Quantum spaces and quantum groups

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I shall try to convince the listeners that quantum groups are quantum spaces endowed with a group structure. Roughly speaking quantum spaces are spaces with non-commutative coordinates. To understand better this statement we start with coordinate description of classical spaces.

Cartesian coordinates $(x^1, x^2, \ldots, x^N)$ label points in $\mathbf{C}^N$. They may also be used to describe points on a closed subsets of $\mathbf{C}^N$. To this end they have to be subject to suitable constrains (relations). Usually the constrains have the form of equations:

$$f^k(x^1, x^2, \ldots, x^N) = 0,$$

where $k = 1, 2, \ldots K$ and f^k are continuous functions. Let a closed subset X of $\mathbf{C}^N$ be described by

$$X \quad \rightsquigarrow \quad \boxed{\begin{array}{ll} \text{Coordinates :} & x^1, x^2, \ldots, x^N \\ \text{Relations :} & \left\{ \begin{array}{c} f^k(x^1, x^2, \ldots x^N) = 0 \\ k = 1, 2, \ldots, K \end{array} \right. \end{array}}$$

and similarly $Y \subset \mathbf{C}^M$ be described by

$$Y \quad \rightsquigarrow \quad \boxed{\begin{array}{ll} \text{Coordinates :} & y^1, y^2, \ldots, y^M \\ \text{Relations :} & \left\{ \begin{array}{c} g^l(y^1, y^2, \ldots, y^M) = 0 \\ l = 1, 2, \ldots, L \end{array} \right. \end{array}}$$

Then the cartesian product is described by

$$X \times Y \quad \rightsquigarrow \quad \boxed{\begin{array}{ll} \text{Coordinates :} & \left\{ \begin{array}{c} x^1, x^2, \ldots, x^N \\ y^1, y^2, \ldots, y^M \end{array} \right. \\ \text{Relations :} & \left\{ \begin{array}{c} f^k(x^1, x^2, \ldots x^N) = 0 \\ k = 1, 2, \ldots, K \\ g^l(y^1, y^2, \ldots, y^M) = 0 \\ l = 1, 2, \ldots, L \end{array} \right. \end{array}}$$

A continuous map $\Phi : X \longrightarrow Y$ is given by a system of M continuous functions φ^m of N-variable satisfying the condition:

$$\left(\begin{array}{c} f^k(x^1, x^2, \ldots x^N) = 0 \\ k = 1, 2, \ldots, K \\ y^m = \varphi^m(x^1, x^2, \ldots x^N) \\ m = 1, 2, \ldots, M \end{array}\right) \Longrightarrow \left(\begin{array}{c} g^l(y^1, y^2, \ldots, y^M) = 0 \\ l = 1, 2, \ldots, L \end{array}\right) \tag{1}$$

In the above scheme the values of coordinates are complex numbers. Replacing then by closed operators acting on Hilbert spaces we obtain the theory of non-commutative spaces (quantum spaces). In this case we have to assume that all functions that are considered respect the symmetry of Hilbert spaces. It means that they behave well with respect to unitary equivalence and direct sums. Considering the cartesian product of classical X and Y we imposed no conditions coupling coordinates on X and Y. In the case of quantum spaces we shall always assume that coordinates describing X commute with the ones related to Y.

Quantum groups are quantum spaces endowed with a group structure. For any quantum group G we have the composition map $G \times G \longrightarrow G$. We shall use different letters (e.g: $\alpha, \beta, \ldots$) to denote the coordinates on G. Consequently coordinates on $G \times G$ will be denoted by the same letters with subscripts 1 and 2 (e.g: $\alpha_1, \beta_1, \ldots, \alpha_2, \beta_2, \ldots$). Composition map on G is introduced, when $\alpha, \beta, \ldots$ are expressed as functions of $\alpha_1, \beta_1, \ldots, \alpha_2, \beta_2, \ldots$.

We shall discuss the following quantum groups:

$S_qU(2) \quad \rightsquigarrow$

Coordinates :	α, γ
Relations :	$\left\{\begin{array}{c} \alpha^*\alpha + \gamma^*\gamma = I \\ \alpha\alpha^* + q^2\gamma^*\gamma = I \\ \alpha\gamma = q\gamma\alpha \\ \alpha\gamma^* = q\gamma^*\alpha \\ \gamma\gamma^* = \gamma^*\gamma \end{array}\right.$
Composition law :	$\left\{\begin{array}{c} \alpha = \alpha_1\alpha_2 - q\gamma_1^*\gamma_2 \\ \gamma = \gamma_1\alpha_2 + \alpha_1^*\gamma_2 \end{array}\right.$

$E_q(2) \quad \rightsquigarrow$

Coordinates :	v, n
Relations :	$\left\{\begin{array}{c} v \text{ is unitary} \\ n \text{ is normal} \\ vn = qnv \end{array}\right.$
Composition law :	$\left\{\begin{array}{c} v = v_1v_2 \\ n = v_1n_2 + n_1v_2^* \end{array}\right.$

Quantum Heisenberg-Lorentz group (c is a deformation parameter):

$$G \quad \rightsquigarrow \quad \boxed{\begin{array}{cc} \text{Coordinates :} & \alpha,\beta,\gamma,\delta \\ \text{Relations :} & \left\{\begin{array}{c} \alpha,\beta,\gamma,\delta \text{ mutually commute} \\ \alpha\delta-\beta\gamma=I,\ [\alpha^*,\delta]=0 \\ \gamma^* \text{ commutes with } \alpha,\beta,\gamma,\delta \\ [\alpha^*,\alpha]=c^2\gamma^*\gamma,\ [\alpha^*,\beta]=c^2\gamma^*\delta \\ [\delta^*,\beta]=-c^2\gamma^*\alpha,\ [\delta^*,\delta]=-c^2\gamma^*\gamma \\ [\beta^*,\beta]=c^2\delta\delta^*-c^2\alpha^*\alpha \end{array}\right. \\ \text{Composition law :} & \left\{\begin{array}{c} \alpha=\alpha_1\alpha_2+\beta_1\gamma_2 \\ \beta=\alpha_1\beta_2+\beta_1\delta_2 \\ \gamma=\gamma_1\alpha_2+\delta_1\gamma_2 \\ \delta=\gamma_1\beta_2+\delta_1\delta_2 \end{array}\right. \end{array}}$$

$$S_qU(1,1) \quad \rightsquigarrow \quad \boxed{\begin{array}{cc} \text{Coordinates :} & \alpha,\gamma \\ \text{Relations :} & \left\{\begin{array}{c} \alpha^*\alpha-\gamma^*\gamma=I \\ \alpha\alpha^*-q^2\gamma^*\gamma=I \\ \alpha\gamma=q\gamma\alpha \\ \alpha\gamma^*=q\gamma^*\alpha \\ \gamma\gamma^*=\gamma^*\gamma \end{array}\right. \\ \text{Composition law :} & \left\{\begin{array}{c} \alpha=\alpha_1\alpha_2+q\gamma_1^*\gamma_2 \\ \gamma=\gamma_1\alpha_2+\alpha_1^*\gamma_2 \end{array}\right. \end{array}}$$

The above formulae come from the theory of Hopf *-algebras. It implies that at least on the level of formal computations the condition (1) is in all cases satisfied. However there is no garantee that this is really the case.

For $S_qU(2)$ and the quantum Heisenberg-Lorentz group the answer is positive. $S_qU(2)$ is compact, coordinates α,γ are bounded operators and no domain problem occours. For the quantum Heisenberg-Lorentz group one has to interprete commutators appearing in the relations in the spirit of Weyl and simple computation shows that everything is O.K.

Already in the $E_q(2)$-group we meet unexpected problem. Relation $nn^* = n^*n$ implies that n is a balanced operator ($D(n^*) = D(n)$). However the operator given by the formula $n = v_1n_2+n_1v_2^*$ is in general not balanced. To obtain desired result one has to complete the relations by adding the following spectral condition: $\mathrm{Sp}(n^*n) \subset q^{\mathbf{Z}}\cup\{0\}$.

The case of $S_qU(1,1)$ is the most difficult. Relations imply that the coordinates are balanced operators. However in 1990 it was shown that the operator $\alpha = \alpha_1\alpha_2 + q\gamma_1^*\gamma_2$ is neither balanced nor have balanced extension. The way out was indicated in 1994 by Korogodski who indicated that the problem is more manageable if one passes from $SU(1,1)$ to its double covering. Using his ideas one may construct quantum deformation of extended $SU(1,1)$-group.

To give the link with the usual theory of locally compact quantum groups we shall discuss the following topics:

1. Operator domains and operator functions.
2. C* algebras of continuous functions on quantum groups. Comultiplication.
3. Compact quantum groups: Axioms, existence of Haar weight, Peter-Weyl theory.
4. Locally compact quantum groups according to Kustermans and Vaes.
5. Multiplicative unitaries. Duality.