole population depends on the competition between these two processes.

We derive sufficient conditions for migration and replication rates such that the whole population will be in the long run in a state in which most individuals occupy only one habitat (the first one for the above described initial conditions) and play the efficient strategy.

Let ϵ be a time step. We allow two subpopulations to replicate with different speeds. We assume that during any time-step ϵ, a fraction ϵ of the first subpopulation and a fraction $\kappa\epsilon$ of the second subpopulation plays the game and receives payoffs which are interpreted as the number of their offspring. Moreover, we allow a fraction of individuals to migrate to a habitat in which their strategies have higher expected payoffs.

Let r_s^i denote the number of individuals which use the strategy $s \in \{A, B\}$ in the subpopulation $i \in \{1, 2\}$. By U_s^i we denote the expected payoff of the strategy s in the subpopulation i:

$$U_A^1 = ax + b(1-x), \quad U_B^1 = cx + d(1-x),$$
$$U_A^2 = ay + b(1-y), \quad U_B^2 = cy + d(1-y),$$

where

$$x = \frac{r_A^1}{r_1}, \quad y = \frac{r_A^2}{r_2}, \quad r_1 = r_A^1 + r_B^1, \quad r_2 = r_A^2 + r_B^2;$$

x and y denote fractions of A-players in the first and second population respectively. We denote by $\alpha = \frac{r_1}{r}$ the fraction of the whole population in the first subpopulation, where $r = r_1 + r_2$ is the total number of individuals.

The evolution of the number of individuals in each subpopulation is assumed to be a result of the replication and the migration flow. In our model, the direction and intensity of migration of individuals with a given strategy will be determined by the difference of the expected payoffs of that strategy in both habitats. Individuals will migrate to a habitat with a higher payoff. The evolution equations for the number of individuals playing the strategy s, $s \in \{A, B\}$, in the habitat i, $i \in \{1, 2\}$, have the following form:

$$r_A^1(t + \epsilon) = R_A^1 + \Phi_A, \tag{7}$$
$$r_B^1(t + \epsilon) = R_B^1 + \Phi_B, \tag{8}$$
$$r_A^2(t + \epsilon) = R_A^2 - \Phi_A, \tag{9}$$
$$r_B^2(t + \epsilon) = R_B^2 - \Phi_B, \tag{10}$$

where all functions on the right-hand sides are calculated at the time t.

Functions R_s^i describe an increase of the number of the individuals playing the strategy s in the subpopulation i due to the replication:

$$R_s^1 = (1 - \epsilon)r_s^1 + \delta U_s^1 r_s^1, \tag{11}$$
$$R_s^2 = (1 - \kappa\epsilon)r_s^2 + \kappa\epsilon U_s^2 r_s^2, \tag{12}$$

The rate of the replication of individuals playing the strategy s in the first subpopulation is given by ϵU_s^1, and in the second subpopulation by $\kappa\epsilon U_s^2$. The parameter κ measures the difference of reproduction speeds in both habitats.

Functions Φ_s, $s \in \{A, B\}$, are defined by

$$\Phi_s = \epsilon\gamma(U_s^1 - U_s^2)[r_s^2\Theta(U_s^1 - U_s^2) + r_s^1\Theta(U_s^2 - U_s^1)], \qquad (13)$$

where Θ is the Heaviside's function,

$$\Theta(x) = \begin{cases} 1, & x \geq 0; \\ 0, & x < 0 \end{cases} \qquad (14)$$

and γ is the migration rate.

Functions Φ_s describe changes of the numbers of the individuals playing strategy s in the relevant habitat due to migration. Φ_s will be referred to as the migration of individuals (who play the strategy s) between two habitats.

Thus, if for example $U_A^1 > U_A^2$, then there is a migration of individuals with the strategy A from the second habitat to the first one:

$$\Phi_A = \delta\gamma r_A^2(U_A^1 - U_A^2), \qquad (15)$$

and since then necessarily $U_B^1 < U_B^2$ [note that $U_A^1 - U_A^2 = (a - b)(x - y)$ and $U_B^1 - U_B^2 = (c - d)(x - y)$], there is a migration flow of individuals with strategy B from the first habitat to the second one:

$$\Phi_B = \epsilon\gamma r_B^1(t)(U_B^1 - U_B^2). \qquad (16)$$

In this case, the migration flow Φ_A describes the increase of the number of individuals which play the strategy A in the first subpopulation due to migration of the individuals playing A in the second subpopulation. This increase is assumed to be proportional to the number of individuals playing A in the second subpopulation and the difference of payoffs of this strategy in both subpopulations. The constant of proportionality is ϵ times the migration rate γ.

The case $\gamma = 0$ corresponds to two separate populations which do not communicate and evolve independently. Our model reduces then to the standard discrete-time replicator dynamics. In this case, the total number of players who use a given strategy changes only due to the increase or decrease of the strategy fitness, as described by functions defined in (11-12).

In the absence of the replication, there is a conservation of the number of individuals playing each strategy in the whole population. This corresponds to our model assumption that individuals can not change their strategies but only habitats in which they live.

For $U_A^1 > U_A^2$ we obtain from (7-10) equations for $r_i(t)$ and $r(t)$:

$$r_1(t+\epsilon) = (1-\epsilon)r_1(t)$$

$$+\delta r_1(t)[\frac{r_A^1 U_A^1 + r_B^1 U_B^1}{r_1} + \gamma \frac{r_A^2(U_A^1 - U_A^2) + r_B^1(U_B^1 - U_B^2)}{r_1}], \qquad (17)$$

$$r_2(t+\epsilon) = (1-\kappa\epsilon)r_2(t)$$

$$+\delta r_2(t)[\kappa \frac{r_A^2 U_A^2 + r_B^2 U_B^2}{r_2} + \gamma \frac{r_A^2(U_A^2 - U_A^1) + r_B^1(U_B^2 - U_B^1)}{r_2}], \qquad (18)$$

$$r(t+\delta) = (1-\epsilon)r_1(t) + (1-\kappa\delta)r_2(t)$$

$$+\delta r(t)[\alpha(\frac{r_A^1}{r_1}U_A^1 + \frac{r_B^1}{r_1}U_B^1) + (1-\alpha)\kappa(\frac{r_A^2}{r_2}U_A^2 + \frac{r_B^2}{r_2}U_B^2)], \qquad (19)$$

where all functions in square brackets depend on t.

Now, like in the derivation of the standard replicator dynamics, we consider frequencies of individuals playing the relevant strategies in both habitats. Thus, we focus on the temporal evolution of the frequencies, x and y, and the relative size of the first subpopulation, α. We divide (7) by (17), (9) by (18), and (17) by (19). Performing the limit $\epsilon \to 0$ we obtain the following differential equations:

$$\frac{dx}{dt} = x[(1-x)(U_A^1 - U_B^1)$$

$$+\gamma[(\frac{y(1-\alpha)}{x\alpha} - \frac{y(1-\alpha)}{\alpha})(U_A^1 - U_A^2) - (1-x)(U_B^1 - U_B^2)]], \qquad (20)$$

$$\frac{dy}{dt} = y[\kappa(1-y)(U_A^2 - U_B^2) + \gamma[(1-y)(U_A^2 - U_A^1) - \frac{(1-x)\alpha}{1-\alpha}(U_B^2 - U_B^1)]], \quad (21)$$

$$\frac{d\alpha}{dt} = \alpha(1-\alpha)[xU_A^1 + (1-x)U_B^1 - (yU_A^2 + (1-y)U_B^2)]$$

$$+\alpha\gamma[\frac{y(1-\alpha)}{\alpha}(U_A^1 - U_A^2) + (1-x)(U_B^1 - U_B^2)]$$

$$+\alpha(1-\alpha)(\kappa-1)(1-yU_A^2 - (1-y)U_B^2). \qquad (22)$$

Similar equations are derived for the case $U_A^1 < U_A^2$ (since our model is symmetric with respect to the permutation of the subpopulations, it is enough to renumerate the relevant indices and redefine the parameter κ).

Assume first that $U_A^1(0) > U_A^2(0)$, which is equivalent to $x(0) > y(0)$. It follows from (7-10) that a fraction of A-players from the subpopulation 2 will migrate to the subpopulation 1 and a fraction of B-players will migrate in the opposite direction. This will cause x to increase and y to decrease. However, if $x(0) < x^*$ and $y(0) < x^*$, then $U_A^1 < U_B^1$ and $U_A^2 < U_B^2$, therefore B-players will have more offspring than A-players. This has the opposite effect on the relative number of A-players in the first subpopulation than migration. If $x(0) < y(0)$, then migration takes place in the reverse directions.

The outcome of the competition between migration and replication depends, for a given payoff matrix, on the relation between $x(0) - y(0)$, γ and κ. We are interested in formulating sufficient conditions for the parameters of the model, for which most individuals of the whole population will play in the long run the efficient strategy A. We prove the following theorem [55].

Theorem 5 *If*

$$\gamma[x(0) - y(0)] > max[\frac{d-b}{d-c}, \frac{\kappa(a-c)}{a-b}],$$

then $x(t) \to_{t\to\infty} 1$ and $y(t) \to_{t\to\infty} 0$.
    If $\kappa < (a-1)/(d-1)$, then $\alpha(t) \to_{t\to\infty} 1$.

If

$$\gamma[y(0) - x(0)] > max[\frac{\kappa(d-b)}{d-c}, \frac{a-c}{a-b}],$$

then $x(t) \to_{t\to\infty} 0$ and $y(t) \to_{t\to\infty} 1$.
    If $\kappa > (d-1)/(a-1)$, then $\alpha(t) \to_{t\to\infty} 0$.

Proof:
Assume first that $x(0) > y(0)$. From (20-21) we get the following differential inequalities:

$$\frac{dx}{dt} > x(1-x)[U_A^1 - U_B^1) + \gamma(U_B^2 - U_B^1)], \tag{23}$$

$$\frac{dy}{dt} < y(1-y)[\kappa(U_A^2 - U_B^2) + \gamma(U_A^2 - U_A^1)], \tag{24}$$

Using explicit expressions for U_s^i we get

$$\frac{dx}{dt} > x(1-x)[(a-c+d-b)x + b - d + \gamma(d-c)(x-y)], \tag{25}$$

$$\frac{dy}{dt} < y(1-y)[\kappa[(a-c+d-b)y + b - d] - \gamma(a-b)(x-y)], \tag{26}$$

We note that if $\gamma(d-c)(x(0) - y(0)) > d - b$ then $\gamma(d-c)(x(0) - y(0)) + b - d + (a-c+d-b)x(0) > 0$, i.e. $dx/dt(0) > 0$.

Analogously, if $\gamma(a-b)(x(0)-y(0)) > \kappa(a-c)$, then $\gamma(a-b)(x(0)-y(0)) > \kappa[(a-c+d-b)+b-d] > \kappa[(a-c+d-b)y(o)+b-d]$, therefore $dy/dt(0) < 0$. Thus, combining both conditions we conclude that $x(t) - y(t)$ is an increasing function so $x(t) > y(t)$ for all $t \geq 0$, hence we may use (20-22) all the time. We get that $x(t) \to_{t\to\infty} 1$ and $y(t) \to_{t\to\infty} 0$, and the first part of the thesis follows. Now from (22) it follows that if $a - d + (\kappa - 1)(1 - d) > 0$, i.e. $\kappa < (a-1)/(d-1)$, then $\alpha(t) \to_{t\to\infty} 1$.

The second part of Theorem 5, corresponding to initial conditions $y(0) > x(0)$, can be proved analogously, starting from eqs. (7-10) written for the case $U_A^1(0) < U_A^2(0)$ and their continuous counterparts. We omit details.

The above conditions for κ mean that the population consisting of just A-players replicates faster (exponentially in $(a-1)t$) than the one consisting of just B-players (exponentially in $(d-1)\kappa t$). The same results would follow if the coefficients of the payoff matrix of the game played in one habitat would differ from those in the second habitat by an additive constant.

We showed that introduction of the mechanism of attraction by the habitat with a higher expected payoff in the standard replicator dynamics helps the whole population to reach the state in which in the long run most individuals play the efficient strategy.

More precisely, we proved that for a given rate of migration, if the fractions of individuals playing the efficient strategy in both habitats are not too close to each other, then the habitat with a higher fraction of such players overcomes the other one in the long run. The fraction of individuals playing the efficient strategy tends to unity in this habitat and consequently in the whole population. Alternatively, we may say that the bigger the rate of migration is, larger is the basin of attraction of the efficient equilibrium. In particular, we showed that for a large range of parameters of our dynamics, even if the initial conditions in both habitats are in the basin of attraction of the risk-dominant equilibrium (with respect to the standard replication dynamics without migration), in the long run most individuals play the efficient strategy.

6 Replicator dynamics with time delay

Here we consider two-player games with two strategies, two pure non-symmetric Nash equilibria, and a unique symmetric mixed one, that is $a < c$ and $d < b$ in a general payoff matrix given in the beginning of Chapter 3. Let us recall that the Hawk-Dove game is of such type.

Recently Tao and Wang [92] investigated the effect of a time delay on the stability of the mixed equilibrium in the replicator dynamics. They showed that it is asymptotically stable if a time delay is small. For sufficiently large delays it becomes unstable.

We construct two models of discrete-time replicator dynamics with a time delay [2]. In the social-type model, players imitate opponents taking into account average payoffs of games played some units of time ago. In the biological-type model, new players are born from parents who played in the past. We show that in the first type of dynamics, the unique symmetric mixed Nash equilibrium is asymptotically stable for small time delays and becomes unstable for large ones when the population oscillates around its stationary state. In the second type of dynamics, however, the Nash equilibrium is asymptotically stable for any time delay. Our proofs are elementary, they do not rely on the general theory of delay differential and difference equations.

6.1 Social-type time delay

Here we assume that individuals at time t replicate due to average payoffs obtained by their strategies at time $t - \tau$ for some delay $\tau > 0$ (see also a discussion after (32)). As in the standard replicator dynamics, we assume that during the small time interval ϵ, only an ϵ fraction of the population takes part in pairwise competitions, that is plays games. Let $r_i(t)$, $i = A, B$, be the number of individuals playing at the time t the strategy A and B respectively, $r(t) = r_A(t) + r_B(t)$ the total number of players and $x(t) = \frac{r_1(t)}{r(t)}$ a fraction of the population playing A.

We propose the following equations:

$$r_i(t + \epsilon) = (1 - \epsilon)r_i(t) + \epsilon r_i(t)U_i(t - \tau); \quad i = A, B. \tag{27}$$

Then for the total number of players we get

$$r(t + \epsilon) = (1 - \epsilon)r(t) + \epsilon r(t)\bar{U}_o(t - \tau), \tag{28}$$

where $\bar{U}_o(t - \tau) = x(t)U_A(t - \tau) + (1 - x(t))U_B(t - \tau)$.

We divide (27) by (28) and obtain an equation for the frequency of the strategy A,

$$x(t + \epsilon) - x(t) = \epsilon \frac{x(t)[U_A(t - \tau) - \bar{U}_o(t - \tau)]}{1 - \epsilon + \epsilon\bar{U}_o(t - \tau)} \tag{29}$$

and after some rearrangements we get

$$x(t + \epsilon) - x(t) = -\epsilon x(t)(1 - x(t))[x(t - \tau) - x^*]\frac{\delta}{1 - \epsilon + \epsilon\bar{U}_o(t - \tau)}, \tag{30}$$

where $x^* = (d - b)/(d - b + a - c)$ is the unique mixed Nash equilibrium of the game.

Now the corresponding replicator dynamics in the continuous time reads

$$\frac{dx(t)}{dt} = x(t)[U_A(t - \tau) - \bar{U}_o(t - \tau)] \tag{31}$$

and can also be written as

$$\frac{dx(t)}{dt} = x(t)(1 - x(t))[U_A(t - \tau) - U_B(t - \tau)]$$

$$= -\delta x(t)(1 - x(t))(x(t - \tau) - x^*). \tag{32}$$

The first equation in (32) can be also interpreted as follows. Assume that randomly chosen players imitate randomly chosen opponents. Then the probability that a player who played A would imitate the opponent who played B at time t is exactly $x(t)(1 - x(t))$. The intensity of imitation depends on

the delayed information about the difference of corresponding payoffs at time $t - \tau$. We will therefore say that such models have a social-type time delay.

Equations (31-32) are exactly the time-delay replicator dynamics proposed and analyzed by Tao and Wang [92]. They showed that if $\tau < c - a + b - d\pi/2(c - a)(b - d)$, then the mixed Nash equilibrium, x^*, is asymptotically stable. When τ increases beyond the bifurcation value $c - a + b - d\pi/2(c - a)(b - d)$, x^* becomes unstable. We have the following theorem [2].

Theorem 6 x^* *is asymptotically stable in the dynamics (30) if τ is sufficiently small and unstable for large enough τ.*

Proof: We will assume that τ is a multiple of ϵ, $\tau = m\epsilon$ for some natural number m. Observe first that if $x(t - \tau) < x^*$, then $x(t + \epsilon) > x(t)$, and if $x(t - \tau) > x^*$, then $x(t + \epsilon) < x(t)$. Let us assume first that there is t' such that $x(t'), x(t' - \epsilon), x(t' - 2\epsilon), ..., x(t' - \tau) < x^*$. Then $x(t)$, $t \geq t'$ increases up to the moment t_1 for which $x(t_1 - \tau) > x^*$. If such t_1 does not exist then $x(t) \to_{t \to \infty} x^*$ and the theorem is proved. Now we have $x^* < x(t_1 - \tau) < x(t_1 - \tau + \epsilon) < \ldots < x(t_1)$ and $x(t_1 + \epsilon) < x(t_1)$ so t_1 is a turning point. Now $x(t)$ decreases up to the moment t_2 for which $x(t_2 - \tau) < x^*$. Again, if such t_2 does not exist, then the theorem follows. Therefore let us assume that there is an infinite sequence, t_i, of such turning points. Let $\eta_i = |x(t_i) - x^*|$. We will show that $\eta_i \to_{i \to \infty} 0$.

For $t \in \{t_i, t_i + \epsilon, \ldots, t_{i+1} - 1\}$ we have the following bound for $x(t + \epsilon) - x(t)$:

$$|x(t + \epsilon) - x(t)| < \frac{1}{4}\eta_i \frac{\epsilon\delta}{1 - \epsilon + \epsilon\bar{U}_o(t - \tau)}. \tag{33}$$

This means that

$$\eta_{i+1} < (m + 1)\epsilon K \eta_i, \tag{34}$$

where K is the maximal possible value of $\frac{\delta}{4(1 - \epsilon + \epsilon\bar{U}_o(t - \tau))}$.
We get that if

$$\tau < \frac{1}{K} - \epsilon, \tag{35}$$

then $\eta_i \to_{i \to \infty} 0$ so $x(t)$ converges to x^*.

Now if for every t, $|x(t + \epsilon) - x^*| < \max_{k \in \{0,1,...,m\}} |x(t - k\epsilon) - x^*|$, then $x(t)$ converges to x^*. Therefore assume that there is t'' such that $|x(t'' + \epsilon) - x^*| \geq \max_{k \in \{0,1,...,m\}} |x(t'' - k\epsilon) - x^*|$. If τ satisfies (35), then it follows that $x(t + \epsilon), ..., x(t + \epsilon + \tau)$ are all on the same side of x^* and the first part of the proof can be applied. We showed that $x(t)$ converges to x^* for any initial conditions different from 0 and 1 hence x^* is globally asymptotically stable.

Now we will show that x^* is unstable for any large enough τ.

Let $\gamma > 0$ be arbitrarily small and consider a following perturbation of the stationary point x^*: $x(t) = x^*, t \leq 0$ and $x(\epsilon) = x^* + \gamma$. It folows from (30) that

$x(k\epsilon) = x(\epsilon)$ for $k = 1, ..., m+1$. Let $K' = \min_{x \in [x^*-\gamma, x^*+\gamma]} \frac{x(1-x)\delta}{4(1-\epsilon+\epsilon \bar{U}_o(t-\tau))}$. If $\frac{m}{2}\epsilon K'\gamma > 2\gamma$, that is $\tau > \frac{4}{K'}$, then it follows from (30) that after $m/2$ steps (we assume without loss of generality that m is even) $x((m+1+m/2)\epsilon) < x^* - \gamma$. In fact we have $x((2m+1)\epsilon) < ... < x((m+1)\epsilon)$ and at least $m/2$ of $x's$ in this sequence are smaller than $x^* - \gamma$. Let $\bar{t} > (2m+1)\epsilon$ be the smallest t such that $x(t) > x^* - \gamma$. Then we have $x(\bar{t}-m\epsilon), ..., x(\bar{t}-\epsilon) < x^* - \gamma < x(\bar{t})$ hence after $m/2$ steps, $x(t)$ crosses $x^* + \gamma$ and the situation repeats itself.

We showed that if

$$\tau > \frac{4}{K'}, \tag{36}$$

then there exists an infinite sequence, $\tilde{t}_i$, such that $|x(\tilde{t}_i) - x^*| > \gamma$ and therefore x^* is unstable. Moreover, $x(t)$ oscillates around x^*.

6.2 Biological-type time delay

Here we assume that individuals born at time $t - \tau$ are able to take part in contests when they become mature at time t or equivalently they are born τ units of time after their parents played and received payoffs. We propose the following equations:

$$r_i(t + \epsilon) = (1 - \epsilon)r_i(t) + \epsilon r_i(t - \tau)U_i(t - \tau); \ \ i = A, B. \tag{37}$$

Then the equation for the total number of players reads

$$r(t + \epsilon) = (1 - \epsilon)r(t)$$

$$+\epsilon r(t)[\frac{x(t)r_A(t-\tau)}{r_A(t)}U_A(t-\tau) + \frac{(1-x(t))r_B(t-\tau)}{r_B(t)}U_B(t-\tau)]. \tag{38}$$

We divide (37) by (38) and obtain an equation for the frequency of the first strategy,

$$x(t+\epsilon) - x(t) = \epsilon \frac{x(t-\tau)U_A(t-\tau) - x(t)\bar{U}(t-\tau)}{(1-\epsilon)\frac{r(t)}{r(t-\tau)} + \epsilon \bar{U}(t-\tau)}, \tag{39}$$

where $\bar{U}(t-\tau) = x(t-\tau)U_A(t-\tau) + (1-x(t-\tau))U_B(t-\tau)$.

We proved in [2] the following

Theorem 7 x^* *is asymptotically stable in the dynamics (39) for any value of the time delay* τ.

We begin by showing our result in the following simple example.

The payoff matrix is given by $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ hence $x^* = \frac{1}{2}$ is the mixed Nash equilibrium which is asymptotically stable in the replicator dynamics without the time delay. The equation (39) now reads

$$x(t+\epsilon) - x(t) = \epsilon \frac{x(t-\tau)(1-x(t-\tau)) - 2x(t)x(t-\tau)(1-x(t-\tau))}{(1-\epsilon)\frac{r(t)}{r(t-\tau)} + 2\epsilon x(t-\tau)(1-x(t-\tau))} \quad (40)$$

After simple algebra we get

$$x(t+\epsilon) - \tfrac{1}{2} + \tfrac{1}{2} - x(t)$$

$$= \epsilon(1-2x(t))\frac{x(t-\tau)(1-x(t-\tau))}{(1-\epsilon)\frac{r(t)}{r(t-\tau)} + 2\epsilon x(t-\tau)(1-x(t-\tau))}, \quad (41)$$

so

$$x(t+\epsilon) - \frac{1}{2} = (x(t) - \frac{1}{2})\frac{1}{1 + \frac{\epsilon r(t-\tau)}{(1-\epsilon)r(t)}2x(t-\tau)(1-x(t-\tau))}$$

hence

$$\left| x(t+\epsilon) - \frac{1}{2} \right| < \left| x(t) - \frac{1}{2} \right|. \quad (42)$$

It follows that x^* is globally asymptotically stable.

Now we present the proof for the general payoff matrix with a unique symmetric mixed Nash equilibrium.

Proof of Theorem 7:

Let $c_t = \frac{x(t)U_A(t)}{\bar{U}(t)}$. Observe that if $x(t) < x^*$, then $c_t > x(t)$, if $x(t) > x^*$, then $c_t < x(t)$, and if $x(t) = x^*$, then $c_t = x^*$. We can write (39) as

$$x(t+\epsilon) - x(t)$$

$$= \epsilon \frac{x(t-\tau)U_A(t-\tau) - c_{t-\tau}\bar{U}(t-\tau) + c_{t-\tau}\bar{U}(t-\tau) - x(t)\bar{U}(t-\tau)}{(1-\epsilon)\frac{p(t)}{p(t-\tau)} + \epsilon\bar{U}(t-\tau)} \quad (43)$$

and after some rearrangements we obtain

$$x(t+\epsilon) - c_{t-\tau} = (x(t) - c_{t-\tau})\frac{1}{1 + \frac{\epsilon p(t-\tau)}{(1-\epsilon)p(t)}\bar{U}(t-\tau)}. \quad (44)$$

We get that at time $t+\epsilon$, x is closer to $c_{t-\tau}$ than at time t and it is on the same side of $c_{t-\tau}$. We will show that c is an increasing or a constant function of x. Let us calculate the derivative of c with respect to x.

$$c' = \frac{f(x)}{(xU_A + (1-x)U_B)^2}, \quad (45)$$

where

$$f(x) = (ac + bd - 2ad)x^2 + 2d(a - b)x + bd. \tag{46}$$

A simple analysis shows that $f > 0$ on $(0, 1)$ or $f = 0$ on $(0, 1)$ (in the case of $a = d = 0$). Hence $c(x)$ is either an increasing or a constant function of x. In the latter case, $\forall_x c(x) = x^*$, as it happens in our example, and the theorem follows.

We will now show that

$$|x(t + \tau + \epsilon) - x^*| < \max\{|x(t) - x^*|, |x(t + \tau) - x^*|\} \tag{47}$$

hence $x(t)$ converges to x^* for any initial conditions different from 0 and 1 so x^* is globally asymptotically stable.

If $x(t) < x^*$ and $x(t + \tau) < x^*$, then $x(t) < c_t \le x^*$ and also $x(t + \tau) < c_{t+\tau} \le x^*$.

From (44) we obtain

$$\begin{cases} x(t + \tau) < x(t + \tau + \epsilon) < c_t \ \ if \ \ x(t + \tau) < c_t \\ x(t) < x(t + \tau + \epsilon) = c_t \ \ if \ \ x(t + \tau) = c_t \\ x(t) < c_t < x(t + \tau + \epsilon) < x(t + \tau) \ \ if \ \ x(t + \tau) > c_t \end{cases}$$

hence (47) holds.

If $x(t) > x^*$ and $x(t + \tau) < x^*$, then $x(t + \tau) < x^* < c_t < x(t)$ and either $x(t + \tau) < x(t + \tau + \epsilon) < x^*$ or $x^* < x(t + \tau + \epsilon) < c_t$ which means that (47) holds.

The cases of $x(t) > x^*$, $x(t + \tau) > x^*$ and $x(t) < x^*$, $x(t + \tau) < x^*$ can be treated analogously. We showed that (47) holds.

7 Stochastic dynamics of finite populations

In the next two chapters we will discuss various stochastic dynamics of populations with a fixed number of players interacting in discrete moments of time. We will analyze symmetric two-player games with two or three strategies and multiple Nash equilibria. We will address the problem of equilibrium selection - which strategy will be played in the long run with a high frequency.

Our populations are characterized either by numbers of individuals playing respective strategies in well-mixed populations or by a complete profile - assignment of strategies to players in spatial games. Let Ω be a state space of our system. For non-spatial games with two strategies, $\Omega = \{0, 1, ..., n\}$, where n is the number of players or $\Omega = 2^\Lambda$ for spatial games with players located on the finite subset Λ of $\mathbf{Z}, \mathbf{Z}^2$, or any other infinite graph, and interacting with their neighbours. In well-mixed populations, in discrete moments of times, some individuals switch to a strategy with a higher mean payoff. In spatial games, players choose strategies which are best responses, i.e. ones which maximize the sum of the payoffs obtained from individual games. The

above rules define deterministic dynamics with some stochastic part corresponding to a random matching of players or a random choice of players who may revise their strategies. We call this mutation-free or noise-free dynamics. It is a Markov chain with a state space Ω and a transition matrix P^0. We are especially interested in absorbing states, i.e. rest points of our mutation-free dynamics. Now, with a small probability, ϵ, players may mutate or make mistakes of not chosing the best reply. The presence of mutatation allows the system to make a transition from any state to any other state with a positive probability in some finite number of steps or to stay indefinitively at any state for an arbitrarily long time. This makes our Markov chains with a transition matrix P^ϵ ergodic ones. They have therefore unique stationary measures. To describe the long-run behavior of stochastic dynamics of finite populations, Foster and Young [22] introduced a concept of stochastic stability. A state of the system is **stochastically stable** if it has a positive probability in the stationary measure of the corresponding Markov chain in the zero-noise limit, that is the zero probability of mistakes or the zero-mutation level. It means that along almost any time trajectory the frequency of visiting this state converges to a positive value given by the stationary measure. Let μ^ϵ be the stationary measure of our Markov chain.

Definition 2 $X \in \Omega$ *is* **stochastically stable** *if* $\lim_{\epsilon \to 0} \mu^\epsilon(X) > 0$.

It is a fundamental problem to find stochastically stable states for any stochastic dynamics of interest. We will use the following tree representation of stationary measures of Markov chains proposed by Freidlin and Wentzell [107, 23], see also [83]. Let (Ω, P^ϵ) be an ergodic Markov chain with a state space Ω, transition probabilities given by the transition matrix $P^\epsilon : \Omega \times \Omega \to [0, 1]$, where $P^\epsilon(Y, Y')$ is a conditional probability that the system will be in the state $Y' \in \Omega$ at the time $t + 1$, if it was in the state $Y \in \Omega$ at the time t, and a unique stationary measure, μ^ϵ, also called a stationary state. A stationary state is an eigenvector of P^ϵ corresponding to the eigenvalue 1, i.e. a solution of a system of linear equations,

$$\mu^\epsilon P^\epsilon = \mu^\epsilon, \tag{48}$$

where μ^ϵ is a row wector $[\mu_1^\epsilon, ..., \mu_{|\Omega|}^\epsilon]$. After specific rearrangements one can arrive at an expression for the stationary state which involves only positive terms. This will be very useful in describing the asymptotic behaviour of stationary states.

For $X \in \Omega$, let an X-tree be a directed graph on Ω such that from every $Y \neq X$ there is a unique path to X and there are no outcoming edges out of X. Denote by $T(X)$ the set of all X-trees and let

$$q^\epsilon(X) = \sum_{d \in T(X)} \prod_{(Y,Y') \in d} P^\epsilon(Y, Y'), \tag{49}$$

where the product is with respect to all edges of d.

We have that

$$\mu^\epsilon(X) = \frac{q^\epsilon(X)}{\sum_{Y \in \Omega} q^\epsilon(Y)} \tag{50}$$

for all $X \in \Omega$.

We assume that our noise-free dynamics, i.e. in the case of $\epsilon = 0$, has at least one absorbing state and there are no absorbing sets (recurrent classes) consisting of more than one state. It then follows from (50) that only absorbing states can be stochastically stable.

Let us begin with the case of two absorbing states, X and Y. Consider a dynamics in which $P^\epsilon(Z, W)$ for all $Z, W \in \Omega$, is of order ϵ^m, where m is the number of mistakes involved to pass from Z to W. The noise-free limit of μ^ϵ in the form (50) has a $0/0$ character. Let m_{XY} be a minimal number of mistakes needed to make a transition from the state X to Y and m_{YX} the minimal number of mistakes to evolve from Y to X. Then $q^\epsilon(X)$ is of the order $\epsilon^{m(YX)}$ and $q^\epsilon(Y)$ is of the order $\epsilon^{m(XY)}$. If for example $m_{YX} < m_{XY}$, then $\lim_{\epsilon \to 0} \mu^\epsilon(X) = 1$ hence X is stochastically stable.

In general, to study the zero-noise limit of the stationary measure, it is enough to consider paths between absorbing states. More precisely, we construct X-trees with absorbing states X^k, $k = 1, ..., l$ as vertices; the family of such X-trees is denoted by $\tilde{T}(X)$. Let

$$q_m(X) = max_{d \in \tilde{T}(X)} \prod_{(Y, Y') \in d} \tilde{P}(Y, Y'), \tag{51}$$

where $\tilde{P}(Y, Y') = max \prod_{(W, W')} P(W, W')$, where the product is taken along any path joining Y with Y' and the maximum is taken with respect to all such paths. Now we may observe that if $lim_{\epsilon \to 0} q_m(X^i)/q_m(X^k) = 0$, for every $i = 1, ..., l$, $i \neq k$, then X^k is stochastically stable. Therefore we have to compare trees with the biggest products in (51); such trees are called maximal.

The above characterisation of the stationary measure was used to find stochastically stable states in non-spatial [40, 105, 76, 100, 106, 45] and spatial games [17, 18]. We will use it below in our examples.

In many cases, there exists a state X such that $\lim_{\epsilon \to 0} \mu^\epsilon(X) = 1$ in the zero-noise limit. Then we say that X was selected in the zero-noise limit of a given stochastic dynamics. However, for any low but fixed mutation level, when the number of players is very large, the frequency of visiting any single state can be arbitrarily low. It is an ensemble of states that can have a probability close to one in the stationary measure. The concept of the ensemble stability is discussed in Chapter 9.

8 Stochastic dynamics of well-mixed populations

Here we will discuss stochastic dynamics of well-mixed populations of players interacting in discrete moments of time. We will analyze two-player games

with two strategies and two pure Nash equilibria. The efficient strategy (also called payoff dominant) when played by the whole population results in its highest possible payoff (fitness). The risk-dominant one is played by individuals averse to risk. The strategy is risk dominant if it has a higher expected payoff against a player playing both strategies with equal probabilities [29]. We will address the problem of equilibrium selection - which strategy will be played in the long run with a high frequency.

We will review two models of dynamics of a population with a fixed number of individuals. In both of them, the selection part of the dynamics ensures that if the mean payoff of a given strategy is bigger than the mean payoff of the other one, then the number of individuals playing the given strategy increases. In the first model, introduced by Kandori, Mailath, and Rob [40], one assumes (as in the standard replicator dynamics) that individuals receive average payoffs with respect to all possible opponents - they play against the average strategy. In the second model, introduced by Robson and Vega-Redondo [76], at any moment of time, individuals play only one or few games with randomly chosen opponents. In both models, players may mutate with a small probability, hence the population may move against a selection pressure. Kandori, Mailath, and Rob showed that in their model, the risk-dominant strategy is stochastically stable - if the mutation level is small enough we observe it in the long run with the frequency close to one [40]. In the model of Robson and Vega-Redondo, the efficient strategy is stochastically stable [76, 100]. It is one of very few models in which an efficient strategy is stochastically stable in the presence of a risk-dominant one. The population evolves in the long run to a state with the maximal fitness.

The main goal of this chapter is to investigate the effect of the number of players on the long-run behaviour of the Robson-Vega-Redondo model [54]. We will discuss a sequential dynamics and the one where each individual enjoys each period a revision opportunity with the same probability. We will show that for any arbitrarily low but a fixed level of mutations, if the number of players is sufficiently large, then a risk-dominant strategy is played in the long run with a frequency closed to one - a stochastically stable efficient strategy is observed with a very low frequency. It means that when the number of players increases, the population undergoes a transition between an efficient payoff-dominant equilibrium and a risk-dominant one. We will also show that for some range of payoff parameters, stochastic stability itself depends on the number of players. If the number of players is below certain value (which may be arbitrarily large), then a risk-dominant strategy is stochastically stable. Only if the number of players is large enough, an efficient strategy becomes stochastically stable as proved by Robson and Vega-Redondo.

Combining the above results we see that for a low but fixed noise level, the population undergoes twice a transition between its two equilibria as the number of individuals increases [57]. In addition, for a sufficiently large number of individuals, the population undergoes another equilibrium transition when the noise decreases.

Let us formally introduce our models. We will consider a finite population of n individuals who have at their disposal one of two strategies: A and B. At every discrete moment of time, $t = 1, 2, ...$ individuals are randomly paired (we assume that n is even) to play a two-player symmetric game with payoffs given by the following matrix:

$$U = \quad \begin{array}{c c c} & A & B \\ A & a & b \\ B & c & d, \end{array}$$

where $a > c, d > b, a > d$, and $a + b < c + d$ so (A, A) is an efficient Nash equilibrium and (B, B) is a risk-dominant one.

At the time t, the state of our population is described by the number of individuals, z_t, playing A. Formally, by the state space we mean the set

$$\Omega = \{z, 0 \leq z \leq n\}.$$

Now we will describe the dynamics of our system. It consists of two components: selection and mutation. The selection mechanism ensures that if the mean payoff of a given strategy, $\pi_i(z_t), i = A, B$, at the time t is bigger than the mean payoff of the other one, then the number of individuals playing the given strategy increases in $t + 1$. In their paper, Kandori, Mailath, and Rob [40] write

$$\pi_A(z_t) = \frac{a(z_t - 1) + b(n - z_t)}{n - 1}, \tag{52}$$

$$\pi_B(z_t) = \frac{cz_t + d(n - z_t - 1)}{n - 1},$$

provided $0 < z_t < n$.

It means that in every time step, players are paired infnitely many times to play the game or equivalently, each player plays with every other player and his payoff is the sum of corresponding payoffs. This model may be therefore considered as an analog of replicator dynamics for populations with a fixed numbers of players.

The selection dynamics is formalized in the following way:

$$z_{t+1} > z_t \ \ if \ \ \pi_A(z_t) > \pi_B(z_t), \tag{53}$$

$$z_{t+1} < z_t \ \ if \ \ \pi_A(z_t) < \pi_B(z_t),$$

$$z_{t+1} = z_t \ \ if \ \ \pi_A(z_t) = \pi_B(z_t),$$

$$z_{t+1} = z_t \ \ if \ \ z_t = 0 \ \ or \ \ z_t = n.$$

Now mutations are added. Players may switch to new strategies with the probability ϵ. It is easy to see that for any two states of the population, there is a positive probability of the transition between them in some finite number

of time steps. We have therefore obtained an ergodic Markov chain with $n+1$ states and a unique stationary measure which we denote by μ_n^ϵ. Kandori, Mailath, and Rob proved that the risk-dominant strategy B is stochastically stable [40]

Theorem 8 $\lim_{\epsilon \to 0} \mu_n^\epsilon(0) = 1$

This means that in the long run, in the limit of no mutations, all players play B.

The general set up in the Robson-Vega-Redondo model [76] is the same. However, individuals are paired only once at every time step and play only one game before a selection process takes place. Let p_t denote the random variable which describes the number of cross-pairings, i.e. the number of pairs of matched individuals playing different strategies at the time t. Let us notice that p_t depends on z_t. For a given realization of p_t and z_t, mean payoffs obtained by each strategy are as follows:

$$\tilde{\pi}_A(z_t, p_t) = \frac{a(z_t - p_t) + bp_t}{z_t}, \tag{54}$$

$$\tilde{\pi}_B(z_t, p_t) = \frac{cp_t + d(n - z_t - p_t)}{n - z_t},$$

provided $0 < z_t < n$. Robson and Vega-Redondo showed that the payoff-dominant strategy is stochastically stable [76].

Theorem 9 $\lim_{\epsilon \to 0} \mu_n^\epsilon(n) = 1$

We will outline their proof.

First of all, one can show that there exists k such that if n is large enough and $z_t \geq k$, then there is a positive probability (a certain realization of p_t) that after a finite number of steps of the mutation-free selection dynamics, all players will play A. Likewise, if $z_t < k$ (for any $k \geq 1$), then if the number of players is large enough, then after a finite number of steps of the mutation-free selection dynamics all players will play B. In other words, $z = 0$ and $z = n$ are the only absorbing states of the mutation-free dynamics. Moreover, if n is large enough, then if $z_t \geq n - k$, then the mean payoff obtained by A is always (for any realization of p_t) bigger than the mean payoff obtained by B (in the worst case all B-players play with A-players). Therefore the size of the basin of attraction of the state $z = 0$ is at most $n - k - 1$ and that of $z = n$ is at least $n - k$. Observe that mutation-free dynamics is not deterministic (p_t describes the random matching) and therefore basins of attraction may overlap. It follows that the system needs at least $k+1$ mutations to evolve from $z = n$ to $z = 0$ and at most k mutations to evolve from $z = 0$ to $z = n$. Now using the tree representation of stationary states, Robson and Vega-Redondo finish the proof and show that the efficient strategy is stochastically stable.

However, as outlined above, their proof requires the number of players to be sufficiently large. We will now show that a risk-dominant strategy is

stochastically stable if the number of players is below certain value which can be arbitrarily large.

Theorem 10 *If $n < \frac{2a-c-b}{a-c}$, then the risk-dominant strategy B is stochastically stable in the case of random matching of players.*

Proof: If the population consists of only one B-player and $n - 1$ A-players and if $c > [a(n-2)+b]/(n-1)$, that is $n < (2a-c-b)/(a-c)$, then $\tilde{\pi}_B > \tilde{\pi}_A$. It means that one needs only one mutation to evolve from $z = n$ to $z = 0$. It is easy to see that two mutations are necessary to evolve from $z = 0$ to $z = n$.

To see stochastically stable states, we need to take the limit of no mutations. We will now examine the long-run behavior of the Robson-Vega-Redondo model for a fixed level of mutations in the limit of the infinite number of players.

Now we will analyze the extreme case of the selection rule (53) - a sequential dynamics where in one time unit only one player can change his strategy. Although our dynamics is discrete in time, it captures the essential features of continuous-time models in which every player has an exponentially distributed waiting time to a moment of a revision opportunity. Probability that two or more players revise their strategies at the same time is therefore equal to zero - this is an example of a birth and death process.

The number of A-players in the population may increase by one in $t + 1$, if a B-player is chosen in t which happens with the probability $(n - z_t)/n$. Analogously, the number of B-players in the population may increase by one in $t + 1$, if an A-player is chosen in t which happens with the probability $(z_t)/n$.

The player who has a revision opportunity chooses in $t+1$ with the probability $1 - \epsilon$ the strategy with a higher average payoff in t and the other one with the probability ϵ. Let

$$r(k) = P(\tilde{\pi}_A(z_t, p_t) > \tilde{\pi}_B(z_t, p_t)) \ \ and \ \ l(k) = P(\tilde{\pi}_A(z_t, p_t) < \tilde{\pi}_B(z_t, p_t)).$$

The sequential dynamics is described by the following transition probabilities:

if $z_t = 0$, then $z_{t+1} = 1$ with the probability ϵ and $z_{t+1} = 0$ with the probability $1 - \epsilon$,

if $z_t = n$, then $z_{t+1} = n - 1$ with the probability ϵ and $z_{t+1} = n$ with the probability $1 - \epsilon$,

if $z_t \neq 0, n$, then $z_{t+1} = z_t + 1$ with the probability

$$r(k)\frac{n - z_t}{n}(1 - \epsilon) + (1 - r(k))\frac{n - z_t}{n}\epsilon$$

and $z_{t+1} = z_t - 1$ with the probability

$$l(k)\frac{z_t}{n}(1-\epsilon) + (1-l(k))\frac{z_t}{n}\epsilon.$$

In the dynamics intermediate between the parallel (where all individuals can revise their strategies at the same time) and the sequential one, each individual has a revision opportunity with the same probability $\tau < 1$ during the time interval of the lenght 1. For a fixed ϵ and an arbitrarily large but fixed n, we consider the limit of the continuous time, $\tau \to 0$, and show that the limiting behaviour is already obtained for a sufficiently small τ, namely $\tau < \epsilon/n^3$.

For an interesting discussion on the importance of the order of taking different limits ($\tau \to 0, n \to \infty$, and $\epsilon \to 0$) in evolutionary models (especially in the Aspiration and Imitation model) see Samuelson [79].

In the intermediate dynamics, instead of $(n - z_t)/n$ and z_t/n probabilities we have more involved combinatorial factors. In order to get rid of these inconvenient factors, we will enlarge the state space of the population. The state space Ω' is the set of all configurations of players, that is all possible assignments of strategies to individual players. Therefore, a state $z_t = k$ in Ω consists of $\binom{n}{k}$ states in Ω'. Observe that the sequential dynamics on Ω' is not anymore a birth and death process. However, we are able to treat both dynamics in the same framework.

We showed in [54] that for any arbitrarily low but fixed level of mutation, if the number of players is large enough, then in the long run only a small fraction of the population plays the payoff-dominant strategy. Smaller the mutation level is, fewer players use the payoff-dominant strategy.

The following two theorems were proven in [54].

Theorem 11 *In the sequential dynamics, for any $\delta > 0$ and $\beta > 0$ there exist $\epsilon(\delta, \beta)$ and $n(\epsilon)$ such that for any $n > n(\epsilon)$*

$$\mu_n^\epsilon(z \le \beta n) > 1 - \delta.$$

Theorem 12 *In the intermediate dynamics dynamics, for any $\delta > 0$ and $\beta > 0$ there exist $\epsilon(\delta, \beta)$ and $n(\epsilon)$ such that for any $n > n(\epsilon)$ and $\tau < \frac{\epsilon}{n^3}$*

$$\mu_n^\epsilon(z \le \beta n) > 1 - \delta.$$

We can combine Theorems 9, 10, and 12 and obtain [57]

Theorem 13 *In the intermediate dynamics, for any $\delta > 0$ and $\beta > 0$ there exists $\epsilon(\delta, \beta)$ such that, for all $\epsilon < \epsilon(\delta, \beta)$, there exist $n_1 < n_2 < n_3(\epsilon) < n_4(\epsilon)$ such that*
if $n < n_1 = \frac{2a-c-b}{a-c}$, then $\mu_n^\epsilon(z = 0) > 1 - \delta$,
if $n_2 < n < n_3(\epsilon)$, then $\mu_n^\epsilon(z = n) > 1 - \delta$,
if $n > n_4(\epsilon)$ and $\tau < \epsilon/n^3$, then $\mu_n^\epsilon(z \le \beta n) > 1 - \delta$.

Small τ means that our dynamics is close to the sequential one. We have that $n_3(\epsilon), n_4(\epsilon), n_3(\epsilon) - n_2$, and $n_4(\epsilon) - n_3(\epsilon) \to \infty$ when $\epsilon \to 0$.

It follows from Theorem 13 that the population of players undergoes several **equilibrium transitions**. First of all, for a fixed noise level, when the number of players increases, the population switches from a B-equilibrium, where most of the individuals play the strategy B, to an A-equilibrium and then back to B one. We know that if $n > n_2$, then $z = n$ is stochastically stable. Therefore, for any fixed number of players, $n > n_4(\epsilon)$, when the noise level decreases, the population undergoes a transition from a B-equilibrium to A one. We see that in order to study the long-run behaviour of stochastic population dynamics, we should estimate the relevant parameters to be sure what limiting procedures are appropriate in specific examples.

Let us note that the above theorems concern an ensemble of states, not an individual one. In the limit of the infinite number of players, that is the infinite number of states, every single state has zero probability in the stationary state. It is an ensemble of states that might be stable [51, 53]. The concept of ensemble stability will be discussed in Chapter 9.

9 Spatial games with local interactions

9.1 Nash configurations and stochastic dynamics

In spatial games, players are located on vertices of certain graphs and they interact only with their neighbours; see for example [62, 63, 64, 5, 17, 106, 18, 44, 41, 7, 86, 87, 89, 30, 31, 32, 33] and a recent review paper [90] and references therein.

Let Λ be a finite subset of the simple lattice $\mathbf{Z}^d$. Every site of Λ is occupied by one player who has at his disposal one of m different pure strategies. Let S be the set of strategies, then $\Omega_\Lambda = S^\Lambda$ is the space of all possible configurations of players, that is all possible assignments of strategies to individual players. For every $i \in \Lambda$, X_i is a strategy of the $i-$th player in the configuration $X \in \Omega_\Lambda$ and X_{-i} denotes strategies of all remaining players; X therefore can be represented as the pair (X_i, X_{-i}). Every player interacts only with his nearest neighbours and his payoff is the sum of the payoffs resulting from individual plays. We assume that he has to use the same strategy for all neighbours. Let N_i denote the neighbourhood of the $i-$th player. For the nearest-neighbour interaction we have $N_i = \{j; |j - i| = 1\}$, where $|i - j|$ is the distance between i and j. For $X \in \Omega_\Lambda$ we denote by $\nu_i(X)$ the payoff of the $i-$th player in the configuration X:

$$\nu_i(X) = \sum_{j \in N_i} U(X_i, X_j), \tag{55}$$

where U is a $m \times m$ matrix of payoffs of a two-player symmetric game with m pure strategies.

Definition 3 $X \in \Omega_\Lambda$ *is a* **Nash configuration** *if for every* $i \in \Lambda$ *and* $Y_i \in S$,

$$\nu_i(X_i, X_{-i}) \geq \nu_i(Y_i, X_{-i})$$

Here we will discuss only coordination games, where there are m pure symmetric Nash equilibria and therefore m homogeneous Nash configurations, where all players play the same strategy.

In the Stag-hunt game in Example 1, we have two homogeneous Nash configurations, X^{St} and X^H, where all individuals play St or H respectively.

We describe now the sequential deterministic dynamics of the **best-response rule**. Namely, at each discrete moment of time $t = 1, 2, ...$, a randomly chosen player may update his strategy. He simply adopts the strategy, X_i^{t+1}, which gives him the maximal total payoff $\nu_i(X_i^{t+1}, X_{-i}^t)$ for given X_{-i}^t, a configuration of strategies of remaining players at the time t.

Now we allow players to make mistakes, that is they may not choose best responses. We will discuss two types of such stochastic dynamics. In the first one, the so-called **perturbed best response**, a player follows the best-response rule with probability $1 - \epsilon$ (in case of more than one best-response strategy he chooses randomly one of them) and with probability ϵ he makes a mistake and chooses randomly one of the remaining strategies. The probability of mistakes (or the noise level) is state-independent here.

In the so called **log-linear dynamics**, the probability of chosing by the i−th player the strategy X_i^{t+1} at the time $t+1$ decreases with the loss of the payoff and is given by the following conditional probability:

$$p_i^\epsilon(X_i^{t+1} | X_{-i}^t) = \frac{e^{\frac{1}{\epsilon}\nu_i(X_i^{t+1}, X_{-i}^t)}}{\sum_{Y_i \in S} e^{\frac{1}{\epsilon}\nu_i(Y_i, X_{-i}^t)}}, \tag{56}$$

Let us observe that if $\epsilon \to 0$, p_i^ϵ converges pointwise to the best-response rule. Both stochastic dynamics are examples of ergodic Markov chains with $|S^\Lambda|$ states. Therefore they have unique stationary states denoted by μ_Λ^ϵ.

Stationary states of the log-linear dynamics can be explicitly constructed for the so-called potential games. A game is called a **potential game** if its payoff matrix can be changed to a symmetric one by adding payoffs to its columns [49]. As we know, such a payoff transformation does not change strategic character of the game, in particular it does not change the set of its Nash equilibria. More formally, we have the following definition.

Definition 4 *A two-player symmetric game with a payoff matrix U is a* **potential game** *if there exists a symmetric matrix V, called a potential of the game, such that for any three strategies $A, B, C \in S$*

$$U(A, C) - U(B, C) = V(A, C) - V(B, C). \tag{57}$$

It is easy to see that every game with two strategies has a potential V with $V(A,A) = a - c$, $V(B,B) = d - b$, and $V(A,B) = V(B,A) = 0$. It follows that an equilibrium is risk-dominant if and only if it has a bigger potential.

For players on a lattice, for any $X \in \Omega_\Lambda$,

$$V(X) = \sum_{(i,j) \subset \Lambda} V(X_i, X_j) \tag{58}$$

is then the potential of the configuration X.

For the sequential log-linear dynamics of potential games, one can explicitly construct stationary measures [106].

We begin by the following general definition concerning a Markov chain with a state space Ω and a transition matrix P.

Definition 5 *A measure μ on Ω satisfies a* **detailed balance condition** *if*

$$\mu(X)P_{XY} = \mu(Y)P_{YX}$$

for every $X, Y \in \Omega$

Lemma

If μ satisfies the detailed balance condition then it is a stationary measure

Proof:

$$\sum_{X \in \Omega} \mu(X)P_{XY} = \sum_{X \in \Omega} \mu(Y)P_{YX} = \mu(Y)$$

The following theorem is due Peyton Young [106]. We will present here his proof.

Theorem 14 *The stationary measure of the sequential log-linear dynamics in a game with the potential V is given by*

$$\mu_\Lambda^\epsilon(X) = \frac{e^{\frac{1}{\epsilon}V(X)}}{\sum_{Z \in \Omega_\Lambda} e^{\frac{1}{\epsilon}V(Z)}}. \tag{59}$$

Proof:

We will show that μ_Λ^ϵ in (59) satisfies the detailed balance condition. Let us notice that in the sequential dynamics, $P_{XY} = 0$ unless $X = Y$ or Y differs fom X at one lattice site only, say $i \in \Lambda$.

Let

$$\lambda = \frac{1}{|\Lambda|} \frac{1}{\sum_{Z \in \Omega_\Lambda} e^{\frac{1}{\epsilon}V(Z)}} \frac{1}{\sum_{Z_i \in S} e^{\frac{1}{\epsilon} \sum_{j \in N_i} U(Z_i, X_j)}}$$

Then

$$\mu_\Lambda^\epsilon(X)P_{XY} = \lambda e^{\frac{1}{\epsilon}(\sum_{(h,k)\subset\Lambda} V(X_h,X_k)+\sum_{j\in N_i} U(Y_i,X_j))}$$

$$= \lambda e^{\frac{1}{\epsilon}(\sum_{(h,k)\subset\Lambda} V(X_h,X_k)+\sum_{j\in N_i}(U(X_i,X_j)-V(X_i,X_j)+V(Y_i,X_j)))}$$

$$= \lambda e^{\frac{1}{\epsilon}(\sum_{(h,k)\subset\Lambda} V(Y_h,Y_k)+\sum_{j\in N_i} U(X_i,X_j))} = \mu_\Lambda^\epsilon(Y)P_{YX}.$$

We may now explicitly perform the limit $\epsilon \to 0$ in (59). In the Stag-hunt game, X^H has a bigger potential than X^{St} so $\lim_{\epsilon\to 0}\mu_\Lambda^\epsilon(X^H) = 1$ hence X^H is stochastically stable (we also say that H is stochastically stable).

The concept of a Nash configuration in spatial games is very similar to the concept of a ground-state configuration in lattice-gas models of interacting particles. We will discuss similarities and differences between these two systems of interacting entities in the next section.

9.2 Ground states and Nash configurations

We will present here one of the basic models of interacting particles. In classical lattice-gas models, particles occupy lattice sites and interact only with their neighbours. The fundamental concept is that of a ground-state configuration. It can be formulated conveniently in the limit of an infinite lattice (the infinite number of particles). Let us assume that every site of the $\mathbf{Z}^d$ lattice can be occupied by one of m different particles. An infinite-lattice configuration is an assignment of particles to lattice sites, i.e. an element of $\Omega = \{1,...,m\}^{\mathbf{Z}^d}$. If $X \in \Omega$ and $i \in \mathbf{Z}^d$, then we denote by X_i a restriction of X to i. We will assume here that only nearest-neighbour particles interact. The energy of their interaction is given by a symmetric $m \times m$ matrix V. An element $V(A,B)$ is the interaction energy of two nearest-neighbour particles of the type A and B. The total energy of a system in the configuration X in a finite region $\Lambda \subset \mathbf{Z}^d$ can be then written as

$$H_\Lambda(X) = \sum_{(i,j)\subset\Lambda} V(X_i, X_j). \tag{60}$$

Y is a **local excitation** of X, $Y \sim X$, $Y, X \in \Omega$, if there exists a finite $\Lambda \subset \mathbf{Z}^d$ such that $X = Y$ outside Λ.

For $Y \sim X$, the **relative energy** is defined by

$$H(Y,X) = \sum_{(i,j)\in\mathbf{Z}^d} (V(Y_i,Y_j) - V(X_i,X_j)), \tag{61}$$

where the summation is with respect to pairs of nearest neighbours on $\mathbf{Z}^d$. Observe that this is the finite sum; the energy difference between Y and X is equal to zero outside some finite Λ.

Definition 6 $X \in \Omega$ *is a* **ground-state configuration** *of V if*

$$H(Y, X) \geq 0 \ \ for \ \ any \ \ Y \sim X.$$

That is, we cannot lower the energy of a ground-state configuration by changing it locally.

The energy density $e(X)$ of a configuration X is

$$e(X) = \liminf_{\Lambda \to \mathbf{Z}^2} \frac{H_\Lambda(X)}{|\Lambda|}, \tag{62}$$

where $|\Lambda|$ is the number of lattice sites in Λ.

It can be shown that any ground-state configuration has the minimal energy density [85]. It means that local conditions present in the definition of a ground-state configuration force the global minimization of the energy density.

We see that the concept of a ground-state configuration is very similar to that of a Nash configuration. We have to identify particles with agents, types of particles with strategies and instead of minimizing interaction energies we should maximize payoffs. There are however profound differences. First of all, ground-state configurations can be defined only for symmetric matrices; an interaction energy is assigned to a pair of particles, payoffs are assigned to individual players and may be different for each of them. Ground-state configurations are stable with respect to all local changes, Nash configurations are stable only with respect to one-player changes. It means that for the same symmetric matrix U, there may exist a configuration which is a Nash configuration but not a ground-state configuration for the interaction matrix $-U$. The simplest example is given by the following matrix:

Example 5

$$U = \begin{array}{c c c} & A & B \\ A & 2 & 0 \\ B & 0 & 1 \end{array}$$

(A, A) and (B, B) are Nash configurations for a system consisting of two players but only (A, A) is a ground-state configuration for $V = -U$. We may therefore consider the concept of a ground-state configuration as a refinement of a Nash equilibrium.

For any classical lattice-gas model there exists at least one ground-state configuration. This can be seen in the following way. We start with an arbitrary configuration. If it cannot be changed locally to decrease its energy it is already a ground-state configuration. Otherwise we may change it locally and decrease the energy of the system. If our system is finite, then after a finite number of steps we arrive at a ground-state configuration; at every step we decrease the energy of the system and for every finite system its possible energies form a finite set. For an infinite system, we have to proceed ad infinitum

converging to a ground-state configuration (this follows from the compactness of Ω in the product of discrete topologies). Game models are different. It may happen that a game with a nonsymmetric payoff matrix may not posess a Nash configuration. The classical example is that of the Rock-Scissors-Paper game. One may show that this game dos not have any Nash configurations on $\mathbf{Z}$ and $\mathbf{Z}^2$ but many Nash configurations on the triangular lattice.

In short, ground-state configurations minimize the total energy of a particle system, Nash configurations do not necessarily maximize the total payoff of a population.

Ground-state configuration is an equilibrium concept for systems of interacting particles at zero temperature. For positive temperatures, we must take into account fluctuations caused by thermal motions of particles. Equilibrium behaviour of the system results then from the competition between its energy V and entropy S (which measures the number of configurations corresponding to a macroscopic state), i.e. the minimization of its free energy $F = V - TS$, where T is the temperature of the system - a measure of thermal motions. At the zero temperature, $T = 0$, the minimization of the free energy reduces to the minimization of the energy. This zero-temperature limit looks very similar to the zero-noise limit present in the definition of the stochastic stability. Equilibrium behaviour of a system of interacting particles can be described by specifying probabilities of occurence for all particle configurations. More formally, it is described by a Gibbs state (see [26] and references therein).

We construct it in the following way. Let Λ be a finite subset of $\mathbf{Z}^d$ and ρ_Λ^T the following probability mass function on $\Omega_\Lambda = (1, ..., m)^\Lambda$:

$$\rho_\Lambda^T(X) = (1/Z_\Lambda^T) \exp(-H_\Lambda(X)/T), \tag{63}$$

for every $X \in \Omega_\Lambda$, where

$$Z_\Lambda^T = \sum_{X \in \Omega_\Lambda} \exp(-H_\Lambda(X)/T) \tag{64}$$

is a normalizing factor.

We define a **Gibbs state** ρ^T as a limit of ρ_Λ^T as $\Lambda \to \mathbf{Z}^d$. One can prove that a limit of a translation-invariant Gibbs state for a given interaction as $T \to 0$ is a measure supported by ground-state configurations. One of the fundamental problems of statistical mechanics is a characterization of low-temperature Gibbs states for given interactions between particles.

Let us observe that the finite-volume Gibbs state in (63) is equal to stationary state μ_Λ^ϵ in (59) if we identify T with ϵ and $V \to -V$.

9.3 Ensemble stability

The concept of stochastic stability involves individual configurations of players. In the zero-noise limit, a stationary state is usually concentrated on one

or at most few configurations. However, for a low but fixed noise and for a sufficiently large number of players, the probability of any individual configuration of players is practically zero. The stationary measure, however, may be highly concentrated on an ensemble consisting of one Nash configuration and its small perturbations, i.e. configurations where most players use the same strategy. Such configurations have relatively high probability in the stationary measure. We call such configurations ensemble stable. Let μ_Λ^ϵ be a stationary measure.

Definition 7 $X \in \Omega_\Lambda$ *is* γ**-ensemble stable** *if* $\mu_\Lambda^\epsilon(Y \in \Omega_\Lambda; Y_i \neq X_i) < \gamma$ *for any* $i \in \Lambda$ *if* $\Lambda \supset \Lambda(\gamma)$ *for some* $\Lambda(\gamma)$.

Definition 8 $X \in \Omega_\Lambda$ *is* **low-noise ensemble stable** *if for every* $\gamma > 0$ *there exists* $\epsilon(\gamma)$ *such that if* $\epsilon < \epsilon(\gamma)$, *then* X *is* γ*-ensemble stable.*

If X is γ-ensemble stable with γ close to zero, then the ensemble consisting of X and configurations which are different from X at at most few sites has the probability close to one in the stationary measure. It does not follow, however, that X is necessarily low-noise ensemble or stochastically stable as it happens in examples presented below [51].

Example 6

Players are located on a finite subset Λ of $\mathbf{Z}^2$ (with periodic boundary conditions) and interact with their four nearest neighbours. They have at their disposal three pure strategies: A, B, and C. The payoffs are given by the following symmetric matrix:

$$
U = \quad
\begin{array}{c|ccc}
 & \text{A} & \text{B} & \text{C} \\
\text{A} & 1.5 & 0 & 1 \\
\text{B} & 0 & 2 & 1 \\
\text{C} & 1 & 1 & 2 \\
\end{array}
$$

Our game has three Nash equilibria: $(A, A), (B, B)$, and (C, C), and the corresponding spatial game has three homogeneous Nash configurations: X^A, X^B, and X^C, where all individuals are assigned the same strategy. Let us notice that X^B and X^C have the maximal payoff in every finite volume and therefore they are ground-state configurations for $-U$ and X^A is not.

The unique stationary measure of the log-linear dynamics (56) is is given by (59) with $U = V$ which is a finite-volume Gibbs state (63) with V replaced by $-U$ and T by ϵ. We have

$$
\sum_{(i,j) \subset \Lambda} U(X_i^k, X_j^k) - \sum_{(i,j) \in \Lambda} U(Y_i, Y_j) > 0,
$$

for every $Y \neq X^B$ and X^C, $k = B, C$, and

$$
\sum_{(i,j) \subset \Lambda} U(X_i^B, X_j^B) = \sum_{(i,j) \subset \Lambda} U(X_i^C, X_j^C).
$$

It follows that $\lim_{\epsilon \to 0} \mu_\Lambda^\epsilon(X^k) = 1/2$, for $k = B, C$ so X^B and X^C are stochastically stable. Let us investigate the long-run behaviour of our system for large Λ, that is for a large number of players.

Observe that

$$\lim_{\Lambda \to \mathbf{Z}^2} \mu_\Lambda^\epsilon(X) = 0$$

for every $X \in \Omega = S^{\mathbf{Z}^2}$.

Therefore, for a large Λ we may only observe, with reasonably positive frequencies, ensembles of configurations and not particular configurations. We will be interested in ensembles which consist of a Nash configuration and its small perturbations, that is configurations, where most players use the same strategy. We perform first the limit $\Lambda \to \mathbf{Z}^2$ and obtain an infinite-volume Gibbs state in the temperature $T = \epsilon$,

$$\mu^\epsilon = \lim_{\Lambda \to \mathbf{Z}^2} \mu_\Lambda^\epsilon. \tag{65}$$

In order to investigate the stationary state of our example, we will apply a technique developed by Bricmont and Slawny [8, 9]. They studied low-temperature stability of the so-called dominant ground-state configurations. It follows from their results that

$$\mu^\epsilon(X_i = C) > 1 - \delta(\epsilon) \tag{66}$$

for any $i \in \mathbf{Z}^2$ and $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ [51].

The following theorem is a simple consequence of (66).

Theorem 15 X^C *is low-noise ensemble stable.*

We see that for any low but fixed ϵ, if the number of players is large enough, then in the long run, almost all players use C strategy. On the other hand, if for any fixed number of players, ϵ is lowered substantially, then B and C appear with frequencies close to $1/2$.

Let us sketch briefly the reason of such a behavior. While it is true that both X^B and X^C have the same potential which is the half of the payoff of the whole system (it plays the role of the total energy of a system of interacting particles), the X^C Nash configuration has more lowest-cost excitations. Namely, one player can change its strategy and switch to either A or B and the potential will decrease by 4 units. Players in the X^B Nash configuration have only one possibility, that is to switch to C; switching to A decreases the potential by 8. Now, the probability of the occurrence of any configuration in the Gibbs state (which is the stationary state of our stochastic dynamics) depends on the potential in an exponential way. One can prove that the probability of the ensemble consisting of the X^C Nash configuration and configurations which are different from it at few sites only is much bigger than the probability of the analogous X^B-ensemble. It follows from the fact that the X^C-ensemble has many more configurations than the X^B-ensemble. On the

other hand, configurations which are outside X^B and X^C-ensembles appear with exponentially small probabilities. It means that for large enough systems (and small but not extremely small ϵ) we observe in the stationary state the X^C Nash configuration with perhaps few different strategies. The above argument was made into a rigorous proof for an infinite system of the closely related lattice-gas model (the Blume-Capel model) of interacting particles by Bricmont and Slawny in [8].

In the above example, X^B and X^C have the same total payoff but X^C has more lowest-cost excitations and therefore it is low-noise ensemble stable. We will now discuss the situation, where X^C has a smaller total payoff but nevertheless in the long run C is played with a frequency close to 1 if the noise level is low but not extremely low. We will consider a family of games with the following payoff matrix:

Example 7

$$
U = \quad
\begin{array}{c c c c}
 & A & B & C \\
A & 1.5 & 0 & 1 \\
B & 0 & 2+\alpha & 1 \\
C & 1 & 1 & 2,
\end{array}
$$

where $\alpha > 0$ so B is both payoff and pairwise risk-dominant.

We are interested in the long-run behavior of our system for small positive α and low ϵ. One may modify the proof of Theorem 15 and obtain the following theorem [51].

Theorem 16 *For every $\gamma > 0$, there exist $\alpha(\gamma)$ and $\epsilon(\gamma)$ such that for every $0 < \alpha < \alpha(\gamma)$, there exists $\epsilon(\alpha)$ such that for $\epsilon(\alpha) < \epsilon < \epsilon(\gamma)$, X^C is γ-ensemble stable, and for $0 < \epsilon < \epsilon(\alpha)$, X^B is γ-ensemble stable.*

Observe that for $\alpha = 0$, both X^B and X^C are stochastically stable (they appear with the frequency $1/2$ in the limit of zero noise) but X^C is low-noise ensemble stable. For small $\alpha > 0$, X^B is both stochastically (it appears with the frequency 1 in the limit of zero noise) and low-noise ensemble stable. However, for an intermediate noise $\epsilon(\alpha) < \epsilon < \epsilon(\gamma)$, if the number of players is large enough, then in the long run, almost all players use the strategy C (X^C is ensemble stable). If we lower ϵ below $\epsilon(\alpha)$, then almost all players start to use the strategy B. $\epsilon = \epsilon(\alpha)$ is the line of the first-order phase transition. In the thermodynamic limit, there exist two Gibbs states (equilibrium states) on this line. We may say that at $\epsilon = \epsilon(\alpha)$, the population of players undergoes a sharp **equilibrium transition** from C to B-behaviour.

9.4 Stochastic stability in non-potential games

Let us now consider non-potential games with three strategies and three symmetric Nash equilibria: $(A, A), (B, B)$, and (C, C). Stationary measures of such games cannot be explicitly constructed. To find stochastically stable states we

will use here the tree representation of stationary measures described in Chapter 7. We will discuss some interesting examples.

Example 8

Players are located on a finite subset of the one-dimensional lattice $\mathbf{Z}$ and interact with their nearest neighbours only. Denote by n the number of players. For simplicity we will assume periodic boundary conditions, that is we will identify the $n+1$-th player with the first one. In other words, the players are located on the circle.

The payoffs are given by the following matrix:

$$U = \begin{array}{c|ccc} & A & B & C \\ \hline A & 1+\alpha & 0 & 1.5 \\ B & 0 & 2 & 0 \\ C & 0 & 0 & 3 \end{array}$$

with $0 < \alpha \leq 0.5$.

As before, we have three homogeneous Nash configurations: X^A, X^B, and X^C. The log-linear and perturbed best-response dynamics for this game were discussed in [52].

Let us note that X^A, X^B, and X^C are the only absorbing states of the noise-free dynamics. We begin with a stochastic dynamics with a state-independent noise. Let us consider first the case of $\alpha < 0.5$.

Theorem 17 *If $0 < \alpha < 0.5$, then X^C is stochastically stable in the perturbed best-response dynamics.*

Proof: It is easy to see that $q_m(X^C)$ is of the order ϵ^2, $q_m(X^B)$ is of the order $\epsilon^{\frac{n}{2}+1}$, and $q_m(X^A)$ is of the order ϵ^{n+2}.

Let us now consider the log-linear rule.

Theorem 18 *If $n < 2 + 1/(0.5 - \alpha)$, then X^B is stochastically stable and if $n > 2 + 1/(0.5 - \alpha)$, then X^C is stochastically stable in the log-linear dynamics.*

Proof: The following are maximal A-tree, B-tree, and C-tree:

$$B \to C \to A, \quad C \to A \to B, \quad A \to B \to C,$$

where the probability of $A \to B$ is equal to

$$\frac{1}{1 + 1 + e^{\frac{1}{\epsilon}(2+2\alpha)}} \Big(\frac{1}{1 + e^{-\frac{2}{\epsilon}} + e^{\frac{1}{\epsilon}(-1+\alpha)}}\Big)^{n-2} \frac{1}{1 + e^{-\frac{4}{\epsilon}} + e^{-\frac{4}{\epsilon}}}, \qquad (67)$$

the probability of $B \to C$ is equal to

$$\frac{1}{1 + 1 + e^{\frac{4}{\epsilon}}} \Big(\frac{1}{1 + e^{-\frac{1}{\epsilon}} + e^{-\frac{1.5}{\epsilon}}}\Big)^{n-2} \frac{1}{1 + e^{-\frac{6}{\epsilon}} + e^{-\frac{3}{\epsilon}}}, \qquad (68)$$

and the probability of $C \to A$ is equal to

$$\frac{1}{1 + e^{-\frac{3}{\epsilon}} + e^{\frac{3}{\epsilon}}}\left(\frac{1}{1 + e^{-\frac{1}{\epsilon}(2.5+\alpha)} + e^{\frac{1}{\epsilon}(0.5-\alpha)}}\right)^{n-2}\frac{1}{1 + e^{-\frac{2}{\epsilon}(1+\alpha)} + e^{-\frac{2}{\epsilon}(1+\alpha)}},$$

(69)

Let us observe that

$$P_{B \to C \to A} = O(e^{-\frac{1}{\epsilon}(7+(0.5-\alpha)(n-2))}),$$

(70)

$$P_{C \to A \to B} = O(e^{-\frac{1}{\epsilon}(5+2\alpha+(0.5-\alpha)(n-2))}),$$

(71)

$$P_{A \to B \to C} = O(e^{-\frac{1}{\epsilon}(6+2\alpha)}),$$

(72)

where $\lim_{x \to 0} O(x)/x = 1$.

Now if $n < 2 + 1/(0.5 - \alpha)$, then

$$\lim_{\epsilon \to 0}\frac{q_m(X^C)}{q_m(X^B)} = \lim_{\epsilon \to 0}\frac{P_{A \to B \to C}}{P_{C \to A \to B}} = 0$$

(73)

which finishes the proof.

It follows that for a small enough n, X^B is stochastically stable and for a large enough n, X^C is stochastically stable. We see that adding two players to the population may change the stochastic stability of Nash configurations. Let us also notice that the strategy C is globally risk dominant. Nevertheless, it is not stochastically stable in the log-linear dynamics for a sufficiently small number of players.

Let us now discuss the case of $\alpha = 0.5$ [52].

Theorem 19 *If $\alpha = 0.5$, then X^B is stochastically stable for any n in the log-linear dynamics.*

Proof:

$$\lim_{\epsilon \to 0}\frac{q_m(X^C)}{q_m(X^B)} = \lim_{\epsilon \to 0}\frac{e^{-\frac{4}{\epsilon}}e^{-\frac{3}{\epsilon}}}{(1/2)^{n-2}e^{-\frac{3}{\epsilon}}e^{-\frac{3}{\epsilon}}} = 0.$$

X^B is stochastically stable which means that for any fixed number of players, if the noise is sufficiently small, then in the long run we observe B players with an arbitrarily high frequency. However, we conjecture that for any low but fixed noise, if the number of players is big enough, the stationary measure is concentrated on the X^C-ensemble. We expect that X^C is ensemble stable because its lowest-cost excitations occur with a probability of the order $e^{-\frac{3}{\epsilon}}$ and those from X^B with a probability of the order $e^{-\frac{4}{\epsilon}}$. We observe this phenomenon in Monte-Carlo simulations.

Example 9

Players are located on a finite subset Λ of $\mathbf{Z}$ (with periodic boundary conditions) and interact with their two nearest neighbours. They have at their

disposal three pure strategies: A, B, and C. The payoffs are given by the following matrix [51]:

$$U = \begin{array}{c|ccc} & A & B & C \\ A & 3 & 0 & 2 \\ B & 2 & 2 & 0 \\ C & 0 & 0 & 3 \end{array}$$

Our game has three Nash equilibria: $(A, A), (B, B)$, and (C, C). Let us note that in pairwise comparisons, B risk dominates A, C dominates B and A dominates C. The corresponding spatial game has three homogeneous Nash configurations: X^A, X^B, and X^C. They are the only absorbing states of the noise-free best-response dynamics.

Theorem 20 X^B *is stochastically stable*

   Proof: The following are maximal A-tree, B-tree, and C-tree:

$$B \to C \to A, \quad C \to A \to B, \quad A \to B \to C.$$

   Let us observe that

$$P_{B \to C \to A} = O(e^{-\frac{6}{\epsilon}}), \tag{74}$$

$$P_{C \to A \to B} = O(e^{-\frac{4}{\epsilon}}), \tag{75}$$

$$P_{A \to B \to C} = O(e^{-\frac{6}{\epsilon}}). \tag{76}$$

The theorem follows from the tree characterization of stationary measures.

   X^B is stochastically stable because it is much more probable (for low ϵ) to escape from X^A and X^C than from X^B. The relative payoffs of Nash configurations are not relevant here (in fact X^B has the smallest payoff). Let us recall Example 7 of a potential game, where an ensemble-stable configuration has more lowest-cost excitations. It is easier to escape from an ensemble-stable configuration than from other Nash configurations.

   Stochatic stability concerns single configurations in the zero-noise limit; ensemble stability concerns families of configurations in the limit of the infinite number of players. It is very important to investigate and compare these two concepts of stability in nonpotential games.

   Non-potential spatial games cannot be directly presented as systems of interacting particles. They constitute a large family of interacting objects not thoroughly studied so far by methods statistical physics. Some partial results concerning stochastic stability of Nash equilibria in non-potential spatial games were obtained in [17, 18, 5, 53, 52].

   One may wish to say that A risk dominates the other two strategies if it risk dominates them in pairwise comparisons. In Example 9, B dominates A, C dominates B, and finally A dominates C. But even if we do not have such a cyclic relation of dominance, a strategy which is pairwise risk-dominant may

not be stochastically stable as in the case of Example 8. A more relevant notion seems to be that of a global risk dominance [45]. We say that A is globally risk dominant if it is a best response to a mixed strategy which assigns probability $1/2$ to A. It was shown in [17, 18] that a global risk-dominant strategy is stochastically stable in some spatial games with local interactions.

A different criterion for stochastic stability was developed by Blume [5]. He showed (using techniques of statistical mechanics) that in a game with m strategies A_i and m symmetric Nash equilibria (A_k, A_k), $k = 1, ..., m$, A_1 is stochastically stable if

$$\min_{k>1}(U(A_1, A_1) - U(A_k, A_k)) > \max_{k>1}(U(A_k, A_k) - U(A_1, A_k)). \qquad (77)$$

We may observe that if A_1 satisfies the above condition, then it is pairwise risk dominant.

9.5 Dominated strategies

We say that a pure strategy is **strictly dominated** by another (pure or mixed) strategy if it gives a player a lower payoff than the other one regardless of strategies chosen by his opponents.

Definition 9 $k \in S$ is strictly dominated by $y \in \Delta$ if $U_i(k, w_{-i}) < U_i(y, w_{-i})$ for every $w \in \Delta^I$.

Let us see that a strategy can be strictly dominated by a mixed strategy without being strictly dominated by any pure strategy in its support.

Example 10

	A	B	C
A	5	1	3
U = B	2	2	2
C	1	5	3

B is strictly dominated by a mixed strategy assigning the probability $1/2$ both to A and C but is strictly dominated neither by A nor by C.

It is easy to see that strictly dominated pure strategies cannot be present in the support of any Nash equilibrium.

In the replicator dynamics (16), all strictly dominated pure strategies are wiped out in the long run if all strategies are initially present [1, 78].

Theorem 21 *If a pure strategy k is strictly dominated, then $\xi_k(t, x^0) \rightarrow_{t \to \infty} 0$ for any $x^0 \in interior(\Delta)$.*

Strictly dominated strategies should not be used by rational players and consequently we might think that their presence should not have any impact on the long-run behaviour of the population. We will show that in the

best-reply dynamics, if we allow players to make mistakes, this may not be necessarily true. Let us consider the following game with a strictly dominated strategy and two symmetric Nash equilibria [51].

Example 11

$$
U = \quad
\begin{array}{c c c c}
 & \text{A} & \text{B} & \text{C} \\
\text{A} & 0 & 0.1 & 1 \\
\text{B} & 0.1 & 2+\alpha & 1.1 \\
\text{C} & 1.1 & 1.1 & 2,
\end{array}
$$

where $\alpha > 0$.

We see that strategy A is strictly dominated by both B and C, hence X^A is not a Nash configuration. X^B and X^C are both Nash configurations but only X^B is a ground-state configuration for $-U$. In the absence of A, B is both payoff and risk-dominant and therefore is stochastically stable and low-noise ensemble stable. Adding the strategy A does not change dominance relations; B is still payoff and pairwise risk dominant. However, Example 11 fulfills all the assumptions of Theorem 16 and we get that X^C is γ-ensemble stable at intermediate noise levels. The mere presence of a strictly dominated strategy A changes the long-run behaviour of the population.

Similar results were discussed by Myatt and Wallace [58]. In their games, at every discrete moment of time, one of the players leaves the population and is replaced by another one who plays the best response. The new player calculates his best response with respect to his own payoff matrix which is the matrix of a common average payoff modified by a realization of some random variable with the zero mean. The noise does not appear in the game as a result of players' mistakes but is the effect of their idiosyncratic preferences. The authors then show that the presence of a strictly dominated strategy may change the stochastic stability of Nash equilibria. However, the reason for such a behavior is different in their and in our models. In our model, it is relatively easy to get out of X^C and this makes X^C ensemble stable. Mayatt and Wallace introduce a strictly dominated strategy in such a way that it is relatively easy to make a transition to it from a risk and payoff-dominant equilibrium and then with a high probability the population moves to a second Nash configuration which results in its stochastic stability.

This is exactly a mechanism present in Examples 8 and 9.

10 Review of other results

We discussed the long-run behaviour of populations of interacting individuals playing games. We have considered deterministic replicator dynamics and stochastic dynamics of finite populations.

In spatial games, individuals are located on vertices of certain graphs and they interact only with their neighbours.

In this paper, we considered only simple graphs - finite subsets of the regular $\mathbf{Z}$ or $\mathbf{Z}^2$ lattice. Recently there appeared many interesting results of evolutionary dynamics on random graphs, Barabasi-Albert free-scale graphs, and small-world networks [88, 91, 101, 102, 90, 80, 81, 82, 3]. Especially the Prisoner's Dilemma was studied on such graphs and it was shown that their heterogeneity favors the cooperation in the population [80, 81, 82, 90].

In well-mixed populations, individuals are randomly matched to play a game. The deterministic selection part of the dynamics ensures that if the mean payoff of a given strategy is bigger than the mean payoff of the other one, then the number of individuals playing the given strategy increases. In discrete moments of time, individuals produce offspring proportional to their payoffs. The total number of individuals is then scaled back to the previous value so the population size is constant. Individuals may mutate so the population may move against a selection pressure. This is an example of a stochastic frequency-dependent Wright-Fisher process [20, 21, 108, 12, 19].

There are also other stochastic dynamics of finite populations. The most important one is the Moran process [50, 12, 19]. In this dynamics, at any time step a single individual is chosen for reproduction with the probability proportional to his payoff, and then his offspring replaces the random chosen individual. It was showed recently that in the limits of the infinite population, the Moran process results in the replicator dynamics [95, 96].

The stochastic dynamics of finite populations has been extensively studied recently [65, 93, 31, 38, 43, 69, 70, 71, 98, 99]. The notion of an evolutionarily stable strategy for finite populations was introduced [65, 93, 61, 104, 15, 97]. One of the important quantity to calculate is the fixation probability of a given strategy. It is defined as the probability that a strategy introduced into a population by a single player will take over the whole population. Recently, Nowak et. al. [65] have formulated the following weak selection 1/3 law. In two-player games with two strategies, selection favors the strategy A replacing B if the fraction of A-players in the population for which the average payoff for the strategy A is equal to the average payoff of the strategy B if is smaller than 1/3, i.e. the mixed Nash equilibrium for this game is smaller than 1/3. The 1/3 law was proven to hold both in the Moran [65, 93] and the Wright-Fisher process [38].

In this review we discussed only two-player games. Multi-player games were studied recently in [42, 10, 11, 53, 28, 74, 39, 75].

We have not discussed at all population genetics in the context of game theory. We refer to [36, 12] for results and references.

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