Stochastic stability in three-player games

Dominik Kamiński
Jacek Miękisz
Marcin Zaborowski
Institute of Applied Mathematics and Mechanics
Warsaw University
ul. Banacha 2
02-097 Warsaw, Poland

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Abstract. Animal behavior and evolution can often be described by game-theoretic models. Although in many situations, the number of players is very large, their strategic interactions are usually decomposed into a sum of two-player games. Only recently evolutionarily stable strategies were defined for multi-player games and their properties analyzed (Broom *et al.,* 1997). Here we study the long-run behavior of stochastic dynamics of populations of randomly matched individuals playing symmetric three-player games. We analyze stochastic stability of equilibria in games with multiple evolutionarily stable strategies. We also show that in some games, a population may not evolve in the long run to an evolutionarily stable equilibrium.

1 Introduction

Long-run behavior of interacting individuals can often be described within game-theoretic models. The basic notion here is that of a Nash equilibrium. This is a state of a population - an assignment of strategies to players - such that no player, for fixed strategies of his opponents, has an incentive to deviate from his current strategy; the change cannot increase his payoff. Maynard Smith and Price (1973) have refined this concept of equilibrium to include the stability of Nash equilibria against mutants (see also Maynard Smith, 1982). They introduced the fundamental notion of an evolutionarily stable strategy. If everybody plays such a strategy, then the small number of mutants playing a different strategy is eliminated from the population.

Although in many models, the number of players is very large, their strategic interactions are usually decomposed into a sum of two-player games. However, truly multi-player games naturally appear in many situations. For example, in the stag-hunt game, there are n players who choose simultaneously one of two actions: either join the stag hunt (strategy S) or go after a hare (strategy H). The strategy H yields a safe but low fixed payoff. The strategy S gives the higher payoff if at least k players join the stag hunt and split the reward. The game has two Nash equilibria: the efficient one, where all players choose S and the one in which players are averse to risks and play H. This is a typical situation with a selection problem. The long-run outcome of such games is the subject of this paper. In particular, we will show that in the stag-hunt game, strategy S is played with a high frequency in the adaptive dynamics in the presence of mutations.

Recently there have appeared some systematic studies of truly multi-player games. Broom *et al.* (1997) defined evolutionarily stable strategies for multi-player games and analyzed their properties. Kim (1996) investigated an asymptotic and stochastic stability of Nash equilibria in multi-player games.

Bukowski and Miękisz (2004) provided a classification of symmetric three-player games with two strategies. For certain payoff parameters, such games have multiple evolutionarily stable strategies. In one class of three-player games, there are two pure evolutionarily stable strategies like in the stag-hunt game. In another class, we have one pure and one mixed evolutionarily stable strategy. In both cases we are faced with a standard problem of equilibrium selection. We will approach this problem from a dynamical point of view.

The dynamical interpretation of the evolutionarily stable strategy was provided by several authors (Taylor and Jonker, 1978; Hofbauer *et al.*, 1979; Zeeman, 1981). They proposed a system of differential or difference equations, the so-called replicator equations, which describe the time-evolution of frequencies of strategies. It is known that any evolutionarily stable strategy is an asymptotically stable stationary point of such dynamics (Weibull, 1995; Hofbauer and Sigmund, 1998 and 2003).

Here we discuss a discrete-time stochastic adaptation dynamics of a population of a fixed number of individuals playing three-player games with two strategies. The selection part of the dynamics ensures that if the mean payoff of a given strategy at the time t is bigger than the mean payoff of the other one, then the number of individuals playing the given strategy should increase in $t+1$. We assume that individuals are randomly matched with their opponents. Such model was introduced and analyzed for the case of two-player games by Robson and Vega-Redondo (1996a, 1996b). Alternatively, if the rate of playing games is much bigger than the rate of adaptation, then one may assume (as in the standard replicator dymanics) that individuals receive average payoffs with respect to all possible opponents - they play against the average strategy (Kandori *et al.*, 1993). In both models, players may mutate with a small probability hence the population may move against a selection pressure. To describe the long-run behavior of stochastic dynamics, Foster and Young (1990) introduced a concept of stochastic stability. A configuration of a system (a number of players playing the first strategy in our models) is stochastically stable if it has a positive probability in the stationary state of the dynamics in the limit of no mutations. It means that in the long run we observe it with a positive frequency. In the Kandori-Mailath-Rob model, the risk-dominant strategy (the one which has a higher expected payoff against a player playing both strategies with equal probabilities) is stochastically stable - if the mutation level is sufficiently small, we observe it in the long run with the frequency close to one. In the model of Robson and Vega-Redondo, the payoff-dominant strategy (also called efficient) is stochastically stable. It is one of very few models in which a payoff-dominant strategy is stochastically stable in the presence of a risk-dominant one.

Here we analyze the model of Robson and Vega-Redondo in the case of symmetric three-player games. We characterize stochastic stability of equilibria in all generic classes of such games. In particular, in games with multiple evolutionarily stable strategies we show that only one is stochastically stable and therefore we resolve the problem of equilibrium selection. We show that stochastic stability may depend on the number of players. We also show that in some games, a population may not evolve in the long run to an evolutionarily stable equilibrium.

In Section 2, we introduce three-player games. In Section 3, we discuss adaptive dynamics with random matching of players. Results are presented in Section 4. Discussion follows in Section 5. In Appendix A, we present a tree representation of stationary states of irreducible Markov chains. Some more technical proofs are given in Appendix B.

2 Three-player games

To characterize a game-theoretic model, one has to specify the set of players, strategies they have at their disposal and payoffs they receive. The payoff of any player depends not only on his strategy but also on strategies played by his opponents. We will discuss symmetric three-player games with two strategies. In symmetric games, all players assume the same role in the game. It is enough therefore to specify payoffs of one player. It is given by two matrices:

$$U = \left(\begin{pmatrix} u_{111} & u_{121} \\ u_{211} & u_{221} \end{pmatrix}, \begin{pmatrix} u_{112} & u_{122} \\ u_{212} & u_{222} \end{pmatrix} \right), \tag{1}$$

where u_{ijk}, $i, j, k = A, B$ is a payoff of the first (row) player when he plays the strategy i when the second (column) player plays the strategy j, and the third (matrix) player plays k.

For example, if we assume that in the three-player stag-hunt game, a stag is worth, say, 6 units and a hare 1 unit and one needs all three hunters to get a stag and share a reward, then the payoffs are given by

$$U_1 = \left(\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right) \tag{2}$$

In general we assume that the payoff of any player depends only on his strategy and numbers of players playing different types of strategies. We consider the following payoffs:

$$U_2 = \left(\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \right) \tag{3}$$

Let $a = a_1 - a_2, b = b_2 - b_1$, and $c = c_2 - c_1$. We will discuss all three classes of generic symmetric three-player games; see (Bukowski and Miękisz, 2004) for the complete classification.

In the first class, we have $a > 0$ and $c > 0$ and therefore two evolutionarily stable strategies: A and B.

A mixed strategy is a probability distribution on the set of pure strategies. It can be represented by $x, 0 \leq x \leq 1$, that is a probability of playing the first strategy A. The payoffs of A and B against x are given by the expected values, $a_1x^2 + 2b_1x(1-x) + c_1(1-x)^2$ and $a_2x^2 + 2b_2x(1-x) + c_2(1-x)^2$ respectively. Any x^* for which the above two expected values are equal is a mixed Nash equilibrium. x^* can also be interpreted as a fraction of the population playing the first strategy in equilibrium.

In the first class of games, in addition to two pure Nash equilibria, there exists also an unstable mixed Nash equilibrium, $x^* = \frac{(b-c)+\sqrt{b^2+ac}}{a+2b-c}$ if $b \neq (c-a)/2$ and $x^* = \frac{c}{a+c}$ otherwise.

In our second class of games, we have $a > 0$ and $c < 0$ (analogous results hold for $a < 0 < c$). Then A is the only pure evolutionarily stable strategy. If $b > \sqrt{|ac|}$, then we have in addition a mixed evolutionarily stable strategy, $y^* = \frac{(b-c)-\sqrt{b^2+ac}}{a+2b-c}$, and an unstable mixed Nash equilibrium x^*.

In the third class of games, $a < 0$ and $c < 0$ and there is a unique mixed evolutionarily stable strategy x^*.

Below we study stability of above evolutionarily stable strategies in adaptive stochastic dynamics with random matching of players.

3 Adaptive dynamics with random matching of players

We consider a finite population of n individuals who have at their disposal one of two strategies: A and B. They are randomly matched in triples (we assume that n is divisible by 3) to play a three-player symmetric game with payoffs given by U_2.

At every discrete moment of time, $t = 1, 2, ...$, the state of the population is described by the number of individuals, z_t, playing A. Formally, by the state space we mean the set $\Omega = \{z, 0 \leq z \leq n\}$. Due to the random nature of matching, average payoffs of strategies depend on the realization of two random variables: p_t - the number of triples in which there are exactly two A-players and q_t - the number of triples with exactly one A-player. It follows that $\frac{z_t-2p_t-q_t}{3}$ is the number of triples, where all individuals play the first strategy.

Now we will describe the dynamics of our system. It consists of two components: selection and mutation. The selection mechanism ensures that if the average payoff of a given strategy, $\pi_i(z_t, p_t, q_t), i = A, B$, at the time t, is bigger than the average payoff of the other one, then the number of individuals playing the given strategy should increase in $t + 1$. We assume that at any time period, each individual has a revision opportunity with a small positive probability τ.

Average payoffs are given by following expressions:

$$\pi_A(z_t,p_t,q_t) = \frac{a_1(z_t - 2p_t - q_t) + 2b_1 p_t + c_1 q_t}{z_t}, \tag{4}$$

$$\pi_B(z_t,p_t,q_t) = \frac{a_2 p_t + 2b_2 q_t + c_2(n - z_t - p_t - 2q_t)}{(n - z_t)},$$

provided $0 < z_t < n$.

The selection dynamics is formalized in the following way:

$$z_{t+1} > z_t \ \ if \ \ \pi_A(z_t,p_t,q_t) > \pi_B(z_t,p_t,q_t), \tag{5}$$

$$z_{t+1} < z_t \ \ if \ \ \pi_A(z_t,p_t,q_t) < \pi_B(z_t,p_t,q_t),$$

$$z_{t+1} = z_t \ \ if \ \ \pi_A(z_t,p_t,q_t) = \pi_B(z_t,p_t,q_t),$$

$$z_{t+1} = z_t \ \ if \ \ z_t = 0 \ \ or \ \ z_t = n.$$

Now mutations are added. At every time period, each player who has a revision opportunity, instead of following the selection rule may adopt the other strategy with a small probability ϵ. It is easy to see, that for any two states of the population, there is a positive probability of the transition between them in some finite number of time steps. We have therefore obtained an irreducible Markov chain with $n+1$ states. It has a unique stationary state (a probability mass function) which we denote by μ_n^ϵ. For any $z \in \Omega$, $\mu_n^\epsilon(z)$ is the frequency of visiting the state z in the long run. The following definition was first introduced by Foster and Young (1990).

Definition $z \in \Omega$ is stochastically stable if $\lim_{\epsilon \to 0} \mu_n^\epsilon(z) > 0$.

In most cases below, there exists a state for which the above limit is actually equal to 1. In particular, unless we explicitly desribe the situation, if $z = n$ ($z = 0$) is stochastically stable, then in the long run, in the limit of no mutations, all individuals play the strategy A (B). We also say that respective strategies are stochastically stable.

4 Stochastic stability of evolutionarily stable strategies

We study here the stability of evolutionarily stable strategies of three-player games in described above adaptive stochastic dynamics with random matching of players. We present our results and give some proofs. All remaining proofs are given in Appendix B. They are based on a certain tree representation of stationary states of irreducible Markov chains (Freidlin and Wentzell, 1970 and 1984; see Appendix A). We assume that at any time period, each individual has a revision opportunity with a small positive probability τ. It follows that $z = 0$ and $z = n$ are the only absorbing states. After a finite number of steps of the mutation-free dynamics we arrive at one of these two states and stay there forever - there are no other recurrence classes. Therefore to obtain a stationary state in the limit of no mutations, it is enough to count a number of mutations the population needs to evolve between these states. If one requires for example fewer mutations to evolve from $z = 0$ to $z = n$ than from $z = n$ to $z = o$, then $z = n$ is stochastically stable. This means that in the long run, if the mutation level is sufficiently low, then almost all individuals play A.

For games in the first class, without a loss of generality, we assume that $a_1 > c_2$. Under an additional condition, we have the following theorem (a proof is given in Appendix B).

Theorem 1 *If $a, c > 0$, $a_1 > b_2, c_2$, and n is sufficiently big, then A is stochastically stable.*

In particular, we see that in the stag-hunt game with payoffs given by U_1, the strategy S is stochastically stable if the number of players is sufficiently large. Observe that for the stag-hunt game with just three players, one needs three mutations to evolve from $z = 0$ to $z = 3$ and only one mutation to evolve from $z = 3$ to $z = 0$, hence H is stochastically stable. We see that the stochastic stability depends on the number of players.

In general, we have the following theorem.

Theorem 2 *If $a, c > 0$, $a_1 > c_2$, $b_1 < c_2$, and $(a_1(n-3) + c_1)/(n-2) < b_2 < a_1$, then B is stochastically stable.*

Proof: We have that $\pi_B(z = 1) > \pi_A(z = 1)$ and $\pi_B(z = n-1) < \pi_A(z = n-1)$ for sufficiently big n. For $z = n-2$, if two B-players are matched with one A-player, then $\pi_B > \pi_A$ and for $z = 2$, for two possible matchings $\pi_B > \pi_A$. Hence the population needs only two mutations to evolve from $z = n$ to $z = 0$ and three mutations to evolve from $z = 0$ to $z = n$. It follows from the tree representation of stationary states (see Appendix A) that B is stochastically stable. □

It follows from Theorem 2 that if $b_2 < a_1$ and $n < n^* = (3a_1 - 2b_2 - c_1)/(a_1 - b_2)$, then B is stochastically stable. Observe that n^* can be arbitrarily big. However, Theorem 1 tells us that A is stochastically stable if n is big enough. We see that when n increases, the population undergoes a transition between its equilibria.

Let us discuss now the case $a_1 < b_2$. If $b_1 < c_2$, then we may repeat the proof of Theorem 2 and obtain

Theorem 3 *If $a, c > 0$, $a_1 > c_2, a_1 < b_2$, and $b_1 < c_2$, then B is stochastically stable if n is sufficiently big.*

If $b_1 > c_2$, then the population needs two mutations to evolve both from $z = n$ to $z = 0$ and from $z = 0$ to $z = n$. We get the following theorem.

Theorem 4 *If $a, c > 0$, $a_1 > c_2, a_1 < b_2$, and $b_1 > c_2$, then both A and B are stochastically stable if n is sufficiently big.*

In games in the second class, the only pure evolutionarily stable strategy is stochastically stable. Without a loss of generality, we assume that $a > 0$ and $c < 0$ and have the following theorem.

Theorem 5 *If $c < 0 < a$ and n is sufficiently big, then A is stochastically stable.*

Proof: It is easy to see that for a big n, the population needs one mutation to evolve from $z = 0$ to $z = n$ and at least two mutations to evolve from $z = n$ to $z = 0$. $\Box$

However, we observe that for a fixed n, if $a_1 < c_2$, b_1 is sufficiently small (negative and big in the absolute value) and b_2 is sufficiently big, then $\pi_B(z) > \pi_A(z)$ for any $z \neq 0, n$ and all matchings. It follows that for such payoffs, B is stochastically stable.

We will now show that for certain payoffs, although both A and B are stochastically stable, for a fixed and sufficiently small ϵ, almost $2/3$ of the population will play B if τ is sufficiently small, that is when our dynamics is a small perturbation of the sequential one. The reason is that for small τ, the probability that many players adapt at the same time period is rather small - for k players it is of order τ^k and moreover to evolve from $z = 0$ to $z = n$, the population must pass through a large region of $z's$, including $\{z, n/3 \leq z \leq 2n/3\}$, where $\pi_B(z) > \pi_A(z)$ for any matching.

Theorem 6 *If $c < 0 < a$, $a_1 < c_2$, $b_1 < a_2$, $b_2 > c_1$, and n is sufficiently big, then for every $\delta > 0$ and a sufficiently small ϵ there exists $\tau(\delta, \epsilon, n)$ such that if $\tau < \tau(\delta, \epsilon, n)$, then $\mu_n^{\epsilon,\tau}(z, z \leq n/3) > 1 - \delta$.*

The proof is provided in Appendix B.

Finally we discuss the case $a < 0$ and $c < 0$.

Theorem 7 *If $a, c < 0$ and n is sufficiently big, then both A and B are stochastically stable.*

Proof: It is easy to see that for a big n, the population needs one mutation to evolve from $z = 0$ to $z = n$ or from $z = n$ to $z = 0$.

As in the previous case, for a fixed n, if $a_1 < c_2$, b_1 is sufficiently small and b_2 is sufficiently big, then $\pi_B(z) > \pi_A(z)$ for any $z \neq 0, n$ and any matching, and therefore B is stochastically stable.

We also have the similar theorem concerning small τ.

Theorem 8 *If $a, c < 0$, $a_1 < c_2$, $b_1 < c_2$, $b_2 > a_1$, and n is sufficiently big, then for every $\delta > 0$ and sufficiently small τ there exists $\epsilon(\delta, \tau, n)$ such that if $\epsilon < \epsilon(\delta, \tau, n)$, then $\mu_n^{\epsilon,\tau}(0) > 1 - \delta$.*

For a proof see Appendix B.

5 Discussion

Multi-player games naturally appear in many situations (see Broom *et al.,* 1997, for a list of biological examples involving leks and communal nests). Here we discussed stochastic adaptive dynamics of populations of individuals playing three-player games. We assumed that players are matched with randomly chosen opponents.

We considered all three classes of generic symmetric three-player games with two strategies. In the first class, there are two pure evolutionarily stable strategies. We showed that only one is stochastically stable and therefore we resolved the problem of equilibrium selection. We also showed that the stochastic stability may depend on the number of players. For some payoff parameters, when the number of players increases, the population may undergo a transition between its equilibria.

In the second class of games, the unique pure evolutionarily stable strategy is also stochastically stable. However, for a fixed low level of mutation, when one considers very small probability of adaptation (when our dynamics is very close to a sequential one), then for certain payoff parameters, almost 2/3 of all individuals will play in the long run the strategy which provides the higher payoff in the homogeneous population, although it is not evolutionarily stable.

In the third class of games, both strategies are stochastically stable but they are not evolutionarily stable. This is a consequence of the fact that they are the only absorbing states and we take the limit of no mutations for a fixed number of players. If we first take the limit of an infinite number of players and then the limit of no mutations, then the long behavior may be different; see (Samuelson, 1997) for a discussion on the order of taking different limits in evolutionary models.

In our models we assumed that individuals are matched completely randomly. We may allow some clustering in the population, either of geographic nature or following for example from the fact that A-players are more likely to meet other A-players. However, if any matching has a positive probability to occur, then our results will not change qualitatively. This is a consequence of taking a limit of no mutations for a fixed number of players. Stochastic stability of spatial three-player games with local interactions was studied in (Miękisz, 2004).

The stochastic stability concept involves the limit of no mutations for a fixed number of players. However, for any arbitrarily low level of mutations, when the number of players is sufficiently big, then the long-run behavior of a population might be different. In fact, it was shown (Miękisz, 2005) that in the case of two-player games with two strategies, in the limit of the infinite number of players in the random matching model, the long run behavior of the population is similar to the one in the Kandori-Mailath-Rob model (Kandori *et al.*, 1993). The assumption of the infinite number of players has the same effect as the limit of infinitely many matchings per time period. The second limit can be justified if the rate of playing games is much bigger than the rate of adaptation. We expect the same situation in the case of three-player games. The stochastic stability in multi-player games in the limit of infinitely many matchings per time period was investigated by Kim (1996).

In order to study the long-run behavior of stochastic population dynamics, we should estimate the relevant parameters to be sure what limiting procedures are appropriate in specific biological examples.

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Appendix A

The following tree representation of stationary distributions of Markov chains was proposed by Freidlin and Wentzell (1970, 1984). Let (Ω, P) be an irreducible Markov chain with a state space Ω and transition probabilities given by $P^{\epsilon} : \Omega \times \Omega \rightarrow [0,1]$. It has a unique stationary distribution, μ^{ϵ}, also called a stationary state. For $X \in \Omega$, let an X-tree be a directed graph on Ω such that from every $Y \neq X$ there is a unique path to X and there are no outcoming edges out of X. Denote by $T(X)$ the set of all X-trees and let

$$q^{\epsilon}(X) = \sum_{d \in T(X)} \prod_{(Y,Y') \in d} P^{\epsilon}(Y, Y'), \tag{6}$$

where the product is with respect to all edges of d. We have that

$$\mu^{\epsilon}(X) = \frac{q^{\epsilon}(X)}{\sum_{Y \in \Omega} q^{\epsilon}(Y)} \tag{7}$$

for all $X \in \Omega$.

Let us assume now that after a finite number of steps of the mutation-free dynamics, i.e. $\epsilon = 0$, we arrive at one of two absorbing states, say X and Y, and stay there forever - there are no other recurrence classes. It follows from the tree representation that any state different from absorbing states has zero probability in the stationary distribution in the zero-mutation limit. Consider a dynamics in which $P^{\epsilon}(Z, W)$ for all $Z, W \in \Omega$, is of order ϵ^{m}, where m is the number of mutations involved to pass from Z to W or is zero. Then one has to compute the minimal number of mutations, m_{XY}, needed to make a transition from the state X to Y and the number of mutations, m_{YX}, to evolve from Y to X. $q(X)$ is of order $\epsilon^{m(YX)}$ and $q(Y)$ is of order $\epsilon^{m(XY)}$. It follows that if $m_{XY} < m_{YX}$, then Y is stochastically stable and moreover $\lim_{\epsilon \rightarrow 0} \mu^{\epsilon}(Y) = 1$.

Appendix B

Proof of Theorem 1:

We begin by examining in detail the mutation-free dynamics. First we show that if $a, c > 0$, then $z = 0$ and $z = n$ are the only absorbing states (for a sufficiently big n) of the mutation-free dynamics even in the sequential dynamics, i.e. when at any time period only one randomly chosen player can adapt to the environment. Let U_A and U_B be the basins of attraction of $z = n$ and $z = 0$ respectively. Let us emphasize that because of the stochastic nature of matchings, they overlap substantially. Observe first that there exists k^* such that if $k \geq k^*$, then $k \in U_A$. If k is divisible by 3 and all A-players and B-players are matched within themselves, then $\pi_A > \pi_B$. If $k = 3m+1$ for some natural number m, then let two pairs of A-players be matched with B-players; if $k = 3m + 2$, then let two A-players be matched with one B-player. In both cases, all other A-players are matched within themselves. Now for a sufficiently big k^* we have

$$a_1(k^* - 4)/k^* > a_2 \tag{8}$$

and therefore the inequality $\pi_A > \pi_B$. Let us observe that for any $z < k^*$ and n big enough, if all A-players are matched with two B-players, then $\pi_A < \pi_B$. We have shown that $U_A \cup U_B = \Omega$. Moreover, there are no other recurrence classes.

Assume now, without a loss of generality, that all payoffs are positive. Let us notice that for $z = n - k^*$, if

$$a_1(n - 2k^*)/(n - k^*) > max\{a_2, b_2, c_2\} \equiv f \tag{9}$$

which is true if

$$n > k^*(2a_1 - f)/(a_1 - f), \tag{10}$$

then $\pi_A > \pi_B$ for any matching. It follows that the size of U_B is at most $n - k^* - 1$ and that of U_A at least $n - k^*$. Hence the population needs at least $k^* + 1$ mutations to evolve from $z = n$ to $z = 0$ and at most k^* mutations to evolve from $z = 0$ to $z = n$. It follows from the tree representation of stationary states (see Appendix A) that $z = n$ is stochastically stable.

Proof of Theorem 6:

For every $z = k$ with k divisible by 3, if all B-players are matched within themselves, then $\pi_B > \pi_A$ and therefore the number of B-players increases in the mutation-free dynamics. It is also true if k is not too small and not too big (independent of n) and only one or two B-players are matched with A-players. We have that $\pi_B(z = 1) < \pi_A(z = 1)$ and $\pi_B(z = n-1) < \pi_A(z = n - 1)$. However, for other small and big $k's$ (depending on n), there is a positive probability that $\pi_B(z = k) > \pi_A(z = k)$. Note that from the assumptions of the theorem follows that $b_1 < a_2, b_2, c_2$ and $b_2 > a_1, b_1, c_1$. Therefore it is enough that all A-players are matched with one B-player and one A-player for small $k's$ and all B-players are matched with one A-player and one B-player for big $k's$. We get that there is a positive probability that $\pi_B(z) > \pi_A(z)$ if $z \neq 0, 1, n-1, n$.

If $k = n/2$ and there are the same number of triples with two A-players and one B-player and two B-players and one A-player, then $\pi_B > \pi_A$ because $2b_1 + c_1 < a_2 + 2b_2$. One may check that the same conclusion follows if $2n/3$ or more individuals are matched in this way and the rest are either A or B-players matched within themselves. If $k = n/3$ and all A-players are matched with two B-players or if $k = 2n/3$ and all B-players are matched with two A-players then again $\pi_B > \pi_A$. It follows that if $z = k$ and $n/3 \leq k \leq 2n/3$, then $\pi_B > \pi_A$ for all possible matchings.

Now for a sufficiently small τ, one considers only trees of order τ^n, where at any time period, only one randomly chosen individual may adapt to the invironment - our dynamics is then a small perturbation of the sequential one. Among these trees there are $(z = k)$-trees with

$0 \leq k \leq n/3$ which are of order ϵ^2. Observe that $\{z \in \Omega, z = k, 0 < k < n/3\}$ is a recurrence class for the sequential dynamics. The thesis follows for sufficiently small ϵ and τ.

Proof of Theorem 8:

Again, as in the previous case, if k is not too small and not too large (independent of n), then there is a positive probability that $\pi_B(z = k) > \pi_A(z = k)$ for any matching. It is easy to see that if $b_2 > a_1$, then $\pi_B(z) > \pi_A(z)$ for all matchings if $z = k$ and k is sufficiently big; $k > \alpha n$ for some $\alpha > 0$. If $c_2 > b_1$, then there is a positive probability that $\pi_B(z) > \pi_A(z)$ for small $z \neq 0, 1$ $z = k$, $k < \beta n$ for some $\beta > 0$. It follows that there is a positive probability that $\pi_B(z) > \pi_A(z)$ for any $z \neq 0, 1, n$.

Now for a sufficiently small ϵ, one considers only $(z = 0)$-trees with just one mutation. Among these trees there is one which is of order τ^{n+2}. In this tree, the number of individuals playing A decreases by one, from $z = n$ until $z = 3$, then it jumps to $z = 1$, then it increases to $z = 2$, and finally it jumps down to $z = 0$. Such tree, for a sufficiently small τ, is the main contribution to $q(z = 0)$. All $(z = n)$-trees with just one mutation are of order at least $\tau^{n(1+\alpha)}$. The thesis follows for sufficiently small τ and ϵ.

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