NONCOMMUTATIVE METHODS IN ALGEBRAIC TOPOLOGY

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The aim of this talk is to define noncommutative "différential forms" on a space X (simplicial set) with a new algebraic structure called "quasi-commutative". Roughly speaking this differential graded algebra of differential forms - called A = $\mathcal{D}^*(X)$ - is given with a submodule $A \overline{\otimes} A$ of the usual tensor product $A \otimes A$, on which the product is commutative. Moreover, the inclusion of the submodule into $A \otimes A$ is a quasi-isomorphism. This submodule $A \overline{\otimes} A$ is called tthe "reduced tensor product" of A by itself and controls in some sense the noncommutativity of the algebra A.
A fascinating prototype of this algebra is given (in degree 0) by the algebra of functions f on the set of integers, constant at $+\infty$ and $-\infty$ (two different constants) ; the differential of f is then given by the formula of the "difference calculus", i.e. $f(x+1) - f(x) = \omega(x)$ (instead of the usual differential). One should note that $\omega(x)$ is 0 at $\pm\infty$.
Starting with this basic example, we build the algebra $\mathcal{D}^*(X)$ which is quasi-isomorphic to the usual cochain algebra C*(X) (over any coherent commutative ring k). However, it has a much richer and interesting structure.
For instance, we show how Steenrod operations and more generally cup i-products can be easily defined in this context (avoiding the heavy simplicial machinery). We also describe explicitly a complex which cohomology is isomorphic to the cohomology of the r-iterated loop space of X (generalizing previous work of Adams and Hilton for r = 1). Finally, using some recent results of M.A. Mandell and assuming some finiteness hypothesis on X, we show how the full (integral) homotopy type of X can be recovered from this new algebraic structure (with the choice of k = **Z**).
If X is a finite complex, the algebra $\mathcal{D}^*(X)$ has other mysterious structures. From its construction, one may define for instance a representation of the braid group $\mathfrak{B}_n$ on $\mathcal{D}^*(X)^{\otimes n}$. This action may be factorized through the usual action of the symmetric group $\mathfrak{S}_n$ only when restricted to the reduced tensor product of n factors $\mathcal{D}^*(X)^{\overline{\otimes} n}$.