Mortar methods for some second and fourth order elliptic equations

Leszek Marcinkowski

Warsaw University
Department of Mathematics, Informatics and Mechanics
Institute of Applied Mathematics
ul. Banacha 2
02-097 Warszawa, Poland
e-mail: lmarcin@mimuw.edu.pl

PhD thesis

Thesis advisor: Prof. Maksymilian Dryja

January, 1999

Abstract

In this thesis, we discuss discretizations on nonmatching triangulations of certain quasilinear and linear elliptic boundary value problems in the two dimensions by the mortar finite element methods. We perform an error analysis and design and analyze some parallel methods for solving the resulting discrete problems. We also present the results of some numerical experiments which confirm some of the theoretical results of first two parts of the thesis.

We first consider a mortar element method which locally uses conforming P_1 elements for elliptic second order problems with the monotone operator. We prove an error bound. We next design and analyze two parallel algorithms for solving the resulting discrete nonlinear problems. The first method combines a Richardson iteration with Newton one, while the second one is a nonlinear domain decomposition method. We also briefly discuss how the discussed mortar methods for the quasilinear problems can be applied to the more general class of problems with an unbounded nonlinearity.

We next discuss a mortar method with locally non-conforming Crouzeix-Raviart elements for linear second order elliptic problems. We establish an error estimate which is optimal. A parallel algorithm for solving the resulting discrete problem is also presented. This algorithm is an Additive Schwarz Method. The described method is almost optimal, i.e. the condition number of the preconditioned system grows only polylogarithmically with the number of elements per subdomain. Due to the non-conformity of the discrete solution in the analysis of the error and the algorithm, a new special operator and new technical tools have been introduced.

In the third part of the thesis, we study certain versions of the mortar method for plate problems. These methods utilize locally both conforming bicubic elements, Hsieh-Clough-Tocher(HCT) and reduced Hsieh-Clough-Tocher macro elements, and non-conforming Adini and Morley plate elements. We establish error estimates for all these methods. There are also designed and analyzed some parallel algorithms for solving the resulting discrete problems. The discussed algorithms, for solving the resulting problems, are parallel and are based on the abstract theory of Additive Schwarz Methods. In the generalization of the mortar finite element method for plate problems, it was necessary to introduce special mortar conditions on the interfaces, some of them, especially for nonconforming methods, involve special interpolants. In the analysis of the error and ASM methods there have been introduced special new operators and technical tools.

Some of the results of the thesis have been published in [88] and [89].

Acknowledgments

I would like to express my greatest gratitude toward my thesis advisor prof. Maksymilian Dryja for his invaluable advice and help.

I would also like to thank my parents and my friends for their support during the preparation of this thesis.

Finally, I would like to thank the referees prof. Dietrich Braess, prof. Yvon Maday, and prof. Krzysztof Moszyński for their careful reading the manuscript and for their comments and remarks which enable me to improve this final version of the thesis.

Contents

List of Figures

Chapter 1

Introduction

Contents

1.1 Overview

Variational methods for decomposing and solving elliptic problems by domain decomposition techniques are extensively analyzed and successfully used in practise. A large problem is split into some smaller ones that can, for example, be solved independently. Most methods use discretization meshes that are first defined globally over the whole domain and then divided into smaller subdomains. However, it might be convenient to consider methods that use approximations that are defined independently in each subdomain and which do not match on the interfaces. This allows, for example, to make local and adaptive changes of grids on one subdomain without modifying the grids of other ones. This approach requires matching conditions on the interfaces between different subdomains to ensure the weak continuity of the traces on the interfaces and a good transmission of information between adjacent subdomains. One way of enforcing these matching conditions is the pointwise continuity of approximation functions on interfaces. For example this is the case if there is one mesh defined globally over the whole domain. Unfortunately, in this case the approximation error of the mortar method is not optimal, therefore it is not used in practice, in general. The other way is to impose some integral conditions which leads to optimal error estimate, and this approach is used in practise.

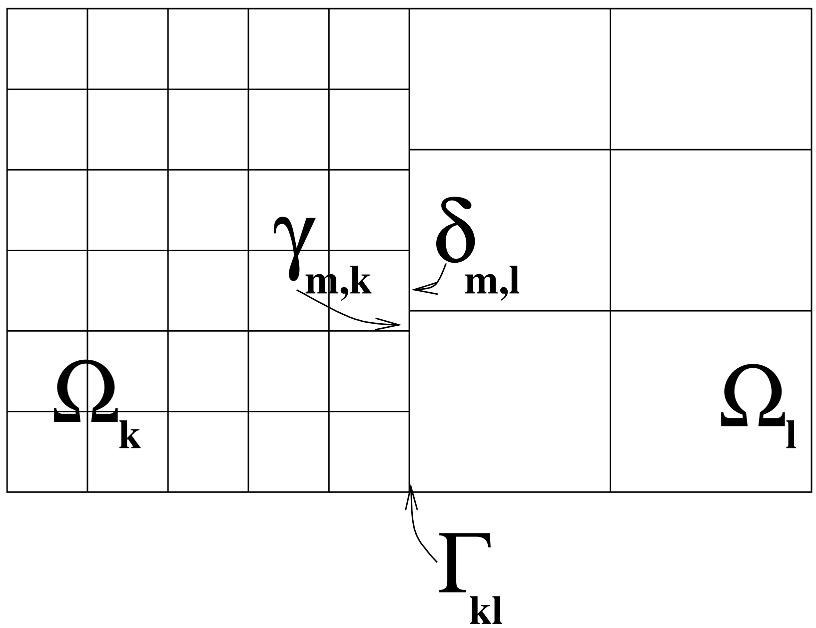

Figure 1.1: Non-matching meshes on the interface Γ_{kl}.

One of the methods which uses the integral conditions is the mortar element method, a domain decomposition method with nonoverlapping subdomains. The mortar method was proposed for second order elliptic problems about ten years ago by Bernardi, Maday, and Patera [19], see also e.g. Bernardi, Debit, and Maday [15] or [14], Ben Belgacem [9], Bernardi, Maday, and Patera [18], Ben Belgacem and Maday [11] or [10].

There are two versions of mortar technique: the geometrically conforming and geometrically nonconforming ones. For the first one, we assume that the decomposition of $\overline{\Omega} = \bigcup_k \overline{\Omega}_k$ into nonoverlapping polygonal subdomains is such that the intersection between the closures of two different subdomains is either the empty set, a vertex or an edge in the two dimensions. The geometrically nonconforming version of the mortar method requires only that there exists a decomposition of the interface

$\overline{\Gamma} = \bigcup_k \partial\Omega_k \setminus \partial\Omega$ into disjoint edges of subdomains, i.e. $\overline{\Gamma} = \bigcup_m \overline{\gamma}_{m,k}$, where $\gamma_{m,k}$ is an open edge of $\partial\Omega_k$. The mortar technique allows us to couple different discretizations methods in the subdomains, but in this thesis, we are concerned only with the case of finite element discretizations. Except of Chapter 2 in this thesis, we are concerned with the geometrically conforming version of the mortar method.

We first introduce locally, i.e. in each subdomain Ω_i, different, independent conforming or nonconforming finite element discretizations over all subdomains. The meshes of two neighboring subdomains do not necessarily match on their common interface, cf. Figure 1.1. We then choose one side of the interface $\overline{\Gamma}_{kl} = \partial\Omega_k \cap \partial\Omega_l$ corresponding to one subdomain as master $\gamma_{m,k}$ and the second as slave $\delta_{m,l}$, cf. Figures 1.1 and 1.2. This choice is arbitrary. Then mortar technique imposes that the trace (or some traces in the case of fourth order problems) of solution on the two neighboring subdomains has the same L^2 projections on the carefully chosen trial mortar space (or spaces) which is defined on their common edge and corresponds to the slave mesh of this interface. In the case of local nonconforming plate discretizations, the mortar (integral) conditions involves special interpolants defined on the corresponding meshes of the common edge, see Chapter 4. The choice of the matching integral conditions depends on the local discretization methods.

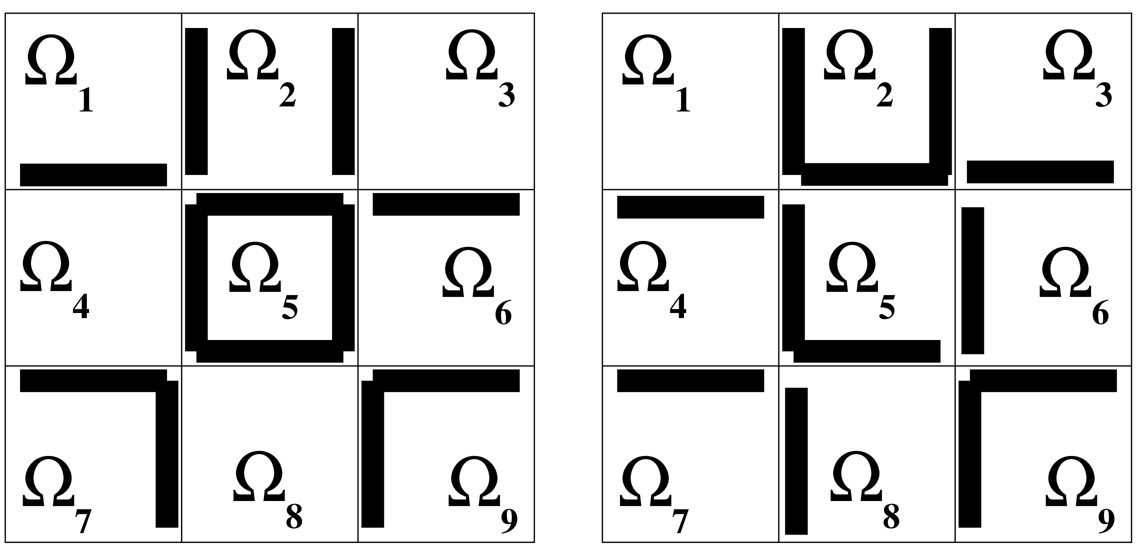

Figure 1.2: Two possible choices of masters.

Finite element discretizations of elliptic problems of second or fourth order, e.g. plate problems, often result in very large, sparse systems of linear (or nonlinear) equations. This is also true in the case of mortar element discretizations. Domain decomposition methods provide a very natural way of introducing parallel algorithms for solving such problems. In each iteration, a number of smaller subproblems which correspond to the restriction of the original problem to subspaces or to a problem with a coarse mesh are solved. Domain decomposition methods can also be viewed as preconditioners for iterative methods like conjugate gradient or GMRES. Domain decomposition methods are now widely analyzed and applied to practical problems, cf. [66], [45], [46], [67], [74], [92], [75], [68], journal papers and Bjørstad, Gropp, and Smith [20], a book.

The first domain decomposition method is the alternating (multiplicative) Schwarz method, proposed in 1869 by H. A. Schwarz [101]. It was designed for the case of two subdomains and for proving the existence of the solution and had a sequential nature. P. L. Lions in [85] interpreted it in terms of a variational framework and considered the Multiplicative Schwarz Method (MSM) in the cases of many subdomains. Later Dryja and Widlund developed an additive version of the Schwarz method, named Additive Schwarz Method (ASM), and proposed the general variational abstract framework that can be used to analyze ASM which have no sequential behavior of the Schwarz alternating method, cf. Dryja and Widlund [61]. In an ASM, we replace an original discrete problem by a preconditioned one which is obtained by subdividing the solution of the original problem into smaller problems whose solutions, which can be computed in parallel, are summed to produce a preconditioned problem. In an alternating Schwarz method, we obtain a preconditioned problem by alternatingly solving smaller problems in subspaces, see Section 1.4. The theory of MSM has been done by Bramble, Pasciak, Wang and Xu [31] and Xu [105].

The second group (family) of domain decomposition methods are iterative substructuring methods, where Ω is decomposed into nonoverlapping subregions. Iterative substructuring methods were developed by Bramble, Pasciak, and Schatz [27], [28], [29], [30], Bjørstad and Widlund [21], Widlund [103], [102], and Dryja [56], [55]. Some of iterative substructuring methods can be analyzed using Additive Schwarz framework. In iterative substructuring methods, we first eliminate the unknowns in the interiors of subdomains and then solve a reduced interface problem.

In this thesis, we prove the error bound of the mortar version of locally conforming and nonconforming finite element discretizations applied to some nonlinear and linear second order elliptic problems and to some fourth order linear elliptic problems. We also design and analyze domain decomposition methods for solving the resulting discrete problems. All domain decomposition algorithms for solving linear discrete problems considered in the thesis are versions of ASM, but for each ASM algorithm, we can straightforwardly develop a multiplicative Schwarz method (MSM) based on the same decomposition of the discrete space, cf. Section 1.4 or [20]. It is also worth to mention that in all previous versions of mortar methods it was usually assumed that local spaces contain conforming functions. For our knowledge the first results for mortar method with locally nonconforming discretizations for second order elliptic problems are given in the Chapter 3 and has been published in [89]. There are independent results for mortar method for plate problems with nonconforming elements given in Lacour [76] and Lacour and Maday [77].

The thesis is organized as follows. In the remainder of Chapter 1, we review some basic definitions of Sobolev spaces used in the thesis. We also briefly discuss conjugate gradient iterative methods and some domain decomposition methods in a form used in this thesis.

The main contribution of our work is given in Chapters 2, 3, and 4.

In Chapter 2, we give a proof of the error bound of the mortar version of finite element discretization applied to some nonlinear problems. Namely,we discuss an application of the mortar element method to a second-order nonlinear elliptic boundary value problem with the strongly monotone and Lipschitz continuous operator in a polygonal region $\Omega \subset \Re^2$ with the Lipschitz boundary. We also briefly discuss, following [94], how the mortar methods for the quasilinear problems which are considered here can be applied to the more general class of problems with an unbounded nonlinearity. For our knowledge there are no results devoted to such topics for nonlinear problems. This work was inspired by the earlier studies of of mortar methods for linear problems in Bernardi, Maday, and Patera [19], Ben Belgacem and Maday [11], Bernardi and Maday [16] (or [17] in French), and Ben Belgacem [9].

We also discuss two domain decomposition methods for solving the discrete problem in the geometrically conforming case. The first method combines the Schwarz method with Newton method, and the second algorithm is a nonlinear domain decomposition method based on the abstract framework introduced in Dryja and Hackbusch [59]. Both algorithms use the same decomposition of the discrete space. Some other parallel algorithms devoted for solving systems of linear equations arising from the mortar version of finite element discretization for linear elliptic problem can be found e.g. in Achdou, Maday, and Widlund [4], Casarin and Widlund [44], Dryja [57].

Chapter 2 is an extension of the paper [88] which has already been published. In the last section of this chapter, we report the results of some numerical experiments.

In Chapter 3, we construct and analyze a new version of the mortar method for second order elliptic problems which locally uses an independent nonconforming Crouzeix-Raviart discretization in each substructure that is a polygonal subregion of Ω. For our knowledge there are no results devoted to those topics, cf. Wohlmuth [104] for some related results, i.e. in [104] the Crouzeix-Raviart finite element method defined on a global triangulation of the domain was interpreted as a mortar finite element method with each element of the triangulation as a subdomain. In all previous versions of mortar methods for second order elliptic problems, it has been assumed that local subspaces contain conforming continuous functions. The mortar technique for locally nonconforming elements imposes that the solution on the two neighboring subdomains has the same L^2 projections on a mortar space that is defined on their common edge. We choose a trial mortar space which has natural L^2 orthogonal basis and leads to simply computations of the matching conditions. For second order elliptic problems, we prove that the error estimate is of the same optimal order as in the standard linear nonconforming finite element method.

We also propose a parallel method for solving the system of linear equations that arises from our discretization. It is described as an additive Schwarz method (ASM) using the general framework of ASM's, see Section 1.4. We give results of the numerical experiments which confirm some of the theoretical results of this chapter.

In this chapter, the error analysis is done for arbitrary polygonal substructures while the additive Schwarz method is considered for a partition of the original 2-D region Ω into triangles that form a coarse triangulation of parameter H. The described method is almost optimal, i.e. the number of iterations required to decrease the energy norm of the error by a conjugate gradient method is proportional to $(1 + \log(\frac{H}{\underline{h}}))$. Here H and h_i are the parameters of the coarse triangulation and the fine one on Ω_i, respectively, and $\underline{h} = \inf_i h_i$. There is a considerable attention focused on the related technique for solving systems of equations arising in nonconforming discretization methods defined on a global triangulation of Ω, e.g. Brenner [33], [34], [36], Sarkis [98], [100], Cowsar [50], Cowsar, Mandel, and Wheeler [51]. The results of Chapter 3, previously presented in a report [89], have been accepted for publication.

Finally, Chapter 4 is devoted to mortar element methods for plate problems. We consider only the case of clamped plate problems, but our results can be straightforwardly applied to plate problems with other boundary conditions, e.g. to simply supported plate problems. In Lacour [76] and Lacour and Maday [77] the approximation of a similar problem by DKT (nonconforming) elements is considered. In Belhachmi [8], a mortar method for the biharmonic problem is discussed, but these results concern only the case of local spectral discretizations. They carried out an error analysis for that case only. We consider locally the conforming bicubic element, the reduced Hsieh-Clough-Tocher (HCT) and the Hsieh-Clough-Tocher macro elements, and the nonconforming Adini and Morley finite element methods. We present an error analysis for all these discretizations and discuss some methods for solving the discrete problems. We restrict ourselves to the geometrically conforming version of mortar method. The mortar technique for plate problems, which we present here, requires the continuity of the solution at the vertices of subdomains and that the solution on the two neighboring subdomain satisfies two mortar conditions of the L^2 type. Those conditions depend on the local discretization methods. For the locally conforming methods (i.e. bicubic, HCT and reduced HCT) the mortar conditions on the common edge of two subdomains are equivalent to the equality of the L^2 projections on two mortar spaces of the solutions and of the normal derivatives of the solutions on these two subdomains. Here the mortar spaces defined on the common interface depend on the local discretization methods. For the both nonconforming methods (i.e. Adini and Morley elements) the mortar conditions are of the same type, but additionally involve some interpolants which are defined locally on each interface.

We also propose four parallel methods for solving some discrete problems. These algorithms are described as additive Schwarz methods (ASM), see Section 1.4. All these methods, except one, are of iterative substructuring type. For additive Schwarz methods and iterative substructuring methods for plate problems with globally conforming or nonconforming discretizations defined on one global triangulation of Ω, we refer e.g. to Brenner [35], Brenner and Sung [39], and Le Tallec, Mandel, and Vidrascu [82].

First two methods considered here are designed for mortar methods with local HCT or reduced HCT discretizations and are based on analogous decompositions of the discrete space. In both cases, we decompose a discrete space as a sum which consists of a coarse space, local one-dimensional spaces associated with degrees of freedom of order one at vertices of subdomains, and of certain local spaces associated with interfaces. The difference between the methods lies in the fact that the first one is of iterative substructuring type, but the second one is not. Additionally, the second method uses a nonstandard outer coarse grid. Therefore we have to introduce a special interpolation operator which maps the coarse grid onto the mortar discrete space. The second algorithm seems easier to be implemented in practise.

Next method is of Neumann-Neumann type, cf. Le Tallec, Mandel, and Vidrascu [82], Dryja and Widlund [64]. Our Neumann-Neumann method is designed for mortar methods built on the decomposition of the domain which satisfies one additional condition: we assume that it is possible to choose master edges of substructures in such way that each subdomain has all its edges either as masters or as slaves. This assumption is due to the property of functions in discrete spaces built by the mortar method. This Neumann-Neumann method is based on the modified abstract scheme of Le Tallec, Mandel, and Vidrascu [82] and can be applied for mortar methods with all conforming local discretizations. This algorithm was easier to analyze due to the quite general scheme of Le Tallec, Mandel, and Vidrascu [82], but its implementation and use in practise seems more complicated than the ones of the first two methods.

The last ASM method is for solving the discrete problem of the mortar method with locally nonconforming Adini discretizations. We distinguish this case because the analysis requires special coarse grid and technical tools necessary to overcome some technical difficulties which are due to the local nonconformity of the solution.

All methods presented in this chapter are almost optimal, i.e. the number of iterations required to decrease the energy norm of the error by a conjugate gradient method is proportional in each case to $(1+\log(\frac{H}{\underline{h}}))$. Here H and h_i are the parameters of the coarse decomposition and the fine triangulation on Ω_i, respectively, and $\underline{h} = \inf_i h_i$.

In the thesis, the following notation is used: $u \asymp v$, $x \succeq y$ and $w \preceq z$ mean that there exist positive constants c and C independent of any h_k-the parameter of the fine triangulation of a subdomain Ω_k, and the number of subdomains such that

$$c\,u \leq \ v \ \leq \ C\,u, \quad x \geq c\,y \quad \text{and} \quad w \leq C\,z, \quad \text{respectively.}$$

1.2 Some function spaces

In this section, we introduce some function spaces which are used in the next chapters of the thesis.

Let N be a positive integer. A vector

$$\alpha = (\alpha_1, \ldots, \alpha_N)$$

with integer components $\alpha_i \geq 0$, is said to be a multi-index of dimension N. The number

$$|\alpha| = \sum_{i=1}^{N} \alpha_i$$

is called the length of multi-index α. We introduce the following notation: for multi-indices α, β, and $x = (x_1, \ldots, x_N) \in \Re^N$,

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_N + \beta_N),$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N},$$

and

$$D^\alpha = \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}.$$

The set of all polynomials in N variables $x_1, \ldots, x_N$ which can be written in the form

$$q(x) = \sum_{i=0}^{n} \sum_{|\alpha|=j} c_\alpha x^\alpha$$

will be denoted by $P_n(\Re^N)$. Similarly, the symbol $Q_n(\Re^N)$ will denote the set of all polynomials in N variables $x_1, x_2, \ldots, x_N$ which have the degree not greater than n with respect to each variable $x_i, i = 1, 2, \ldots, N$.

If $G \subset \Re^N$ then the symbol $P_n(G)$ denotes the set of restrictions of polynomials from $P_n(\Re^N)$ to G.

We now introduce Sobolev spaces $W_p^k(\Omega)$, cf. [6]. Let $p \in [1, \infty]$, k be an integer and Ω be a domain (i.e. an open connected set) in $\Re^N$. Let us denote by $W_p^k(\Omega)$ the set of all functions $u \in L^p(\Omega)$ whose (distributional) derivatives $\partial^\alpha u$ are elements of $L^p(\Omega)$ for all α such that $|\alpha| \leq k$.

These sets form Banach spaces with norms

$$\|u\|_{W_p^k(\Omega)} := \left\{ \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right\}^{1/p} \qquad 1 \leq p < \infty,$$

$$\|u\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)},$$

where $\|\cdot\|_{L^p(\Omega)}$ denotes the standard L^p norm, i.e.

$$\|u\|_{L^p(\Omega)} = \left\{\int_\Omega |u|^p \, dx\right\}^{1/p} \quad 1 \leq p < \infty,$$

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| = \inf_{\operatorname{mes}(E) = 0} \left\{ \sup_{x \in \Omega \setminus E} |f(x)| \right\}.$$

Here the integrals are taken with respect to Lebesgue measure, cf. [73] or [96], and the derivatives are understood in the distributional sense, cf. [95].

We shall also use the following seminorm: if $u \in W_p^k(\Omega)$ and $0 \leq j \leq k$, then we set

$$|u|_{W_p^k(\Omega)} := \left\{\sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p(\Omega)}^p\right\}^{1/p} \quad 1 \leq p < \infty$$

and

$$|u|_{W_\infty^k(\Omega)} := \max_{|\alpha| = k} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

For the spaces $W_2^k(\Omega)$, a special notation is used

$$H^k(\Omega) := W_2^k(\Omega).$$

We also introduce the spaces $H^s(\Omega)$ for all real positive s that are not integer, e.g. see [71]. The space $H^s(\Omega)$, where $s = k + t$, $0 < t < 1$, k is an integer, and Ω is a domain in $\Re^N$, is defined as a subspace of $H^k(\Omega)$ formed by all functions $v \in H^k(\Omega)$ for which

$$|v|_{H^s(\Omega)}^2 := \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{N+2t}} \, dx \, dy < \infty.$$

We are primarily interested in the case of $N = 1$. The norm of $H^s(\Omega)$ is defined by

$$\|v\|_{H^s(\Omega)}^2 := \|v\|_{H^k(\Omega)}^2 + |v|_{H^s(\Omega)}^2,$$

$|v|_{H^s(\Omega)}$ is called the H^s seminorm. Note that $H^s, s \geq 0$ are Hilbertian subspaces of $H^0 = L^2$.

We also define $H_0^s(\Omega)$ a subspace of $H^s(\Omega)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm of $H^s(\Omega)$.

We next introduce the concept of real interpolation between Hilbert spaces with help of K-functionals, cf. [84]. If H_0, H_1 are Hilbert spaces with $H_1 \subset H_0$ then for any $u \in H_0$ and $t > 0$, define

$$K(t,u) := \inf_{v \in H_1} \left(\|u - v\|_{H_0} + t\|v\|_{H_1} \right).$$

Then for $0 < \theta < 1$, define a norm

$$\|u\|_{[H_0,H_1]_\theta} := \left(\int_0^\infty t^{-1-2\theta} K(t,u)^2 \, dt \right)^{1/2}.$$

The set

$$[H_0, H_1]_\theta = \left\{ u \in H_0 : \|u\|_{[H_0,H_1]_\theta} < \infty \right\}$$

form a Hilbert space with the norm $\| \cdot \|_{[H_0,H_1]_\theta}$, e.g. see [13].

If Ω has a Lipschitz boundary, then

$$[H^m(\Omega), H^k(\Omega)]_\theta = H^{(1-\theta)m+\theta k}(\Omega)$$

and the norms are equivalent, for any real numbers m and k, cf. [13] or Theorem 12.2.7 [38].

We also introduce the space $H_{00}^{1/2}([a,b]), -\infty < a < b < \infty$ as a subspace of $H^{1/2}([a,b])$ formed by those functions v whose trivial extension $\hat{v}$ by zero to all R belong to $H^{1/2}(R)$, cf . [84], i.e. those for which

$$\|v\|_{H_{00}^{1/2}([a,b])} := \|\tilde{v}\|_{H^{1/2}(R)} < \infty.$$

The space $H_{00}^{1/2}([a,b])$ is a strictly embedded subspace of $H_0^{1/2}([a,b])$. The norm $\|v\|_{H_{00}^{1/2}([a,b])}$ is equivalent to the one defined by, see [84],

$$|||v|||_{H_{00}^{1/2}([a,b])} := \left\{ |v|^2_{H^{1/2}([a,b])} + \int_a^b \frac{|u(x)|^2}{|x-a|} \, dx + \int_a^b \frac{|u(x)|^2}{|b-x|} \, dx \right\}^{1/2}.$$

The space $H_{00}^{1/2}([a,b])$ can be also characterized as an interpolation space between $L^2([a,b])$ and $H_0^1([a,b])$, cf. [84],

$$H_{00}^{1/2}([a,b]) = [L^2([a,b]), H_0^1([a,b])]_{1/2}.$$

We will not further distinguish between equivalent norms.

We also briefly recall the following interpolation argument which is standard, e.g. see Proposition 12.1.5 in [38]. Suppose that $V_1 \subset V_0$ and $H_1 \subset H_0$ are two pairs of Hilbert spaces, and that T is a bounded linear operator from V_i to H_i with norms c_i, $i = 0, 1$, respectively. Then T is also a bounded linear operator that maps the interpolation space $[V_0, V_1]_\theta$ to $[H_0, H_1]_\theta$ with the norm bounded by $c_0^{1-\theta}\, c_1^\theta$, that is

$$\|Tu\|_{[H_0,H_1]_\theta} \leq c_0^{1-\theta}\, c_1^\theta \|u\|_{[V_0,V_1]_\theta} \;\; \forall u \in [V_0, V_1]_\theta.$$

1.3 Conjugate gradient method

If we want to solve a linear system of equations

$$Au = f$$

with A a real, symmetric, positive definite matrix. We can apply a *direct* or an *iterative* method. If this system of equations arises from the discretization of elliptic problem by a mortar element method, or in general by any finite element method, then the matrix A is sparse and of a very large dimension.

Direct methods give exact solutions. The best known is based on a triangular factorization of the matrix A. These method can be impractical if the dimension of A is large.

Alternative to the direct methods are iterative ones. They produced a sequence of approximate solutions u_k that converges to the solution u. Usually those methods involve only the vector multiplication by the matrix A. There are several well known examples like Jacobi, Gauss-Seidel, SOR, Block Jacobi, Block Gauss-Seidel, Chebyshev semi-iterative, and conjugate gradient methods (CGs), cf. Golub and Van Loan [69].

When A is not well conditioned, we can introduce a preconditioner B, if it exists, and consider a preconditioned linear system

$$B^{-1}Au = B^{-1}f,$$

i.e. we multiply the linear system with the matrix A by a nonsingular matrix B^{-1} in order to get a new preconditioned system with the matrix $B^{-1}A$ that can be easier to solve by an iterative method.

The preconditioner B should be chosen according to the following properties:

- the condition number $\text{cond}(\tilde{A})$ should be small, i.e independent (logarithmically dependent) on the dimension of $\tilde{A}$,
- the application of B^{-1} to a vector should be easy to compute on scalar or parallel machines.

Here $\tilde{A} = B^{-1/2}AB^{-1/2}$ and

$$\text{cond}(\tilde{A}) = \frac{\lambda_{max}(\tilde{A})}{\lambda_{min}(\tilde{A})}.$$

Domain decomposition methods allow to construct preconditioners which have those properties. We are interested for preconditioners which are well suited to parallel computing.

All algorithms studied in this thesis are variations of preconditioned conjugate gradient methods (PCGs). Since this algorithm is well known we only give a short description of PCGs. The very important feature of the CG and PCG for domain decomposition is that they need only a multiplication of the matrix to a given vector, we do not need an explicit representation of the matrix in general.

Preconditioned Conjugate Gradient Algorithm

Set $k = 0$; $x_0 = 0$; $r_0 = f$;
while $\|r_k\|_2 \geq \epsilon \|r_0\|_2$

Compute $z_k = B^{-1}r_k$
$k = k + 1$
if $k = 1$

$p_1 = z_0$

else

$\beta_k = (r_{k-1}, z_{k-1})/(r_{k-2}, z_{k-2})$
$p_k = z_{k-1} + \beta_k p_{k-1}$

end $\alpha_k = (r_{k-1}, z_{k-1})/(p_k, Ap_k)$
$x_k = x_{k-1} + \alpha_k p_k$
$r_k = r_{k-1} - \alpha_k Ap_k$

end

Then by a well known formula for the reduction in the energy norm of the error after k steps of the CG iterations, e.g. cf. [72], we obtain

$$\|x - x_k\|_A \leq 2 \left(\frac{\sqrt{\text{cond}(\tilde{A})} - 1}{\sqrt{\text{cond}(\tilde{A})} + 1} \right)^k \|x - x_0\|_A.$$

Here $\tilde{A} = B^{-1/2} A B^{-1/2}$.

1.4 Abstract Schwarz theory

In this section, we describe and state the convergence results of abstract Schwarz methods, generalizations of the alternating method of Schwarz [101]. This theory has been developed previously for the additive case by Dryja and Widlund [61], [62], and [63], see also Bjørstad, Gropp and Smith [20], Griebel and Oswald [70], Nepomnyschikh [91] and Zhang [107] and [108]. The main contributors for the theory of the *multiplicative* Schwarz methods are Bramble, Pasciak, Wang and Xu [31] and Xu [105]; cf. also Cai and Widlund [42], for a variant of the theory for nonsymmetric and indefinite problems.

1.4.1 Additive Schwarz method

Let V be a finite dimensional space with the inner product $a(\cdot, \cdot)$. We consider the abstract problem

$$a(u^*, v) = f(v) \ \ \forall v \in V, \tag{1.1}$$

where f is a linear functional on V. We next consider a decomposition of V into a sum of subspaces

$$V = V_0 + V_1 + \ldots + V_N.$$

This is not necessarily direct sum of subspaces, i.e. the representation of an element $u \in V$ in terms of components of these subspaces can be not unique. The first subspace with the subscript zero represents a special coarse space.

We next need $b_i(\cdot, \cdot)$, $i = 1, \ldots, N$, symmetric, positive definite bilinear forms on $V_i \times V_i$. We introduce operators $T_i : V \rightarrow V_i$, by

$$b_i(T_i u, v) = a(u, v) \ \ \forall v \in V_i \tag{1.2}$$

and define

$$T = T_0 + \sum_{i=1}^{N} T_i.$$

If we choose $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ then $T_i = P_i$, the orthogonal projections (in terms of $a(\cdot,\cdot)$) onto V_i. We replace (1.1) by an equation with the same solution if T is invertible

$$Tu^* = g, \quad g = \sum_{i=0}^{N} g_i, \quad g_i = T_i u^*. \tag{1.3}$$

The right hand side g is obtained by solving N independent problems

$$b_i(g_i, v) = a(u^*, v) = f(v) \quad \forall v \in V_i.$$

The equation (1.3) can be solved by a conjugate gradient method, using $a(\cdot,\cdot)$ as the inner product.

We now state the main abstract theorem, the proof can be found in [20] or [64].

Theorem 1.4.1 *Let there exist*

(i) a constant C_0 such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^{N} u_i$, $u_i \in V_i$, such that

$$\sum_{i=0}^{N} b_i(u_i, u_i) \le C_0^2 a(u, u),$$

(ii) constants ϵ_{ij}, $i, j = 1, \dots, N$, such that

$$a(u_i, u_j) \le \epsilon_{ij} a(u_i, u_i)^{1/2} a(u_j, u_j)^{1/2} \quad \forall u_i \in V_i \quad \forall u_j \in V_j,$$

(iii) a constant ω such that for $i = 0, \dots, N$,

$$a(u, u) \le \omega b_i(u, u) \quad \forall u \in V_i.$$

Then T is invertible and

$$C_0^{-2} a(u, u) \le a(Tu, u) \le (\rho(\mathcal{E}) + 1)\, \omega a(u, u),$$

where $\rho(\mathcal{E})$ is the spectral radius of the matrix $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

In applications the constants C_0, $\rho(\mathcal{E})$ and ω should be independent or logarithmically dependent on N.

1.4.2 Multiplicative Schwarz method

The multiplicative Schwarz method is defined for the abstract problem (1.1) as follows:

- u^0 arbitrary,
- For $i = 0$ until convergence,
  - $w = u^i$,
  - For $j = 0$ to N,
    $w = w + T_j(u^* - w)$,
  - End j.
  - $u^{i+1} = w$,
- End i.

Here T_i, $i = 0, \ldots, N$ are from (1.2).

For the error $u^{n+1} - u^*$ we obtain

$$u^{n+1} - u^* = E_N(u^n - u^*) = E_N^{n+1}(u^0 - u^*).$$

Here

$$E_N = (I - T_N) \ldots (I - T_0)$$

is the error propagation operator. We see that $\|I - T_i\|_a > 1$ if $\|T_i\|_a > 2$, hence the assumption that $\omega < 2$ is necessary. If ω is too large we can scale the bilinear forms $b_i(\cdot, \cdot)$. Of course it increases the parameter C_0.

If we take $u^0 = 0$, we can regard the algorithm as a simple iterative method for solving the equation

$$T_{ms} u^* = g_{ms},$$

where $T_{ms} = I - E_N$. This nonsymmetric operator equation can be solved by GMRES or a similar iterative methods, cf. [97].

We can also consider the symmetrized multiplicative Schwarz method:

- u^0 arbitrary,
- For $i = 0$ until convergence,

  - $w = u^i$.
  - For $j = 0$ to N,
    $w = w + T_j(u^* - w)$,
  - End j.
  - For $j = N$ to 0,
    $w = w + T_j(u^* - w)$,
  - End j.
  - $u^{j+1} = w$,

- End i.

If we take $u^0 = 0$ and want to use this method as a preconditioner for conjugate gradient method, then the preconditioned operator is given by

$$T_{sms} = I - E_N^T E_N.$$

For further discussion, we refer to [40] and [60].

The result that we state in Theorem 1.4.2 states an upper bound for the spectral radius, or norm of the error propagation operator. The proof can be found in [20] or [64].

Theorem 1.4.2 *The error propagation operator of the multiplicative Schwarz method satisfies*

$$\|E_N\|_a \leq \sqrt{1 - \frac{(2-\hat{\omega})}{(1+2\hat{\omega}^2\rho(\mathcal{E})^2)C_0^2}}.$$

Here $\hat{\omega} = \max(1, \omega)$.

The next corollary is an immediate consequence of Theorem 1.4.2. This result was also formulated in [60].

Corollary 1.4.1 *The abstract symmetric multiplicative Schwarz method satisfies*

$$\frac{(2-\hat{\omega})}{(1+2\hat{\omega}^2\rho(\mathcal{E})^2)C_0^2} a(u,u) \leq a(T_{sms}u, u) \leq a(u,u) \quad \forall u \in V.$$

Here $\hat{\omega} = \max(1, \omega)$.

Chapter 2

A mortar method for quasilinear elliptic boundary value problems

Contents

2.1 Introduction

The goal of this chapter of the thesis is to give an analysis of the error of the mortar version of finite element discretization applied to some nonlinear problems and to discuss two domain decomposition methods for solving the discrete problem in geometrically conforming case. We refer for a general presentation of the mortar method for linear problems to [19], [11] and [16] (or [17] in French) and for a presentation of the matching constraints in terms of Lagrange multipliers to [9]. Recently, there is a development of parallel algorithms devoted for solving systems of linear equations arising from the mortar version of finite element discretization for linear elliptic problems, see cf. [2], [4], [44], [57] and many others.

Namely, in this chapter, we discuss the application of the mortar element method to a second-order nonlinear elliptic boundary value problem with the strongly monotone and Lipschitz continuous operator in a polygonal region $\Omega \subset \Re^2$ with the Lipschitz boundary. We also briefly discuss, following [94], how the mortar method for the quasilinear problems which are considered here can be applied to the more general class of problems with an unbounded nonlinearity. For our knowledge there are no results devoted to such topics for nonlinear problems. We consider first the geometrically conforming case of the mortar element method, i.e. the intersection of the closures of two subdomains can be the empty set, an edge or a vertex. Next we discuss a more general case - the geometrically nonconforming one, i.e. we do not impose the above condition. In the first case, we see that the estimate of the error is of the same order as in the standard conforming piecewise linear finite element discretization provided that the solution of the differential problem is in $H^2(\Omega)$. In the latter one, we have to strengthen our regularity assumption. Namely we shall assume that the solution of the differential problem belongs to the space $H^\sigma(\Omega), 2 \leq \sigma < 5/2$ to derive the error estimate of order $h^{\sigma-3/2}$. The technique that we use to obtain these estimates is a generalization of that used for linear problems, cf. [19] and [11]. We propose two algorithms for solving the discrete problem in the geometrically conforming case. The first method combines the Schwarz one with Newton's method and the second algorithm is a nonlinear domain decomposition method based on the abstract framework introduced in [59]. Both algorithms use the same decomposition of the discrete space.

The first method combines a preconditioner constructed in terms of ASM for a linear problem and Richardson iteration for the discretized nonlinear problem considered in this chapter. The implementation of our preconditioner can be done in parallel and consists of solving one global linear problem on a coarse grid and two types of local problems. The global problem is the standard conforming finite element one defined for Poisson's equation. The first class of local problems consists of these which are one dimensional and are associated with the vertices of subdomains. The second class includes problems that are defined on two substructures with the common edge.

The outline of this chapter is as follows. In Section 2.2, we formulate the differential problem. Section 2.3 is devoted to presenting the mortar element method and analyzing the error estimate in the geometrically conforming case. In Section 2.4, we present the geometrically nonconforming case of the mortar method. In Section 2.5, we discuss an application of Richardson-Newton method to the system of nonlinear equations which arise from discretization of the boundary value problem by the geometrically conforming mortar element method. We show that this method is almost optimal convergent. Section 2.6 is devoted to a nonlinear decomposition method of solving the nonlinear discrete problem. Finally, in Section 2.7, we discuss how to apply the results presented in the previous sections to an approximation of certain problems with an unbounded nonlinearity. There are some numerical experiments presented in Section 2.8.

2.2 Differential problem

In this section, we formulate the differential problem.

We consider the following differential equation

$$-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}a_i(x,u^*,\nabla u^*)+a_0(x,u^*,\nabla u^*)=f(x) \quad in \;\; \Omega \tag{2.1}$$

with the homogeneous Dirichlet boundary condition, where $\Omega \subset \Re^2$ is Lipschitz continuous bounded polygonal region. The weak formulation is of the form:
Find $u^* \in H_0^1(\Omega)$ such that

$$b(u^*,v)=f(v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$b(u,v)=\int_\Omega (\sum_{i=1}^{2} a_i(x,u,\nabla u\ )\, v_{x_i}+a_0(x,u,\nabla u)\, v)\, dx, \quad f(v)=\int_\Omega fv\, dx \tag{2.3}$$

Here $f \in \ L^2(\Omega)$, $\nabla u=(u_{x_1},u_{x_2})^T$ and $u_{x_i}=\frac{\partial u}{\partial x_i}, i=1,2$. Let $p=(p_0,p_1,p_2)$ and $a_i(x,p_0,p_1,p_2)=a_i(x,u,u_{x_1},u_{x_2})$. We assume that the functions $a_i:\Omega\times\Re^3\rightarrow\Re, i=$

$0, 1, 2$, satisfy the following conditions: For some positive constants M, μ_0,

$$a_i \in C^1(\Omega \times \Re^3), \quad i = 0, 1, 2 \tag{2.4}$$

$$\max\{|a_i(x, 0, 0, 0)|, |\frac{\partial a_i}{\partial x_k}(x, p)|, \ |\frac{\partial a_i}{\partial p_j}(x, p)|\} \leq \ M, \ \ for \ \ i, j = 0, 1, 2; \ \ k = 1, 2; \tag{2.5}$$

$$\sum_{i,j=0}^{2} \frac{\partial}{\partial p_j} a_i(x, p) \xi_i \xi_j \geq \mu_0 \sum_{i=1}^{2} \xi_i^2 \tag{2.6}$$

for any $\xi = (\xi_0, \xi_1, \xi_2) \in \Re^3$, and max is taken over $\Omega \times \Re^3$.

As a direct consequence of the above assumptions, we obtain that for all $p, q \in \Re^3$ there is a positive constant L such that in Ω

$$\forall p, q \in \Re^3 \ \ |a_k(x, p) - a_k(x, q)| \leq L(\sum_{i=0}^{2} |p_i - q_i|^2)^{1/2} \quad k = 0, 1, 2. \tag{2.7}$$

Under the above assumptions it can be proven that the form $b(\cdot, \cdot)$ is strongly monotone and Lipschitz continuous and that the problem (2.2) has a unique solution, see [78] or Theorem 32.6, page 240 in [106].

2.3 Geometrically conforming case

In this section, we present an analysis of mortar element method for the differential problem of Section 2.2 in the geometrically conforming case.

2.3.1 Discrete mortar space

We now define a discrete space $V^h \subset L^2(\Omega)$ that is not a subspace of $H_0^1(\Omega)$. In that sense, our method is nonconforming. In this section, we consider a geometrically conforming version of the mortar method. In the next section, we will consider the geometrically nonconforming one. We consider a partition of Ω into polygonal subdomains i.e.

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \quad with \ \ \Omega_i \cap \Omega_j = \emptyset \ \ if \ \ i \neq j$$

that are arranged in such a way that the intersection of $\overline{\Omega}_k \cap \overline{\Omega}_l$ for $k \neq l$ is either the empty set, an edge or a vertex. We call this partition geometrically conforming.

We assume the shape regularity of the decomposition, i.e. we assume, cf. Section 2, p.5 in [37], that there exists a fixed number of reference polygonal subdomains D_k with unit diameters, such that for each Ω_j there is an affine mapping $g_j = G_j + \mathbf{b}_j$ which maps that subdomain to $D_{k(j)}$, one of the reference domains. Here G_j is an invertible 2×2 matrix and $\mathbf{b}_j \in \Re^2$ a vector. We assume that

$$\|G_j\|_\infty \preceq H_j^{-1}, \quad \|G_j^{-1}\|_\infty \preceq H_j,$$

where $H_j = \mathrm{diam}(\Omega_j)$, cf. also [47].

A more general case is considered in the next section. The mortar element method first deals with the union of all edges (interfaces).

$$\Gamma = \bigcup_{k=1}^{N} \partial\Omega_k \setminus \partial\Omega \tag{2.8}$$

and consists of choosing one of the decomposition of Γ, that is made of disjoint open segments (that are edges of subdomains) called masters (mortars), denoted by γ_m, $1 \leq m \leq M$ i.e.

$$\overline{\Gamma} = \bigcup_{m=1}^{M} \overline{\gamma}_m, \quad \gamma_k \cap \gamma_l = \emptyset \;\; if \;\; k \neq l.$$

We denote the common, open edge to Ω_i and Ω_j by Γ_{ij}. By $\gamma_{m,i}$ we denote an edge of Ω_i that is a master (mortar) and by $\delta_{m,j}$ an edge of Ω_j that occupies geometrically the same place as $\gamma_{m,i}$ and it is called slave (nonmortar). There is no rule in selecting $\gamma_{m,i}$ as a master, cf. Figure 1.2.

Let us introduce some notation.

With each Ω_k we associate a quasiuniform triangulation $T_h(\Omega_k)$ made of elements that are triangles, cf. [47] or Definition 4.4.13, p.106 in [38], i.e. we assume that there are two constants c and C such that the ratio of the diameter of any element of this triangulation to the diameter of the ball inscribed in this element is bounded from above by c and that the diameter of any element is bounded from below by $C\,h_k$, where h_k is a parameter of this triangulation - the maximum over the diameters of the triangles. Note that the resulting triangulations are non-matching across Γ. We can now give the definition of the finite element functions on the introduced triangulations. Let us assume that we work with the simple generic case of linear finite elements. We first define the finite element functions locally and introduce the space

$$X_h(\Omega_k) = \{v_k \in C(\Omega_k) : v_{k|\partial\Omega} = 0,\; v_{k|\tau} \in P_1(\tau) \;\; \forall\, \tau \in T_h(\Omega_k)\},$$

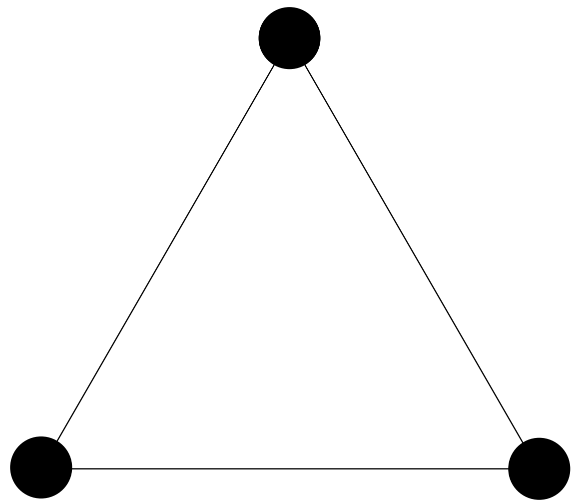

Figure 2.1: Conforming P_1 element.

where $P_1(\tau)$ is the set of all linear polynomials over the triangle $\tau \in T_h(\Omega_k)$, cf. Figure 2.1.

We also introduce space $W^{h_j}(\Gamma_{ij})$ as the restriction of $X_h(\Omega_j)$ to Γ_{ij} and a global space X_h as

$$X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k)$$

with a seminorm $|v|_{H^1_H(\Omega)} = (\sum_{i=1}^N |v_i|^2_{H^1(\Omega_i)})^{1/2}$ and a norm $\|v\|_{H^1_H(\Omega)} = (\|v\|^2_{L^2(\Omega)} + |v|^2_{H^1_H(\Omega)})^{1/2}$.

Note that, since the triangulations on two adjacent subdomains are independent, the interface $\Gamma_{ij} = \gamma_{k,i} = \delta_{k,j}$ is provided with two different and independent (1D) the h_i and h_j triangulations and two different spaces $W^{h_i}(\gamma_{k,i})$ and $W^{h_j}(\delta_{k,j})$. Additionally, we define an auxiliary test space $M^{h_j}_{0,1}(\delta_{k,j})$ of all functions which continuous and piecewise linear on the h_j triangulation of $\delta_{k,j}$ and are constant on elements which intersect the ends of $\delta_{k,j}$. If $\overline{\delta}_{k,j} \cap \partial\Omega = \emptyset$, then the space $M^{h_j}_{0,1}(\delta_{k,j})$ is a subspace of the slave space $W^{h_j}(\delta_{k,j})$ and its dimension is equal to the dimension of $W^{h_j}(\delta_{k,j})$ minus two. The double subscript of $M^{h_j}_{0,1}$ says that this space is contained in $C^0(\delta_{k,j})$ and that is based on polynomials of degree one. This subscript is not necessary in this chapter, but it will be useful for fourth order problems considered in one of the following chapters.

In what follows, we express the matching condition that is sufficient to ensure the optimality of the global approximation and define our discrete space V^h:

$$V^h = \{u \in X_h(\Omega) : \forall\, \delta_{m,j} \subset \Gamma,$$

$$\forall \psi \in M_{0,1}^{h_j}(\delta_{m,j}) \quad \int_{\Gamma_{ij}} (u_{i|\gamma_{m,i}} - u_{j|\delta_{m,j}})\psi \, ds = 0 \quad \}, \tag{2.9}$$

where $u_{i|\gamma_{m,i}}$ and $u_{j|\delta_{m,j}}$ in the integral are the traces of u_i and u_j onto $\gamma_{m,i} = \delta_{m,j} = \Gamma_{ij}$, the common edge to Ω_i and Ω_j. The condition (2.9) is called mortar.

2.3.2 Discrete problem

Our discrete problem is to find $u_h^* \in V^h$ such that

$$b_H(u_h^*, v) = f(v) \quad \forall v \in V^h, \tag{2.10}$$

where

$$b_H(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} \{\sum_{i=1}^{2} a_i(x, u_k, \nabla u_k\,)\, D_i v_k + a_0(x, u_k, \nabla u_k)\, v_k\}\, dx$$

and

$$f(v) = \sum_{k=1}^{N} \int_{\Omega_k} f\, v_k\, dx.$$

We have to introduce the new form $b_H(\cdot,\cdot)$ because the discrete space $V^h \not\subset H_0^1(\Omega)$. The subscript H is associated with $H = \max_i H_i$, where $H_i = \mathrm{diam}(\Omega_i)$, and thus indicates that this form corresponds to the coarse partition of Ω.

Remark 2.3.1 *If $u, v \in H_0^1(\Omega)$, then $b(u,v) = b_H(u,v)$.*

With the help of the Lemma 2.3.2 and Lemma 2.3.3, which are proven below in this section, we can prove that under our assumptions (2.4)-(2.6) there exists a unique solution of (2.10), cf. Theorem 32.6, p.240 in [106]. It can also be seen from Theorem 2.5.2 (see below for $M = I$).

2.3.3 Error estimate

We now state the main theorem of this section.

Theorem 2.3.1 *Let u^* and u_h^* be the solutions of (2.2) and (2.10), respectively and let $u^* \in H^2(\Omega)$. Then we have*

$$\|u^* - u_h^*\|_{H_H^1(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 \left\{ H_k^2 + \|u^*\|_{H^2(\Omega_k)}^2 \right\} \right)^{1/2} \preceq \overline{h}(|\Omega| + \|u^*\|_{H^2(\Omega)}),$$

where c_i are constants independent of any h_k and $\overline{h} = \max_k h_k$.

To prove this theorem we first state and prove some auxiliary lemmas.

We first define an auxiliary operator associated with the form (2.3) which is a generalization of normal derivative in the case of linear operators.

Definition 2.3.1 *Let $\gamma \subset \partial\Omega_i$ be a segment. Then let $l_i : H^2(\Omega_i) \to L^2(\gamma)$ be defined as*

$$(l_i u)(x) = \sum_{k=1}^{2} a_k(x, u(x), \nabla u(x)) n_k \quad on \ \ \gamma, \tag{2.11}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit normal to γ (outward to $\partial\Omega_i$).

It is easy to show that under assumptions (2.4)-(2.5) $l_i u$ is well defined in $L^2(\gamma)$, cf. [106].

The next lemma states the continuity of l_i across γ, the part of the interface.

Lemma 2.3.1 *Let $\gamma \subset \partial\Omega_i \cap \partial\Omega_j$ be a segment and $\overline{\Omega}_{ij} = \overline{\Omega}_i \cup \overline{\Omega}_j$ and $l_i : H^2(\Omega_i) \to L^2(\gamma)$ and $l_j : H^2(\Omega_j) \to L^2(\gamma)$ be defined by (2.11). Then for $u \in H^2(\Omega_{ij})$, we have*

$$l_j u_j = -l_i u_i \quad a.e. \ \ on \ \ \gamma,$$

i.e.

$$\sum_{k=1}^{2} a_k(x, u_i(x), \nabla u_i(x)) n_k = \sum_{k=1}^{2} a_k(x, u_j(x), \nabla u_j(x)) n_k \ \ a.e. \ \ on \ \ \gamma,$$

where $\mathbf{n} = (n_1, n_2)$ is the normal unit outward to $\gamma \cap \Omega_i$ and u_i, u_j are the restrictions of u to Ω_i and Ω_j, respectively.

Proof. First we prove that l_i is Lipschitz continuous. We may assume that γ is parallel to the OX axis and then we have $l_i u(x) = a_2(x, u(x), \nabla u(x))$. We use $\frac{\partial}{\partial x_0} u(x) = u(x)$ to simplify the notation. Let now $u, v \in H^2(\Omega_{ij})$, then we conclude that

$$\int_\gamma |a_2(x, u, \nabla u) - a_2(x, v, \nabla v)|^2 \, dx \preceq \int_\gamma \sum_{i=0}^{2} |\frac{\partial}{\partial x_i}(u - v)|^2 \, dx \preceq \|u - v\|^2_{H^2(\Omega_{ij})}$$

We have used the standard trace theorem, cf. e.g. Theorem 1.5.2.1, p.42 in [71], and (2.7). The statement of the lemma obviously is satisfied for $u \in C^\infty(\overline{\Omega}_{ij})$. Using the density of $C^\infty(\overline{\Omega}_{ij})$ in $H^2(\Omega_{ij})$ and Lipschitz continuity of l_i and l_j, we get the conclusion.

□

The next corollary can be proven using the same ideas as in the proof of the previous lemma, cf. also [106].

Corollary 2.3.1 *Let $\gamma \subset \partial\Omega_i$ be a segment and $l_i : H^2(\Omega_i) \to L^2(\gamma)$ be defined by (2.11). Then under the assumptions (2.4)-(2.5): $l_i u \in H^{1/2}(\gamma)$ with $|l_i u|_{H^{1/2}(\gamma)} \preceq (H_i + \|u\|^2_{H^2(\Omega_i)})^{1/2}$, where $H_i = \mathrm{diam}(\Omega_i)$.*

Now we define the restriction of the form $b(\cdot,\cdot)$ to $H^1(\Omega_i)$.

Definition 2.3.2 *Let $b_i(\cdot,\cdot) : H^1(\Omega_i) \times H^1(\Omega_i) \to \Re$ be the bilinear form defined as*

$$b_i(u, v) = \int_{\Omega_i} (\sum_{i=1}^{2} a_i(x, u, \nabla u\ ) \, v_{x_i} + a_0(x, u, \nabla u) \, v) \, dx \quad \forall u, v \in H^1(\Omega_i)$$

Remark 2.3.2 *Note that for all $u, v \in X_h(\Omega)$ $b_H(u, v) = \sum_{i=1}^N b_i(u_i, v_i)$.*

Using the assumptions (2.4) - (2.6) and the results of [16] (or [17]) we can prove that the form $b_H(u, v)$ is strongly monotone and Lipschitz continuous in V^h. We state that in the two following lemmas.

Lemma 2.3.2 *The form $b_H(\cdot,\cdot)$ is strongly monotone in V^h, i.e. for all $u, v \in V^h$ we have*

$$b_H(u, u - v) - b_H(v, u - v) \succeq \|u - v\|^2_{H^1_H(\Omega)}$$

Proof. Let $u = \{u_k\}, v = \{v_k\} \in V^h$. We see that u_k, v_k are in $H^1(\Omega_k)$. In [106], see Theorem 32.6, page 240, is proven that

$$\forall u, v \in H^1(\Omega_i) \;\; b_i(u, u-v) - b_i(v, u-v) \succeq |u-v|^2_{H^1(\Omega_i)}$$

From that, we deduce that

$$b_H(u, u-v) - b_H(v, u-v) = \sum_{k=1}^{N} b_k(u_k, u_k - v_k) - b_k(v_k, u_k - v_k) \succeq$$

$$\succeq \sum_{k=1}^{N} |u_k - v_k|^2_{H^1(\Omega_k)} = |u-v|^2_{H^1_H(\Omega)} \succeq \|u-v\|^2_{H^1_H(\Omega)}$$

The last estimate follows from the fact that for $u \in V^h$, $|u|_{H^1_H(\Omega)}$ is estimated from below by $C\,\|u\|_{H^1_H(\Omega)}$, with a constant C independent of all h_k and the number of subdomains, see Proposition 2.1 in [16]. □

Lemma 2.3.3 *The form $b_H(\cdot,\cdot)$ is Lipschitz continuous in $X_h(\Omega)$ (and thus also in V^h), i.e. for all $u, v, w \in X_h(\Omega)$ we have*

$$|b_H(u,w) - b_H(v,w)| \preceq \|u-v\|_{H^1_H(\Omega)}\,\|w\|_{H^1_H(\Omega)}$$

Proof. In [106], see Theorem 32.6, page 240, is proven that

$$\forall\, u, v, w \in H^1(\Omega_i) \quad |b_i(u,w) - b_i(u,v)| \preceq \|u-v\|^2_{H^1(\Omega_i)}\,\|w\|^2_{H^1(\Omega_i)}$$

Summing over all subdomains, using the above result for $u, v, w \in X_h(\Omega)$ and Schwarz inequality (for the standard inner product in $\Re^N$) we end the proof. □

We now formulate and prove a lemma which is a generalization of the second Strang lemma for the boundary value problem considered in this paper, cf. [12] or Lemma 8.1.9, p.198 in [38] for the linear case.

Lemma 2.3.4 *Let u^* and u_h^* be the solutions of (2.2) and (2.10), respectively. Let $u^* \in H^2(\Omega)$. Under the assumptions (2.4) - (2.6) we have*

$$\|u^* - u_h^*\|_{H^1_H(\Omega)} \preceq \left\{ \inf_{v \in V^h} \|u^* - v\|_{H^1_H(\Omega)} + \sup_{w \neq 0} \sum_{\delta_m \subset \Gamma} \frac{\int_{\delta_m} l_m u^*\,[w] ds}{\|w\|_{H^1_H(\Omega)}} \right\}, \quad (2.12)$$

where $[w]$ is a jump of w across $\Gamma_{ij} = \delta_{m,j} = \gamma_{m,i}$, $l_m u^*_{|\delta_m} = l_{i(m)} u^*$ is defined in (2.11) and the sum is taken over all slaves $\delta_{m,j} = \gamma_{m,i} = \Gamma_{ij}$.

Proof. We have for all $v \in V^h$

$$\|u^* - u_h^*\|_{H^1_H(\Omega)} \leq \|u^* - v\|_{H^1_H(\Omega)} + \|u_h^* - v\|_{H^1_H(\Omega)}. \tag{2.13}$$

Let $w = u_h^* - v$, then Lemma 2.3.2 and (2.10) yield that

$$\|u_h^* - v\|^2_{H^1_H(\Omega)} \preceq b_H(u_h^*, w) - b_H(v, w) = f(w) - b_H(v, w). \tag{2.14}$$

Note that by (2.1)

$$f(w) = \sum_{k=1}^{N} \int_{\Omega_k} \left(\sum_{i=1}^{2} (-\frac{\partial}{\partial x_i} a_i(x, u^*, \nabla u^*)) + a_0(x, u^*, \nabla u^*) \right) w_k \, dx =$$

$$= b_H(u^*, w) - \sum_{k=1}^{N} \int_{\partial\Omega_k} \sum_{i=1}^{2} a_i(s, u^*, \nabla u^*) n_i \, w_k \, ds =$$

$$= b_H(u^*, w) + \sum_{\gamma_m \subset \Gamma} \int_{\gamma_m} l_m u^* \, [w] ds.$$

The last equality follows from Lemma 2.3.1. Using this and Lemma 2.3.3 in (2.14) we get

$$\|u_h^* - v\|^2_{H^1_H(\Omega)} \preceq b_H(u^*, w) - b_H(v, w) + \sum_{\gamma_m \subset \Gamma} \int_{\gamma_m} l_m u^* \, [w] ds$$

$$\preceq \|u^* - v\|_{H^1_H(\Omega)} \|w\|_{H^1_H(\Omega)} + \sum_{\gamma_m \subset \Gamma} \int_{\gamma_m} l_m u^* \, [w] ds.$$

Dividing by $\|w\|_{H^1_H(\Omega)}$ and substituting the resulting inequality in (2.13) we complete the proof. □

The first term in (2.12) is known as the approximation error and the second one we call the consistency error which is a consequence of the discontinuities of the elements of V^h through the interface. We can now turn to the proof of Theorem 2.3.1.

Proof. We use Lemma 2.3.4. Let us first consider the approximation error. In [19], see Inequality (5.5), Section 5.2 there, it was proven that if $v \in H^1_0(\Omega)$ with $v_{|\Omega_k} \in H^2(\Omega_k)$, then

$$\inf_{w \in V_h} \|v - w\|^2_{H^1_H(\Omega)} \preceq \sum_{k=1}^{N} h_k^2 \, |v_{|\Omega_k}|^2_{H^2(\Omega_k)}. \tag{2.15}$$

Let us turn to the consistency term. We now prove that

$$|\sum_{\delta_m}\int_{\delta_m} l_m u^* \,[w] ds\,| \preceq \sum_{k=1}^{N} h_k \, \left\{|\Omega_k| + \|u^*\|^2_{H^2(\Omega_k)}\right\}^{1/2} \|w_k\|_{H^1(\Omega_k)}. \tag{2.16}$$

It can be done in a similar way to [11]. Let us consider one interface Γ_{ij} common to Ω_i and Ω_j. Let $\gamma_{m,i} = \Gamma_{ij}$ be its master, then $\delta_{m,j} = \Gamma_{ij}$ is a slave. From (2.9) we have

$$\forall \psi \in M^{h_j}_{0,1}(\delta_{m,j}) \quad \int_{\delta_{m,j}} l_m u^* \,[w]\, ds = \int_{\delta_{m,j}} (l_m u^* - \psi)\,(w_j - w_i)\, ds.$$

Hence

$$|\int_{\delta_{m,j}} l_m u^* \,[w] ds\,| \leq \inf_{\psi \in M^{h_j}_{0,1}(\delta_{m,j})} \|l_m u^* - \psi\|_{[H^{1/2}(\delta_{m,j})]'} \|w_j - w_i\|_{H^{1/2}(\delta_{m,j})}.$$

Note that from the mortar condition (2.9) follows that average values of u over a master $\gamma_{m,i}$ and a slave $\delta_{m,j}$ that geometrically coincides, are equal to each other. Using the trace theorem, cf. Theorem 1.5.2.1, p.42 in [71], this observation and the Poincaré's inequality we have

$$|\int_{\delta_{m,j}} l_m u^* \,[w] ds\,| \preceq \inf_{\psi \in M^{h_j}_{0,1}} \|l_m u^* - \psi\|_{[H^{1/2}(\delta_{m,j})]'} \,(|w_i|_{H^1(\Omega_i)} + |w_j|_{H^1(\Omega_j)}).$$

As in [19], see Lemma 4.1 there, we can deduce that

$$\inf_{\psi \in M^{h_j}_{0,1}(\delta_{m,j})} \|l_m u^* - \psi\|_{[H^{1/2}(\delta_{m,j})]'} \preceq h_j \, |l_m u^*|_{H^{1/2}(\delta_{m,j})}. \tag{2.17}$$

It can be proven combining a duality method, an approximation property of $M^{h_j}_{0,1}(\delta_{m,j})$ and an interpolation argument. We now sum over all slaves $\delta_{m,j} = \Gamma_{ij}$ and use Corollary 2.3.1 what proves (2.16). Combining (2.16) and (2.15) completes the proof of the theorem. □

2.4 Geometrically nonconforming case

In this section, we consider the geometrically nonconforming version of the mortar finite element method, i.e. when a vertex of a subdomain Ω_i is not necessarily a vertex of one of neighboring substructures.

2.4.1 Discrete problem

We assume that Ω is divided into disjoint polygonal subregions Ω_k of diameter of order one, and we introduce in each subdomain a quasi-uniform triangulation. We also assume the shape regularity of this decomposition, cf. Section 2.3. Let $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. The local spaces $X_h(\Omega_k)$ and the global space $X_h(\Omega)$ are the same as in Section 2.3. The mortar element method consists of choosing one of the decomposition of Γ defined in (2.8), made of masters γ_m that are disjoint i.e.

$$\overline{\Gamma} = \bigcup_{m=1}^{M} \overline{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \quad m \neq n$$

and that satisfy the assumption that each master is an edge of one subdomain i.e. $\gamma_m = \gamma_{m,i}$ is an edge of Ω_i. We assume that there is at least one such decomposition.

The master sides of Ω_i we denote by $\gamma_{m,i}$ and the slave sides (the edges that are not masters) we denote by $\delta_{k,i}$. For each edge that is not a master we define a space of traces $W^{h_j}(\delta_{k,j})$ and a test space $M_{0,1}^{h_j}(\delta_{k,j})$ in the same way as in the previous section, i.e. $W^{h_j}(\delta_{k,j})$ is the restriction of $X_h(\Omega_j)$ to $\delta_{k,j}$ and $M_{0,1}^{h_j}(\delta_{k,j})$ is the subspace of $W^{h_j}(\delta_{k,j})$ formed by all functions which are constant on elements which intersect the ends of $\delta_{k,j}$.

Now we define our discrete space as

$$V^h = \{\, v \in X_h(\Omega) : \; \forall \delta_{k,j} - \textit{not a master}$$

$$\forall \psi \in M_{0,1}^{h_j}(\delta_{k,j}) \quad \int_{\delta_{k,j}} (v_{j\,|\delta_{k,j}} - \sum v_{i\,|\delta_{k,j} \cap \gamma_{m,i}})\, \psi \, ds \,\}, \tag{2.18}$$

where the sum is taken over Ω_i with $\gamma_{m,i}$ such that $\gamma_{m,i} \cap \delta_{k,j} \neq \emptyset$.

The discrete problem is to find $u_h \in V^h$ such that

$$b_H(u_h, v) = f(v) \;\; \forall v \in V^h, \tag{2.19}$$

where the form $b_H(\cdot, \cdot)$ is defined as in Section 2.3.

We have that the norm $\|\cdot\|_{H^1_H(\Omega)}$ and seminorm $|\cdot|_{H^1_H(\Omega)}$ are equivalent, see Proposition A.3, Appendix A in [19], and that analogous results to statements of Lemmas 2.3.2 and 2.3.3, are also valid in this case. Thus there exists a unique solution of (2.19), cf. e.g. Theorem 34.1, p.245 in [106].

2.4.2 Error estimate

We now state the main theorem of this section that yields the error estimate.

Theorem 2.4.1 *Let u^* and u_h be the solutions of (2.2) and (2.19), respectively. Let $u^* \in H^\sigma(\Omega), 2 \leq \sigma < 5/2$. Then under the hypotheses of (2.4) - (2.6) we have*

$$\|u^* - u_h\|_{H^1_H(\Omega)} \leq \left(\sum_{k=1}^{N} C_k h_k^{2\sigma-3}\right)^{1/2} \preceq \overline{h}^{\sigma-3/2},$$

where C_k, are constants independent of any h_k (dependent on $H^\delta(\Omega_k)$ norm of $u^*_{|\Omega_k}$). Here $\overline{h} = \max_k h_k$.

To prove Theorem 2.4.1, we state an analogue of Lemma 2.3.4 for the geometrically nonconforming case. The proof is similar to that of Lemma 2.3.4.

Lemma 2.4.1 *Let u^* and u_h be the solutions of (2.2) and (2.19), respectively. Let $u^* \in H^2(\Omega)$. Under the assumptions (2.4)-(2.6) we have*

$$\|u^* - u_h\|_{H^1_H(\Omega)} \preceq \inf_{v \in V^h} \|u^* - v\|_{H^1_H(\Omega)} + \sup_{w \in V^h} \sum_{\delta_m \subset \Gamma} \frac{\int_{\delta_m} l_m u^* \, [w] ds}{\|w\|_{H^1_H(\Omega)}}, \quad (2.20)$$

where $[w]$ is the jump of w across δ_m and $l_m u^*_{|\delta_m} = l_{i(m)} u^*$ is defined in (2.11). The sum is taken over all δ_m, sides of substructures that are not masters.

We now prove Theorem 2.4.1

Proof. The first term, the approximation error, can be estimated from (2.15) whose statement is also true in this case, cf. Inequality (5.5), Section 5.2 in [19].

It remains to estimate the second term, the consistency error. We first prove that

$$| \int_{\delta_{m,j}} l_m u^* \, [w] ds \, | \preceq h_j \, |l_m u^*|_{H^1(\delta_{m,j})} \, (\|w_j\|_{H^1(\Omega_j)} + \sum_k \|w_k\|_{H^1(\Omega_k)}) \quad (2.21)$$

which is an analogue to (2.16). The sum is taken over all Ω_k such that $\partial\Omega_k \cap \delta_{m,j} \neq \emptyset$. Proceeding as in the proof of (2.16) we see that from (2.18) for all $\psi \in M^{h_j}_{0,1}(\delta_{m,j})$

$$| \int_{\delta_{m,j}} l_m u^* \, [w] ds \, | = | \int_{\delta_{m,j}} l_m u^* \, (w_j - \sum_{k:\partial\Omega_k \cap \delta_{m,j} \neq \emptyset} w_{k|\partial\Omega_k \cap \delta_{m,j}}) \, ds| =$$

$$= |\int_{\delta_{m,j}} (l_m u^* - \psi)\,(w_j - \sum_{k:\partial\Omega_k \cap \delta_{m,j} \neq \emptyset} w_{k|\partial\Omega_k \cap \delta_{m,j}})\, ds|.$$

Using Schwarz inequality, we have

$$|\int_{\delta_{m,j}} l_m u^*\,[w] ds\,| \leq \inf_{\psi \in M_{0,1}^{h_j}} \|l_m u^* - \psi\|_{L^2(\delta_{m,j})} \|w_j - \sum_{k:\partial\Omega_k \cap \delta_{m,j} \neq \emptyset} w_{k|\partial\Omega_k \cap \delta_{m,j}}\|_{L^2(\delta_{m,j})}.$$

Combining the trace theorem and the approximation result for $l_m u^* \in H^{\sigma-3/2}(\delta_{m,j})$, cf. (2.17) and Lemma 4.1 in [19], we prove (2.21). As $u^* \in H^\sigma(\Omega), 2 \leq \sigma < 5/2$, we can obtain an analogous result to that of Corollary 2.3.1. Combining this with (2.21) and summing over all $\delta_{m,j}$ we finish the proof of Theorem 2.4.1. □

Remark 2.4.1 *The technique of our proofs does not enable us to obtain the optimal error estimate in the geometrically non-conforming version of the mortar method. In the case of linear elliptic second order problems, there is the optimal error estimate, see [19].*

2.5 Richardson-Newton method

In this section, we propose a method for solving the problem (2.10) arising from discretization of the boundary value problem (2.2) by the geometrically conforming mortar finite element method. It is not clear how to generalize this method to the geometrically nonconforming case.

2.5.1 Description

We have to add a new assumption, namely, we assume that the master (mortar) side of an interface Γ_{ij} which is a common edge to Ω_i and Ω_j is chosen in such a way that $h_i \leq h_j$. We could construct a similar method without this assumption, but it would be more complicated, namely, instead of simply basis functions corresponding to vertices we would have to use more complex ones, see Remark 2.5.1, below.

For the simplicity of presentation, we describe the method with one additional assumption that the subdomains Ω_i are triangles and form a shape regular triangulation with a parameter $H = \max_k H_k$ with $H_k = \mathrm{diam}(\Omega_k)$, cf. [47].

To define our method, we first have to introduce some special functions and subspaces of V^h. We will primarily work with nodal basis of V^h. They satisfy the mortar condition and are associated with the following sets of nodes:

- all interior nodes of the substructures Ω_j,
- all interior nodes to the masters $\gamma_{m,i} \subset \Gamma$,
- all vertex nodes of Ω_j except those on $\partial\Omega$.

We associate a basis function with each node of these sets. A function ϕ_k corresponding to a node in the interior of a substructure is a standard nodal basis function as in a conforming finite element discretization. A function ϕ_k associated with a node x_k interior to a master $\gamma_{m,i}$, we define as follows. It is equal to one at x_k and zero at the remaining nodes defined above, i.e. nodes interior to all substructures, vertices of all substructures and all nodes in the interiors of the masters except x_k. The values of this function at the interior nodes of slaves $\delta_{m,j} = \gamma_{m,i}$ are determined by the mortar condition (2.9), with zero values at the ends of $\delta_{m,j}$. We now define basis functions ϕ_k associated with the vertices of the substructures. We first denote by $\mathcal{V}(\Omega_k)$ the set of vertices of the substructure Ω_k that are associated with degrees of freedom of V^h, i.e. those which are not on $\partial\Omega$. We also introduce $\mathcal{V} := \bigcup_{k=1}^{N} \mathcal{V}(\Omega_k)$. Each crosspoint c_r of Γ belongs to several subdomains and therefore corresponds to several nodes of $\mathcal{V}$, to one degree of freedom for each of subregions that meet at that point. These nodes are in the same geometrically position, but are assigned to different subdomains. Let $x_k \in \mathcal{V}$ be a vertex of Ω_i. Then the basis function associated with $x_k \in \mathcal{V}$ we denote by ϕ_k. It is defined as equal to one at x_k, and to zero at all other vertices of $\mathcal{V}$ and at all interior nodes of all substructures and masters. We now define this function on Γ, i.e. on all masters and slaves. There are three possible situations: the vertex x_k can be the common end of two masters: $\gamma_{n,i}$ and $\gamma_{m,i}$, the common end of two slaves $\delta_{l,i}$ and $\delta_{k,i}$ or the common end of a slave $\delta_{s,i}$ and a master $\gamma_{p,i}$. In the first case, ϕ_k restricted to $\gamma_{n,i}$ and $\gamma_{m,i}$ is a standard nodal function corresponding to x_k, i.e. is one at x_k and zero at the remaining nodes of the both masters. On slaves $\delta_n = \gamma_{n,i}$ and $\delta_m = \gamma_{m,i}$, this function is determined by the mortar condition (2.9) with zero values at the ends of δ_n and δ_m, respectively. In the second case, ϕ_k restricted to the masters $\gamma_l = \delta_{l,i}$ and $\gamma_k = \delta_{k,i}$ is zero, and on $\delta_{l,i}$ and $\delta_{k,i}$ is determined by the mortar condition with the value equal to one at x_k and zero at the other ends of $\delta_{l,i}$ and $\delta_{k,i}$. In the last case, ϕ_k is defined on the master $\gamma_{p,i}$ (and $\delta_p = \gamma_{p,i}$) as in the first case while on the slave $\delta_{s,i}$ (and $\gamma_s = \delta_{s,i}$) analogously to the second case. In all cases, ϕ_k

is defined as zero on the remaining masters and slaves. It is obvious that all those basis functions form a basis of the discrete space V^h, i.e.

$$V^h = \text{span}\{\phi_1, \dots, \phi_n\}.$$

Note that there are no basis functions associated with interior nodal points of a slave $\delta_{m,j}$.

Let the solution of (2.10) be represented as $u_h^* = \sum_{i=1}^n \alpha_i \phi_i, \alpha_i = u(x_i)$ and introduce

$$k_i(\alpha_1, \dots, \alpha_n) = b_H \left(\sum_{j=1}^n \alpha_j \phi_j, \phi_i \right), \quad f_i = (f, \phi_i)_{L^2(\Omega)}.$$

Let $B = (k_1, \dots, k_n)^T$, $\mathbf{u_h^*} = (\alpha_1, \dots, \alpha_n)^T$ and $\mathbf{f} = (f_1, \dots, f_n)^T$. With these notations we rewrite the problem (2.10) as a system of nonlinear algebraic equations

$$B(\mathbf{u}_h^*) = \mathbf{f}. \tag{2.22}$$

Here and below if u is a function in V^h, then $\mathbf{u}$ denotes the vector representation of u in terms of the nodal basis, i.e. if $u = \sum_{i=1}^n a_i \phi_i$, then $\mathbf{u} = (a_1, \dots, a_n)^T$.

Additionally, we introduce a bilinear form on $V^h \times V^h$

$$a_\triangle(u, v) := \sum_{i=1}^N \int_{\Omega_i} \nabla u_i \, \nabla v_i \, dx. \tag{2.23}$$

Let D be its matrix representation, i.e. $D = \{a_\triangle(\phi_k, \phi_l)\}_{k,l=1,\dots,n}$. We should also point out that $a_\triangle(u, u)^{1/2} = (D\mathbf{u}, \mathbf{u})_{\Re^n}^{1/2} = |u|_{H_H^1(\Omega)}$, therefore $a_\triangle(\cdot, \cdot)$ is positive definite over V^h.

We solve (2.22) by a method that combines the additive Schwarz preconditioning technique, see Section 1.4, with Newton's method. The Schwarz method is determined by subspaces of V^h, bilinear forms defined on these subspaces and projections defined by these subspaces and bilinear forms, cf. Section 1.4 or [60]. In our case, those forms are equal to $a_\triangle(\cdot, \cdot)$. We now define subspaces that form the decomposition of V^h. Let V_0 be the coarse space of continuous, piecewise linear functions on the coarse triangulation which are equal to zero on $\partial\Omega$, of course $V_0 \subset V^h$. We next define one dimensional vertex spaces V_k^i which are associated with vertices $x_k \in \mathcal{V}(\Omega_i)$: $V_k^i = \text{span}\{\phi_k\}$, where ϕ_k is the basis function associated with a vertex x_k, see above.

Remark 2.5.1 *We can remove the assumption that master side of an interface Γ_{ij} is $\gamma_{m,i} \subset \partial\Omega_i$ if the parameter h_i is not greater than h_j - the parameter of the slave $\delta_{m,j} \subset \partial\Omega_j$. We would have to replace a vertex basis function ϕ_k corresponding to a vertex $x_k \in \partial\Omega_i$ by a modified vertex function $\hat{\phi}_k$ which is equal to ϕ_k over Γ and Ω_i, but is defined as discrete harmonic part of ϕ_k in all subdomains $\Omega_j, j \neq i$. For these modified function the statement of Lemma 2.5.1 is also true and thus all results concerning convergence of the Algorithm 2.5.1, see below, are also valid.*

Finally, we introduce V_{ij} spaces associated with all pairs of two subdomains Ω_i and Ω_j that have the common edge Γ_{ij} which is the master $\gamma_{m,i} \subset \partial\Omega_i$ and the slave $\delta_{m,j} \subset \partial\Omega_j$. We define V_{ij} as a subspace of V^h such that its functions can be nonzero at the interior nodes of Ω_i and Ω_j and at the interior nodes of $\gamma_{m,i}$ and $\delta_{m,j}$. It is easy to see that

$$V^h = V_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} V_k^i + \sum_{\Gamma_{ij} \subset \Gamma} V_{ij}.$$

We then define operators $T_0 : V^h \to V_0$, $T_k^i : V^h \to V_k^i$ and $T_{ij} : V^h \to V_{ij}$, by

$$a_\triangle(T_0(u), v) = b_H(u, v) \quad \forall v \in V_0, \tag{2.24}$$

$$a_\triangle(T_k^i(u), v) = b_H(u, v) \quad \forall v \in V_k^i \tag{2.25}$$

and

$$a_\triangle(T_{ij}(u), v) = b_H(u, v) \quad \forall v \in V_{ij}. \tag{2.26}$$

The vector representation of these operators are denoted by the same symbols. They have the following form: $T_0(\mathbf{u}) = R_0^T D_0^{-1} R_0 B(\mathbf{u})$, $T_k^i(\mathbf{u}) = (R_k^i)^T (D_k^i)^{-1} R_k^i B(\mathbf{u})$ and $T_{ij}(\mathbf{u}) = R_{ij}^T D_{ij}^{-1} R_{ij} B(\mathbf{u})$, where D_0, D_k^i and D_{ij} are the matrix representations of $a_\triangle(\cdot,\cdot)$ in the corresponding subspaces and $R_0 : V^h \to V_0$, $R_k^i : V^h \to V_k^i$ and $R_{ij} : V^h \to V_{ij}$ are the restrictions operators defined as in [20], see p.44,124 there. In subspaces, we can use natural nodal basis, i.e. $V_0 = \text{span}\{\phi_{c_r}^{coarse}\}_{c_r \in \Gamma}$, $V_k^i = \text{span}\{\phi_k\}$ and $V_{ij} = \text{span}\{\phi_k\}_{x_k \in \gamma_{m,i} \cup \Omega_i \cup \Omega_j}$, where $\phi_{c_r}^{coarse}$ is a nodal coarse function corresponding to a crosspoint c_r, i.e. $\phi_{c_r}^{coarse}$ is a continuous function which is piecewise linear on coarse triangulation, equal to one at c_r and equal to zero at all remaining crosspoints. We note that T_0, T_k^i and T_{ij} are nonlinear in general. To define an additive Schwarz method , let

$$T = T_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} T_k^i + \sum_{\Gamma_{ij} \subset \Gamma} T_{ij}. \tag{2.27}$$

We replace the problem (2.22) by the problem of finding $u \in V^h$ such that

$$T(u_h^*) = g, \tag{2.28}$$

where $g = g_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} g_k^i + \sum_{\Gamma_{ij} \subset \Gamma} g_{ij}$ with $g_0 = T_0(u_h^*)$, $g_k^i = T_k^i(u_h^*)$ and $g_{ij} = T_{ij}(u_h^*)$. Here u_h^* is the solution of (2.22). We show that problems (2.28) and (2.22) have the same unique solution. These g_i can be pre-computed without knowing the exact solution u. In the implementation, we usually do not do that.

Introducing a linear operator

$$M^{-1} = R_0^T D_0^{-1} R_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} (R_k^i)^T (D_k^i)^{-1} R_k^i + \sum_{\Gamma_{ij} \subset \Gamma} R_{ij}^T D_{ij}^{-1} R_{ij} \tag{2.29}$$

we have

$$T = M^{-1} B.$$

For solving (2.28), we use the following algorithm:

Algorithm 2.5.1 *Let $u^0 \in V^h$ be arbitrary and τ as in Theorem 2.5.2, see below.*

- *Iterate for $n = 0, 1, \ldots$ until convergence,*
  - *Compute $r_0^n = T_0(u^n) - g_0$ solving*

    $$a_\triangle(r_0^n, v) = b_H(u^n, v) - (f, v) \quad \forall v \in V_0.$$

  - *Compute $r_k^{i,n} = T_k^i(u^n) - g_k^i$ for $x_k \in \mathcal{V}(\Omega_i), i = 1, \ldots, N$, solving*

    $$a_\triangle(r_i^n, v) = b_H(u^n, v) - (f, v) \quad \forall v \in V_k^i.$$

  - *Compute $r_{ij}^n = T_{ij}(u^n) - g_{ij}$ for all masters $\gamma_{m,i} = \Gamma_{ij} \subset \Gamma$ solving*

    $$a_\triangle(r_{ij}^n, v) = b_H(u^n, v) - (f, v) \quad \forall v \in V_{ij}.$$

  - *Compute u^{n+1} as*

    $$u^{n+1} = u^n - \tau \, (r_0^n + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} r_k^{i,n} + \sum_{\Gamma_{ij} \subset \Gamma} r_{ij}^n) = u^n - \tau \, (T(u^n) - g). \tag{2.30}$$

- *End n.*

2.5.2 Analysis of the convergence

To prove the convergence of the algorithm we need the following auxiliary result.

Theorem 2.5.1 *For any $u \in V^h$*

$$(1+\log(H/\underline{h}))^{-2}(D\mathbf{u},\mathbf{u})_{\Re^n} \preceq (DM^{-1}D\mathbf{u},\mathbf{u})_{\Re^n} \preceq (D\mathbf{u},\mathbf{u})_{\Re^n},$$

where all constants are independent of H, h_i, $H = \max_i H_i$ and $\underline{h} = \inf_i h_i$.

This theorem will be proved in the last part of this section. Theorem 2.5.1 yields that (2.28) has a unique solution equal to the solution of (2.22) and that M^{-1} is invertible. The next corollary plays an important role in the proof of convergence of the algorithm.

Corollary 2.5.1 *For any $u, v \in V^h$ holds*

$$(B(\mathbf{u}) - B(\mathbf{v}), \mathbf{u}-\mathbf{v})_{\Re^n} \geq \delta_0 \left\|\mathbf{u}-\mathbf{v}\right\|_M^2$$

and

$$\|B(\mathbf{u}) - B(\mathbf{v})\|_{M^{-1}} \leq \delta_1 \|\mathbf{u}-\mathbf{v}\|_M,$$

where M was defined in (2.29), $\delta_0 = C\ (1+\log(H/\underline{h}))^{-2}$, and C, δ_1 are positive constants independent of H, h_i.

Proof. Theorem 2.5.1 yields that

$$(D\mathbf{u},\mathbf{u})_{\Re^n} \preceq (M\mathbf{u},\mathbf{u})_{\Re^n} \preceq (1+log(H/\underline{h}))^2(D\mathbf{u},\mathbf{u})_{\Re^n} \quad \forall u \in V^h$$

cf. e.g. [65] or Chapter 1 in [26].

In other words, we have the following equivalence of the energetic norms $\|\cdot\|_D \preceq \|\cdot\|_M \preceq (1+log(H/\underline{h}))^{-2}\|\cdot\|_D$ in $\Re^n$. Then we have

$$(B(\mathbf{u}) - B(\mathbf{v}), \mathbf{u}-\mathbf{v})_{\Re^n} = b_H(u, u-v) - b_H(v, u-v) \succeq |u-v|^2_{H^1_H(\Omega)} =$$

$$= (D(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v})^2_{\Re^n} \succeq (1+log(H/\underline{h}))^{-2}(M(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v})^2_{\Re^n}.$$

We have used Lemma 2.3.2. Let $\mathbf{g} = B(\mathbf{u}) - B(\mathbf{v})$. We next deduce that

$$\|B(\mathbf{u}) - B(\mathbf{v})\|_{M^{-1}} = \|M^{-1}\mathbf{g}\|_M = \sup_{\|\mathbf{x}\|_M = 1} |(M^{-1}\mathbf{g}, \mathbf{x})_M| =$$

$$= \sup_{\|\mathbf{x}\|_M = 1} |(B(\mathbf{u}) - B(\mathbf{v}), \mathbf{x})_{\Re^n}| = \sup_{\|\mathbf{x}\|_M = 1} |b_H(u,x) - b_H(v,x)| \preceq \|\mathbf{u} - \mathbf{v}\|_D.$$

We have used Lemma 2.3.3 and the equivalence of broken H_H^1 seminorm and broken H_H^1 norm over V^h, see Proposition 2.1 in [16]. □

We now state the main theorem of this section that can be proven in the standard way using Corollary 2.5.1, cf. [41] or [87].

Theorem 2.5.2 *If we choose $0 < \tau < 2\delta_0/\delta_1^2$, where δ_0 and δ_1 are defined in Corollary 2.5.1, then Algorithm 2.5.1 is convergent in the sense that*

$$\|\mathbf{u}^n - \mathbf{u}_h^*\|_M \leq \rho(\tau)^n \|\mathbf{u}^0 - \mathbf{u}_h^*\|_M,$$

where $\rho(\tau)^2 = 1 - \delta_1^2\tau\,(2\delta_0/\delta_1^2 - \tau) < 1$. The optimal parameter $\tau_{opt} = \delta_0/\delta_1^2$ and $\rho_{opt}^2 = 1 - (\delta_0/\delta_1)^2$.

Proof. By (2.30) and Corollary 2.5.1 we have

$$\|\mathbf{u}^{n+1} - \mathbf{u}_h^*\|_M^2 = \|\mathbf{u}^n - \mathbf{u}_h^*\|_M^2 - 2\tau\,(M^{-1}(B(\mathbf{u}^n) - B(\mathbf{u}_h^*), \mathbf{u}^n - \mathbf{u}_h^*)_M +$$
$$+ \|M^{-1}(B(\mathbf{u}^n) - B(\mathbf{u}_h^*))\|_M^2 \preceq (1 - 2\delta_0\tau + \delta_1^2\tau^2)\|\mathbf{u}^n - \mathbf{u}_h^*\|_M^2$$

□

We need one additional technical lemma.

Lemma 2.5.1 *If we assume that for an interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, $\gamma_{m,i} \subset \partial\Omega_i$ is its master if $h_i \leq h_j$, then for a vertex basis function ϕ_k which corresponds to the vertex $x_k \in \mathcal{V}$ holds*

$$|\phi_k|_{H_H^1(\Omega)} \leq C,$$

where C is a positive constant independent of the number of subdomains and any h_k.

Proof. Let c_r be a crosspoint with which the vertex $x_k \in \mathcal{V}(\Omega_i)$ geometrically coincides. Note that from definition, ϕ_k is zero at all nodal interior points thus we can deduce that

$$|\phi_k|_{H_H^1(\Omega)}^2 = \sum_{j: c_r \in \partial\Omega_j} |\phi_k|_{H^1(\Omega_j)}^2 \preceq \sum_{j: c_r \in \partial\Omega_j} h_j^{-1} \|\phi_k\|_{L^2(\partial\Omega_j)}^2.$$

The vertex $x_k \in \mathcal{V}(\Omega_i)$ can be the common end of two slaves, two masters and of a master and a slave of Ω_i. In the first case, $\phi_{k\,|\Gamma}$ is nonzero only over these slaves and the statement of the lemma follows from Propositions 3.1 in [4], cf. also the proof of Lemma 4 in [44]. In the second case, $\phi_{k\,|\Gamma}$ is nonzero over those two masters denoted by $\gamma_{m,i}$ and $\gamma_{s,i}$ and over two corresponding slaves $\delta_{m,j}$ and $\delta_{s,l}$. Then as in the proof of Lemma 5 in [44] we obtain

$$|\phi_k|^2_{H^1_H(\Omega)} \preceq 1 + h_i/h_j + h_i/h_l.$$

The assumption that the parameters h_j and h_l of slaves are not greater than the respective parameters of masters, here equal to h_i, ends the proof of this case. The last case is proved in the same way. □

We now prove Theorem 2.5.1 using the general theory of ASM, see Theorem 1.4.1 in Section 1.4, cf. also [20] or [60].

Proof. By Theorem 1.4.1 the proof reduces to check three key assumptions.

Assumption (i)
We want to prove that for all $u \in V^h$, there exist functions $u_0 \in V_0$, $u^i_k \in V^i_k$ and $u_{ij} \in V_{ij}$ such that $u = u_0 + \sum_{i=1}^N \sum_{x_k \in \mathcal{V}(\Omega_i)} u^i_k + \sum_{\Gamma_{ij} \subset \Gamma} u_{ij}$ and

$$a_\triangle(u_0,u_0) + \sum_{i=1}^N \sum_{x_k \in \mathcal{V}(\Omega_i)} a_\triangle(u^i_k,u^i_k) + \sum_{\Gamma_{ij} \subset \Gamma} a_\triangle(u_{ij},u_{ij}) \preceq (1+\log(H/h))^2 a_\triangle(u,u) \tag{2.31}$$

We first select $u_0 \in V_0 = V^H$ by making $u_0(c_r) = \overline{u}_{c_r}$, where $c_r \in \Gamma$ is a crosspoint and $\overline{u}_{c_r}$ is the average value of u at the vertices of $\mathcal{V}$ that coincide geometrically with c_r. We further denote $\mathcal{V}(c_r)$ as the set of these vertices. Let N_{c_r} be the number of vertices in $\mathcal{V}(c_r)$. Thus we have that $\overline{u}_{c_r} = (1/N_{c_r}) \sum_{x \in \mathcal{V}(c_r)} u(x)$. Let now define $u^i_k \in V^i_k$ by the pointwise interpolation of $u - u_0$ at v, i.e.

$$u^i_k = (u-u_0)(x_k)\,\phi_k$$

Note that w defined as

$$w = u - u_0 - \sum_{i=1}^N \sum_{x_k \in \mathcal{V}(\Omega_i)} u^i_k$$

vanishes at all vertices of $\mathcal{V}$.

We now decompose w in Ω_i as

$$w_i = w_{|\Omega_i} = P_i w_i + H_i w_i,$$

where $H_i w_i$ is the discrete harmonic part of w_i and $P_i w_i$ is the $H_0^1(\Omega_i)$ projection onto $\overset{\circ}{X}_h(\Omega_i) = X_h(\Omega_i) \cap H_0^1(\Omega_i)$, i.e. $H_i w_i = w_i$ on $\partial\Omega_i$ and

$$a_\triangle(H_i w_i, \psi) = 0, \qquad a_\triangle(P_i w_i, \psi) = a_\triangle(w_i, \psi) \quad \forall \psi \in \overset{\circ}{X}_h(\Omega_i) \tag{2.32}$$

Both $P_i w_i$ and $H_i w_i$, we extend as zero off $\overline{\Omega}_i$.

With each space V_{ij} we associate the common edge $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ which is the master $\gamma_{m,i} \subset \partial\Omega_i$ and the slave $\delta_{m,j} \subset \partial\Omega_j$. Then let $w_{ij} \in V_{ij}$ be equal to w on $\gamma_{m,i}$ and $\delta_{m,j}$, be zero on $\partial\Omega_i \setminus \gamma_{m,i}$, $\partial\Omega_j \setminus \delta_{m,j}$ and be extended as the discrete harmonic function in Ω_i and Ω_j. Note that w_{ij} is zero off $\overline{\Omega}_i \cup \overline{\Omega}_j$ because $w_{ij} \in V_{ij}$.

We finish the decomposition of u by setting

$$u_{ij} = w_{ij} + (1/N_e(i))P_i w_i + (1/N_e(j))P_j w_j,$$

where $N_e(k)$ is a number of edges $\Gamma_{kl} \subset \Gamma \cap \partial\Omega_k$ and $N_e(k)$ is equal to 3 if $\partial\Omega_k \cap \partial\Omega = \emptyset$ and 2 or 1 otherwise. Note that

$$u = u_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} u_k^i + \sum_{\Gamma_{ij} \subset \Gamma} u_{ij}$$

We first estimate $a_\triangle(u_0, u_0)$. Let $\overline{u}_i$ be the average value of u over Ω_i. Using an inverse inequality we have

$$a_\triangle(u_0, u_0) = \sum_{i=1}^{N} |u_0|^2_{H^1(\Omega_i)} = \sum_{i=1}^{N} |u_0 - \overline{u}_i|^2_{H^1(\Omega_i)} \preceq \sum_{i=1}^{N} \sum_{x \in \mathcal{V}(\Omega_i)} |u_0(x) - \overline{u}_i|^2 \tag{2.33}$$

We consider one vertex y of Ω_i which geometrically coincides with a crosspoint c_r and we have that $u_0(y) = u_0(c_r)$. Hence

$$|u_0(y) - \overline{u}_i|^2 = |\frac{1}{N_{c_r}} \sum_{x \in \mathcal{V}(c_r)} u(x) - \overline{u}_i|^2 \preceq \sum_{x \in \mathcal{V}(c_r)} |u(x) - \overline{u}_i|^2$$

Note that from the mortar condition (2.9) follows that average values of u over a master $\gamma_{m,i}$ and a slave $\delta_{m,j}$ that geometrically coincide, are equal to each other. Using this, the standard Sobolev-like inequality for finite elements, see e.g. Lemma 7, p.170 in [20], and the Poincaré's inequality, we obtain

$$|u_0(y) - \overline{u}_i|^2 \preceq \sum_{k:\, x \in \mathcal{V}(c_r) \cap \mathcal{V}(\Omega_k)} (1 + \log(H/h_k))|u_k|^2_{H^1(\Omega_k)} \tag{2.34}$$

The sum is taken over all subdomains with the common vertex c_r. Summing over all vertices of Ω_i we obtain the estimate

$$|u_0|^2_{H^1(\Omega_i)} \preceq \sum_{j:\partial\Omega_j\cap\partial\Omega_i\neq\emptyset} (1+\log(H/\underline{h}))|u_j|^2_{H^1(\Omega_j)} \tag{2.35}$$

Summing over all subdomains gives the estimate of $a_\triangle(u_0,u_0)$.

We now estimate $\sum_{i=1}^N \sum_{x_k\in\mathcal{V}(\Omega_i)} a_\triangle(u_k^i,u_k^i)$. Let c_r be the crosspoint that coincides geometrically with a vertex x_k. By Lemma 2.5.1, we deduce that

$$a_\triangle(u_k^i,u_k^i) = |u(x_k)-u_0(x_k)|^2|\phi_k|^2_{H^1_H(\Omega)} \preceq |u(x_k)-u_0(c_r)|^2.$$

We further use the same arguments as for the estimate of $a_\triangle(u_0,u_0)$ to get

$$a_\triangle(u_k^i,u_k^i) \preceq \sum_{x\in\mathcal{V}(c_r)\cap\mathcal{V}(\Omega_k)} (1+\log(H/\underline{h}))|u_k|^2_{H^1(\Omega_k)} \tag{2.36}$$

The sum is taken over all subdomains that have a vertex $x\in\mathcal{V}(c_r)$. Summing over all subdomains and their vertices we obtain

$$\sum_{i=1}^N \sum_{x_k\in\mathcal{V}(\Omega_i)} a_\triangle(u_k^i,u_k^i) \preceq \sum_{i=1}^N (1+\log(H/\underline{h}))|u_i|^2_{H^1(\Omega_i)} \tag{2.37}$$

We now estimate $\sum_{\Gamma_{ij}\subset\Gamma} a_\triangle(u_{ij},u_{ij})$. We deduce that

$$a_\triangle(u_{ij},u_{ij}) \preceq \left\{|w_{ij}|^2_{H^1(\Omega_j)} + |w_{ij}|^2_{H^1(\Omega_i)} + |P_iw_i|^2_{H^1(\Omega_i)} + |P_jw_j|^2_{H^1(\Omega_j)}\right\}$$

We first estimate $|P_iw_i|^2_{H^1(\Omega_i)}$ ($|P_jw_j|^2_{H^1(\Omega_j)}$ can be estimated in the same way). Using (2.35) and (2.36) we get

$$|P_iw_i|^2_{H^1(\Omega_i)} \preceq |u_i|^2_{H^1(\Omega_i)} + |u_0|^2_{H^1(\Omega_i)} + \sum_{x_k\in\mathcal{V}(\Omega_i)} |u_k^i|^2_{H^1(\Omega_i)} \preceq |u_i|^2_{H^1(\Omega_i)} +$$

$$+ \sum_{j:\partial\Omega_j\cap\partial\Omega_i\neq\emptyset} (1+\log(H/\underline{h}))\,|u_j|^2_{H^1(\Omega_j)},$$

where the last sum is taken over all substructures that have the common vertex to Ω_i.

We now estimate the norms of w_{ij} the discrete harmonic part of u_{ij}. We first note that

$$|w_{ij}|^2_{H^1_H(\Omega)} = |H_iw_i|^2_{H^1(\Omega_i)} + |H_jw_j|^2_{H^1(\Omega_j)} \preceq \|w_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})} +$$

$$+\|w_j\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq \|w_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})}$$

The first inequality follows from the extension property of discrete harmonic functions, see Lemma 5.1, p.1112 in [21], and the second one from $H^{1/2}_{00}$ stability of functions in V^h over each edge $\Gamma_{ij} \subset \Gamma$, see Lemma 1 in [9]. Thus it remains to prove the estimate of $\|w_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})}$.

Let us denote by x_1, x_2 the ends of $\gamma_{m,i}$. Then let $\phi_{x_j} \in V^i_k, j = 1, 2$, denote a vertex function associated with the vertex x_j and we deduce that

$$w_{i|\gamma_{m,i}} = z - \sum_{j=1}^{2} z(x_j)\phi_{x_j} - z_0 + \sum_{j=1}^{2} z_0(x_j)\phi_{x_j},$$

where $z = u_i - \overline{u}_i$ and $z_0 = u_0 - \overline{u}_i$, where $\overline{u}_i$ is the average value of u_i over $\gamma_{m,i} = \Gamma_{ij}$. Note that ϕ_{x_j} is equal to standard nodal function corresponding to $x_j, j = 1, 2$, on $\gamma_{m,i}$. From this we obtain

$$\|w_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})} \preceq \{\|z - \sum_{j=1}^{2} z(x_j)\phi_{x_j}\|^2_{H^{1/2}_{00}(\gamma_{m,i})} +$$

$$+\|z_0 - \sum_{j=1}^{2} z_0(x_j)\phi_{x_j}\|^2_{H^{1/2}_{00}(\gamma_{m,i})}\}.$$

We can estimate the first term by $c(1+\log(H/h_i))^2|u_i|^2_{H^1(\Omega_i)}$, this result is well known, see e.g. p.11 [44], but it also follows from Lemma 4.5.1, see below, the property of standard nodal functions and Poincaré's inequality. Simple computations give the estimate of the second term by

$$\|z_0 - \sum_{j=1}^{2} z_0(x_j)\|^2_{H^{1/2}_{00}(\gamma_{m,i})} \preceq \sum_{j=1}^{2}(1 + \log(H/h_i))|u_0(x_j) - \overline{u}_i|^2,$$

cf. also Lemma 4.5.1, below. Now using (2.34) we deduce that

$$\|w_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})} \preceq (1 + \log(H/\underline{h}))^2 \sum_{k} |u_k|^2_{H^1(\Omega_k)},$$

where the sum is taken over all indices of subdomains that has a vertex that geometrically coincides with one of the ends of $\gamma_{m,i}$.

Summing over all subspaces V_{ij} we have

$$\sum_{\Gamma_{ij}\subset\Gamma} a_{\triangle}(u_{ij}, u_{ij}) \preceq (1 + \log(H/\underline{h}))^2|u|^2_{H^1_H(\Omega)} = (1 + \log(H/\underline{h}))^2 a_{\triangle}(u, u).$$

Combining this, (2.35) and (2.37), we get (2.31) what ends the proof of Assumption (i).

Assumptions (iii)

It is obviously satisfied with $\omega = 1$ as all local forms equals $a_{\triangle}(\cdot, \cdot)$.

Assumption (ii)

It is satisfied with $\rho(\mathcal{E}) \leq C$ because functions from local spaces V_{ij} and V_k^i have local supports. □

2.6 Nonlinear domain decomposition method

In this section, we present a nonlinear domain decomposition method of solving problem (2.10). This method is based on the abstract framework, developed by Dryja and Hackbusch [59] which is a generalization of the one of ASM.

We consider the nonlinear equation

$$F(\mathbf{u}_h^*) = B(\mathbf{u}_h^*) - \mathbf{f} = 0, \tag{2.38}$$

where u_h^* is the solution of (2.22).

We have also to consider the linear system with matrix $A = F'(\mathbf{u_h}^*)$

The symmetry of A needs additional assumptions on the coefficients $a_i(x, p)$, i.e.

$$\frac{\partial a_i(x,p)}{\partial p_j} = \frac{\partial a_j(x,p)}{\partial p_i}, \quad i, j = 0, 1, 2. \tag{2.39}$$

The method for solving (2.38) is based on the subspace decomposition defined in the previous section and is defined as

$$\mathbf{u}^{n+1} = \Phi(\mathbf{u}^n) = \mathbf{u}^n + \omega \mathbf{r}(\mathbf{u}^n), \tag{2.40}$$

where

$$\mathbf{r}(\mathbf{u}^n) = R_0^T \mathbf{r}_0 + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} (R_k^i)^T \mathbf{r}_k^i + \sum_{\gamma_{ij} \subset \Gamma} R_{ij}^T \mathbf{r}_{ij}$$

and $\mathbf{r}_0 \in V_0$, $\mathbf{r}_k^i \in V_k^i$ and $\mathbf{r}_{ij} \in V_{ij}$ are the respective solutions of

$$R_0 F(\mathbf{u}^n - R_0^T \mathbf{r}_0) = 0$$

$$R_k^i F(\mathbf{u}^n - (R_k^i)^T \mathbf{r}_k^i) = 0$$

and

$$R_{ij}F(\mathbf{u}^n - R_{ij}^T \mathbf{r}_{ij}) = 0$$

Here R_0, R_k^i and R_{ij} are the restriction operators, the same as in the previous section, see p.44,124 in [20]. The damping parameter ω will be set later. These local problems can be solved in parallel.

We now analyze the method (2.40). The first problem is uniqueness and existence results for local problems. But under Assumptions (2.4)-(2.6) it can be shown, see e.g. Theorem 1.5 p.3 and Section 2.3 from [59], that there exist solutions of the local problems and that they are unique.

When we apply the subspace iteration (2.40) to the linear problem $G(\mathbf{u}) = A\mathbf{u} - \mathbf{b} = 0$ with $A = F'(\mathbf{u}_h^*)$ and $\mathbf{b} = A\mathbf{u}_h^*$, we obtain a linear iteration with the following iteration matrix

$$M_\omega := I - \omega \left(R_0^T A_0^{-1} R_0 A + \sum_{i=1}^{N} \sum_{x_k \in \mathcal{V}(\Omega_i)} (R_k^i)^T (A_k^i)^{-1} R_k^i A + \sum_{\gamma_{ij} \subset \Gamma} R_{ij}^T A_{ij}^{-1} R_{ij} A \right),$$

where $A_0 = R_0 A R_0^T, A_k^i = R_k^i A (R_k^i)^T$ and $A_{ij} = R_{ij} A R_{ij}^T$. Note that this linear iteration is built analogously to the iteration applied to $M^{-1}D$ in Theorem 2.5.1 with parameter ω. Additionally, it is easy to prove that under Assumptions (2.4)-(2.6) the matrix A is spectrally equivalent to D, e.g. cf. Inequalities (14) and (15) p.7 in [87] or p. Inequalities (15) and (16) p.11 in [86]. Thus using Theorem 2.5.1 we have

$$\|M_\omega\| \leq \zeta < 1$$

for some damping parameter ω and ζ is only dependent on $(1 + \log(H/\underline{h})^2$.

Hence by Theorem 1.7 from [59] we can conclude that

Proposition 2.6.1 *The method (2.40) for the nonlinear problem (2.38) has the same asymptotic convergence rate ζ as the linearized system with $A = F'(\mathbf{u}_h^*)$, dependent only on $C\,(1 + log(H/\underline{h}))^2$, where C is a positive constant independent of H and any h_k, i.e. for any $\zeta' \in (\zeta, 1)$ there exists a neighborhood U' of $\mathbf{u}_h^*$ such that*

$$\|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\| \leq \zeta' \quad \forall \mathbf{u}, \mathbf{v} \in U'.$$

2.7 Problems with unbounded nonlinearities

In the previous sections, we have considered the nonlinear problems with bounded nonlinearities. In this section, we show how the methods for the problems with bounded nonlinearities can be applied to the problems with unbounded nonlinearities, cf. [93] or [94].

Here we consider a problem of type (2.2)-(2.3). We assume that instead of (2.5)-(2.6), the functions $a_i : \Omega \times \Re^3 \to \Re, i = 0, 1, 2$ satisfy the following conditions:

$$\nu(|p|^2) \sum_{i=0}^{2} \xi_i^2 \leq \sum_{i,j=0}^{2} \frac{\partial}{\partial p_j} a_i(x,p)\xi_i\xi_j \leq \mu(|p|^2) \sum_{i=0}^{2} \xi_i^2 \ \ \forall \xi \in \Re^3, \tag{2.41}$$

$$\max\{|a_i(x,0,0,0)|, |\frac{\partial a_i}{\partial x_k}(x,p)|\} \leq \ M_0, \ \ for \ \ i = 0,1,2; \ \ k = 1,2, \tag{2.42}$$

where $\nu(s), \mu(s)$, $0 \leq s < \infty$, are positive smooth (continuous and with continuous derivative) functions.

We further assume that the solution u^* of the respective problem satisfies

$$\sqrt{|u^*|^2 + |\nabla u^*|^2} \leq M. \tag{2.43}$$

The class of problems with solutions which satisfy (2.43) is set in [94] or [78], see e.g. Theorems 1 and 2 in [94], Chapter 3, §8 in [78].

It shows that if the solution of problem with an unbounded nonlinearity satisfies (2.43), then it is possible to construct an auxiliary quasilinear elliptic problem with a bounded nonlinearity equivalent to the original one, i.e. the solution of the new auxiliary problem coincides with the solution u^* of the original one.

We can next approximate the solution of the auxiliary problem with the mortar method presented in the previous sections of this chapter.

Following [94] we show how to construct the auxiliary problem. This problem is of the same form as the problem (2.2)-(2.3). Hence it suffices to define coefficients of the auxiliary problem, i.e. functions $\hat{a}_i(x,p) : \Omega \times \Re^3 \to \Re, i = 0, 1, 2$. Let $\xi \in C^2([0,\infty))$ be a nonnegative, non-increasing function such that

$$\xi(s) = \begin{cases} 1 & 0 \leq s \leq 1 \\ 0 & s \geq 2 \end{cases} .$$

We also introduce a function $\theta : \Re_+ \to \Re_+$ defined as

$$\theta(s) = c\,\frac{\mu^2(s)}{\nu(s)},$$

where c is a certain positive constant and $\mu(\cdot), \nu(\cdot)$ are functions from (2.41). Then we define

$$\hat{a}_i(x,p) = \begin{cases} a_i(x,p)\xi\left(|\frac{p}{M}|^2\right) + \theta(|p|^2)p_i\left(1 - \xi\left(|\frac{p}{M}|^2\right)\right) & |p| \leq 2M \\ \theta(4M^2)p_i & |p| \geq 2M \end{cases}$$

for $i = 0, 1, 2$. Here M is the constant from (2.43).

This auxiliary problem with the coefficients defined above, satisfies (2.5) and (2.6), see [94] or [93].

2.8 Numerical Experiments

In this section, we present some results of numerical experiments. We carry out a few

Table 2.1: Nine subdomains, nonconforming meshes.

$N = H^{-1}$	h_1	cond$(M^{-1}D)$
9	$H/10$	13.6
9	$H/20$	17.3
9	$H/40$	22.3
9	$H/80$	27.6
9	$H/160$	33.2

experiments to test the ASM preconditioner M^{-1}, see (2.29) and Theorem 2.5.1. We apply this preconditioner in PCG method for solving a linear problem with a discrete Laplacian on a unit square, i.e. to the linear system of equations with the matrix D, see (2.23). Our algorithm has been implemented in PETSCs 2.0 (the Portable, Extensible Toolkit for Scientific Computation) in C on Sun Sparc Workstation. The region Ω is the unit square $(0,1) \times (0,1)$ divided into $N * N$ adjacent squares of diameter $H = 1/N$. Each substructure Ω_k is divided into a grid of smaller $n_k * n_k$ squares. These small squares are then divided into two triangles by drawing the lines from bottom left to top right. The resulting meshes do not match on the interface. For each interface Γ_{ij}, which is the common side of two coarse squares Ω_i and Ω_j, we assign as a master the side of this edge for which the mesh parameter is smaller, i.e. if $h_i \leq h_j$, then the master is $\gamma_{m,i}$. Here $h_k = 1/n_k$.

Table 2.2: Nine subdomains, conforming meshes.

$N = H^{-1}$	h	cond$(M^{-1}D)$
9	$H/10$	15.5
9	$H/20$	19.1
9	$H/40$	23.3
9	$H/80$	28.1
9	$H/160$	33.6

In Table 2.1, we refer to results of the following experiments: we set the number of subdomains to nine, i.e. $N = 9$, set $n_i = n_1 - i + 1, i = 1, \ldots, 9$, and in successive experiments increase n_1.

In Table 2.2, we give results of the similar experiments for conforming meshes, i.e. $n_i = n_j$ for all $i, j = 1, \ldots, 9$. The meshes are conforming, but the functions are not continuous at the crosspoints, thus the method is also nonconforming.

In Tables 2.3 and 2.4, we give results of analogous experiments but for sixteen subdomains.

Table 2.3: Sixteen subdomains, nonconforming meshes.

$N = H^{-1}$	h_1	cond$(M^{-1}D)$
16	$H/20$	17.0
16	$H/40$	22.1
16	$H/80$	27.4
16	$H/160$	33.1

Table 2.4: Sixteen subdomains, conforming meshes.

$N = 1/H$	h	cond$(M^{-1}D)$
16	$H/20$	19.3
16	$H/40$	23.6
16	$H/80$	28.3
16	$H/160$	33.7

The results of the experiments confirm the theoretical statement of Theorem 2.5.1.

Chapter 3

A mortar method with locally nonconforming elements

Contents

3.1 Introduction

The goal of this part of the thesis is to construct and analyze a new version of the mortar method for second order elliptic problems. We consider discretization with nonconforming elements in each substructure. For our knowledge, there are no results devoted to those topics, cf. [104] for some related results. In all previous versions of the mortar methods, it has been assumed that local subspaces contain conforming continuous functions.

The mortar technique for locally nonconforming elements imposes that the solution on the two neighboring subdomains has the same L^2 projections on the mortar space that is defined on their common edge. We choose the mortar space that has natural L^2 orthogonal basis and leads to simple computations of the matching conditions.

For second order elliptic problems, we prove that the error estimate is of the same optimal order as in the standard linear nonconforming finite element method. For the simplicity of presentation, we consider only the Poisson equation.

We also propose a parallel method for solving the system of linear equations that arises from our discretization. It is described as an additive Schwarz method (ASM) using the general framework of ASM's, see Section 1.4, (cf. [20], [64] or [60]). In this chapter, the error analysis is done for arbitrary polygonal substructures while the additive Schwarz method is considered for a partition of the original 2-D region Ω into triangles that form a coarse triangulation of parameter H. The described ASM uses a standard coarse space defined on the coarse mesh, i.e. $V_0 = V^H$, the space of piecewise linear continuous functions which vanish on $\partial\Omega$. The remaining spaces are local and are associated with all mortar edges and with some subdomains. The problems in these subspaces are independent and can be solved in parallel.

The described method is almost optimal, i.e. the number of iterations required to decrease the energy norm of the error by a conjugate gradient method is proportional to $(1 + \log(\frac{H}{\underline{h}}))$. Here H and h_i are the parameters of the coarse triangulation and the fine one on Ω_i, respectively, and $\underline{h} = \inf_i h_i$.

Iterative methods for solving linear systems of equations of locally conforming versions of the mortar finite elements have been described and analyzed in several papers, e.g. see [2], [4], [44], [56], [88]. In [100], methods for solving systems of equations of linear nonconforming elements defined on global triangulation of Ω are described.

The outline of the chapter is as follows. In Section 3.2, we present the mortar element method that locally uses the Crouzeix-Raviart linear elements. Section 3.3 is devoted to study the error estimate. In Section 3.4, we describe and analyze the ASM method. In Section 3.5, we present some numerical results.

3.2 Discrete problem

We consider the Poisson equation as a model problem:
Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \quad \forall\, v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a(u, v) = \int_\Omega \nabla u \, \nabla v \, dx, \quad f(v) = \int_\Omega f v \, dx$$

Here $f \in L^2(\Omega)$, $\nabla u = (u_{x_1}, u_{x_2})^T$ and Ω is a polygonal region in $\Re^2$.

We now define our discrete space V^h which is a finite element subspace of the space $L^2(\Omega)$, but it is not contained in $H_0^1(\Omega)$, and in that sense our method is nonconforming. We consider a geometrically conforming version of the mortar method, i.e. Ω is divided into polygonal substructures Ω_i

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$$

with $\overline{\Omega}_i \cap \overline{\Omega}_j$ being the empty set, a vertex or an edge for $i \neq j$. Thus $\{\Omega_k\}$ form a decomposition of Ω. We assume the shape regularity of that decomposition, cf. Section 2.3.

With each Ω_k, we associate a quasiuniform triangulation made of elements that are triangles. The mesh parameter h_k is equal to the maximum over all the diameters of elements and let $T_h(\Omega_k)$ denote this triangulation. Let Γ_{ij} be the open edge that is common to Ω_i and Ω_j, i.e. $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$.

Let the union of all interfaces between the subdomains be denoted by Γ, i.e. $\Gamma = \bigcup \partial\Omega_i \setminus \partial\Omega$. Each edge Γ_{ij} inherits two triangulations made of segments that are edges of elements of the triangulations of Ω_i and Ω_j, respectively. In this way, each Γ_{ij} is provided with two independent and different the 1-D meshes which are denoted by $T_h^i(\Gamma_{ij})$ and $T_h^j(\Gamma_{ij})$. We define the *CR nodal points* as the nonconforming nodal points, i.e. the midpoints of the edges of elements in $T_h(\Omega_k)$. The set of *CR nodal points* belonging to $\overline{\Omega}_k, \partial\Omega_k$ and $\partial\Omega$ are denoted by $\Omega_{k,h}^{CR}, \partial\Omega_{k,h}^{CR}$ and $\partial\Omega_h^{CR}$, respectively. Finally, let $\Omega_{k,h}$ and $\partial\Omega_{k,h}$ denote the sets of vertices of the triangulations $T_h(\Omega_k)$ that are in $\overline{\Omega}_k$ and $\partial\Omega_k$, respectively.

As the triangulation $T_h(\Omega_k)$ is chosen over each Ω_k, we can give the definition of local finite element spaces.

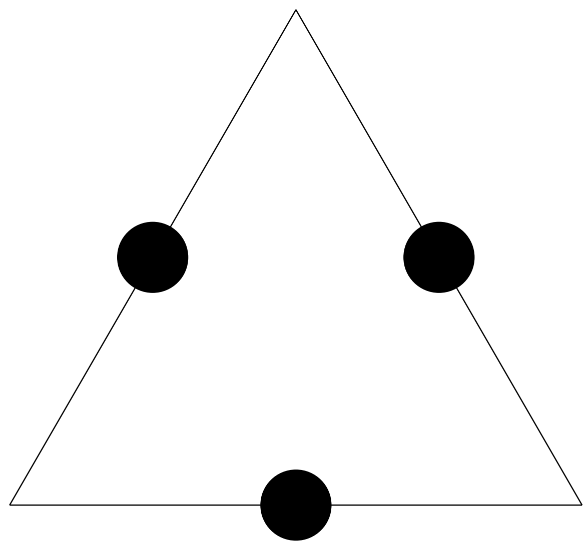

Figure 3.1: Crouzeix-Raviart element.

We choose locally a nonconforming finite element method that is best suited to the local properties of the solution. First we define finite element functions locally and introduce $X_h(\Omega_k)$ as the local nonconforming P_1 (Crouzeix-Raviart) space, i.e. the space formed by all functions which are piecewise linear in each triangle of $T_h(\Omega_k)$ and are continuous at the CR nodes of $\Omega_{k,h}^{CR} \setminus \partial\Omega_{k,h}^{CR}$, and are equal to zero at the CR nodes of $\partial\Omega_h^{CR}$, cf. [52] and Figure 3.1.

The degrees of freedom of Crouzeix-Raviart element are the values at the midpoints of edges of a triangular element.

We also introduce for all $u_k \in X_h(\Omega_k)$ the so called broken norm and the broken seminorm:

$$\|u_k\|_{H^1_h(\Omega_k)} = \left(\sum_{\tau \in T_h(\Omega_k)} \|u_k\|^2_{H^1(\tau)} \right)^{\frac{1}{2}}, \quad |u_k|_{H^1_h(\Omega_k)} = \left(\sum_{\tau \in T_h(\Omega_k)} |u_k|^2_{H^1(\tau)} \right)^{\frac{1}{2}}.$$

We can now introduce a global space X_h as

$$X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k)$$

with the following broken norm $\|u\|_{H^1_H(\Omega)} = (\sum_{k=1}^N \|u\|^2_{H^1_h(\Omega_k)})^{1/2}$ and the broken seminorm $|u|_{H^1_H(\Omega)} = (\sum_{k=1}^N |u|^2_{H^1_h(\Omega_k)})^{1/2}$. This space can be considered as a subspace of $L^2(\Omega)$ formed by all functions which locally restricted over Ω_k are in $X_h(\Omega_k)$.

As each interface is provided with two independent meshes, we must enforce some matching conditions over the interface. In what follows, we express a condition that is sufficient to ensure the optimality of the global approximation.

We define one of the sides of Γ_{ij} as a master (mortar) one, denoted by $\gamma_{m,i}$ and the other one as a slave (nonmortar) denoted by $\delta_{m,j}$. Let the master side for Γ_{ij} be chosen by the condition: $h_i \le h_j$, (i.e. here, the master side is the i-th one). There are two sets of *CR nodal points* belong to Γ_{ij}: the midpoints of elements belonging to $T_h^i(\gamma_{m,i})$ and to $T_h^j(\delta_{m,j})$, denoted by $\gamma^{CR}_{m,i,h}$ and $\delta^{CR}_{m,j,h}$, respectively. As $h_i \le h_j$ and both triangulations are quasiuniform, we can assume that the two elements of the slave triangulation $T_h^j(\delta_{m,j})$ of Γ_{ij} that touch the ends of $\delta_{m,j}$ have longer lengths than the respective elements of the master triangulation $T_h^i(\gamma_{m,i})$ of Γ_{ij}. The choice of the master side is due to the technique that is used to prove the results of Section 4, i.e for ASM. The error estimates of Section 3 are independent of that assumption. We believe that this condition is not necessary at all.

Additionally, we define an auxiliary test (mortar) space $M^{h_j}_{-1,0}(\delta_{m,j})$ being a subspace of $L^2(\Gamma_{ij})$ of all functions which are piecewise constant on elements of the slave triangulation of Γ_{ij}, i.e. inherited from the 2-D triangulation of Ω_j (j-th is the slave side of Γ_{ij}). The dimension of $M^{h_j}_{-1,0}(\delta_{m,j})$ is equal to the number of midpoints on $\delta_{m,j}$, i.e. to the number of elements on $\delta_{m,j}$.

We introduce the L^2 orthogonal projection $Q_m : L^2(\Gamma_{ij}) \rightarrow M^{h_j}_{-1,0}(\delta_{m,j})$ for each slave $\delta_{m,j} = \Gamma_{ij} \subset \Gamma$ defined as

$$(Q_m u, \psi)_{L^2(\delta_{m,j})} = (u, \psi)_{L^2(\delta_{m,j})} \quad \forall \psi \in M^{h_j}_{-1,0}(\delta_{m,j}) \tag{3.2}$$

We can now define our discrete space V^h as

$$V^h = \{u_h \in X_h(\Omega) : \forall\, \delta_{m,j} = \gamma_{m,i} \subset \Gamma, \quad Q_m u_j = Q_m u_i\} \tag{3.3}$$

The condition of the equality of the L^2 projection of traces onto the test space for each interface can also be called *the mortar condition.* Note that $V^h \not\subset H_0^1(\Omega)$.

Since functions in V^h are not continuous, we have to use a "broken" variational form $a_H(\cdot,\cdot)$ in the discretized problem.

Let $a_{h,k}(\cdot,\cdot)$ be the bilinear form defined on triangles belonging to Ω_k which is a subregion of Ω

$$a_{h,k}(u,v) = \sum_{\tau \in T_h(\Omega_k)} \int_\tau \nabla u\, \nabla v\, dx \tag{3.4}$$

and let $a_H(u,v) = \sum_{k=1}^N a_{h,k}(u,v)$.

The form $a_H(\cdot,\cdot)$ is positive-definite on V^h by the argument that $a_H(v,v) = 0$ for $v = \{v_k\} \in V^h$ implies that v_k is constant over each element of $T_h(\Omega_k)$, then the continuity of v_k at midpoints yields that v_k constant in Ω_k and finally by the mortar condition and discrete boundary conditions, we get $v = 0$.

In Section 3.2.1 below, we show that the H^1-broken seminorm which is also equal to the norm induced by the bilinear form $a_H(\cdot,\cdot)$, is equivalent to the L^2 norm on V^h with constants independent of any parameters h_k and the number and size of subdomains, see Lemma 3.2.1 below.

The discrete problem is of the form:
Find $u_h^* \in V^h$ such that

$$a_H(u_h^*, v) = f(v) \quad \forall v \in V^h \tag{3.5}$$

This problem has a unique solution since $a_H(\cdot,\cdot)$ is positive-definite on V^h.

3.2.1 Ellipticity of the discrete problem

In this subsection, we prove that $a_H(\cdot,\cdot)$ is elliptic on the discrete space V^h with constant independent of h and, what is also important, number of subdomains. These results are analogous to those for mortar methods for second order elliptic problems with locally conforming discretization proved in [17]. The proofs of the results of this section are based in part on the one of Proposition 2.1 in [17].

Lemma 3.2.1 *There exists a constant C independent of h_k and the number of subdomains such that for $u \in V^h$*

$$\sum_{k=1}^{N} \|u\|^2_{H^1_h(\Omega_k)} \le C \sum_{k=1}^{N} |u|^2_{H^1_h(\Omega_k)}.$$

Proof. It reduces to show $\sum_{k=1}^N \|u\|^2_{L^2(\Omega_k)} \preceq \sum_{k=1}^N |u|^2_{H^1_h(\Omega_k)}$. Let $u = (u_1, \dots, u_N) \in V^h \subset X_h(\Omega)$. For each subdomain Ω_k let $\tilde{u}_k = \mathcal{M}_k u_k$ and $\tilde{u} = \{\tilde{u}_k\} \in \prod_{k=1}^N H^1(\Omega_k)$, where $\mathcal{M}_k$ is defined below in Definition 3.3.1. By Lemma 3.3.2, see below, we have

$$|u_k|^2_{H^s_h(\Omega_k)} \asymp |\tilde{u}_k|^2_{H^s(\Omega_k)} \quad s = 0, 1. \tag{3.6}$$

We have for any $(x_1, x_2) \in \Omega$

$$\tilde{u}(x_1, x_2) = \int_a^{x_1} \tilde{u}_{x_1}(t, x_2)\, dt + \sum_{t_{kl} \in [a, x_1] \cap \Gamma_{kl}} [\tilde{u}](t_{kl}, x_2),$$

where $[\cdot]$ denotes the jump over Γ_{kl} at point t_{kl} and $(a, x_2) \in \partial\Omega$. Here t_{kl} is the point of intersection of a segment $[a, x_1]$ and an interface $\Gamma_{kl} \subset \partial\Omega_k \cap \partial\Omega_l$.

The first term is estimated by

$$|\int_a^{x_1} \tilde{u}(t, x_2)\, dt| \le \int_a^b |\tilde{u}_{x_1}(t, x_2)|\, dt \le (b-a)^{1/2} (\int_a^b |\tilde{u}_{x_1}(t, x_2)|^2\, dt)^{1/2},$$

where b satisfies $(b, x_2) \in \partial\Omega$. The second term is estimated by

$$| \sum_{t_{kl} \in [a, x_1] \cap \Gamma_{kl}} [\tilde{u}](t_{kl}, x_2)| \le \sum_{t_{kl} \in [a, b] \cap \Gamma_{kl}} |[\tilde{u}](t_{kl}, x_2)| \le$$

$$\le \left(\sum_{t_{kl} \in [a,b] \cap \Gamma_{kl}} |\Gamma_{kl}| \right)^{1/2} \left(\sum_{t_{kl} \in [a,b] \cap \Gamma_{kl}} |\Gamma_{kl}|^{-1} |[\tilde{u}](t_{kl}, x_2)|^2 \right)^{1/2}.$$

In [17], see Lemma 2.2 there, it was proved that the shape regularity of our decomposition yields that $\sum_{t_{kl} \in [a,b] \cap \Gamma_{kl}} |\Gamma_{kl}| \preceq |b - a|$. Thus we obtain

$$|\tilde{u}(x_1, x_2)|^2 \preceq \int_a^b |\tilde{u}_{x_1}(t, x_2)|^2\, dt + \sum_{t_{kl} \in [a,b] \cap \Gamma_{kl}} |\Gamma_{kl}|^{-1} |[\tilde{u}](t_{kl}, x_2)|^2.$$

We now integrate over dx_2 and dx_1 and get

$$\int_\Omega |\tilde{u}|^2\, dx \preceq \int_\Omega |\nabla \tilde{u}|^2\, dx + \sum_{\Gamma_{kl} \subset \Gamma} \int_{\Gamma_{kl}} |\Gamma_{kl}|^{-1} |\tilde{u}_k - \tilde{u}_l|^2\, ds.$$

We have also used the fact that $ds = \sqrt{dx_1^2 + dx_2^2}$, hence $\int_{\Gamma_{kl}} |\tilde{u}_k - \tilde{u}_l|^2 \, ds \geq \int_{\Gamma_{kl}} |\tilde{u}_k - \tilde{u}_l|^2 \, dx_2$. We now have to estimate the second sum. Each term of the second sum we estimate separately. By (3.2) u_k and u_l have equal the average values over an interface Γ_{kl} as $\int_{\Gamma_{kl}} (u_k - u_l) \, ds = 0$. Thus the standard trace theorem, see Theorem 1.5.2.1, p.42 in [71], Lemma 3.3.2 and the version of Poincaré's inequality for CR elements, see Lemma 5, p.392 in [99], yield that

$$\int_{\Gamma_{kl}} |\tilde{u}_k - \tilde{u}_l|^2 \, ds \preceq \sum_{s=0}^{1} H_k^{2s-1} |\mathcal{M}_k u_k|^2_{H^s(\Omega_k)} + \sum_{s=0}^{1} H_l^{2s-1} |\mathcal{M}_l u_l|^2_{H^s(\Omega_l)}$$

$$\preceq \sum_{s=0}^{1} H_k^{2s-1} |u_k|^2_{H_h^s(\Omega_k)} + \sum_{s=0}^{1} H_l^{2s-1} |u_l|^2_{H_h^s(\Omega_l)} \preceq H_k |u_k|^2_{H_h^1(\Omega_k)} + H_l |u_l|^2_{H_h^1(\Omega_l)}.$$

We remind that by definition $\tilde{u}_j = \mathcal{M}_j u_j$. Hence summing over all interfaces and using (3.6) ends the proof of estimate of the L^2 norm of u. □

3.3 Error estimate

In this section, we estimate the error between the discrete solution of (3.5) and the solution of (3.1). We show that the error is of the same order as in the standard nonconforming version of the finite element method.

We now state our main result concerning the error estimate:

Theorem 3.3.1 *Let u^*, u_h^* be the solutions of (3.1) and (3.5), respectively. Let $u^* \in H^2(\Omega)$. Then*

$$|u^* - u_h^*|_{H_H^1(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 |u^*|^2_{H^2(\Omega_k)} \right)^{1/2}.$$

For the proof we need several auxiliary results, one of them is the second Strang lemma, see [12] and Lemma 8.1.9, p.198 in [38], which we now remind.

Lemma 3.3.1 *Under the assumptions of Theorem 3.3.1 we have*

$$|u^* - u_h^*|_{H_H^1(\Omega)} \leq \inf_{v \in V^h} |u^* - v|_{H_h^1(\Omega)} \quad + \tag{3.7}$$

$$+ \sup_{w \in V^h} \sum_{k=1}^{N} \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau} \frac{\partial u^*}{\partial n} (w/|w|_{H^1_H(\Omega)})\, ds.$$

The first term in (3.7) is known as the approximation error while the second one is called the consistency error and is a consequence of the discontinuities of the functions of V^h through the edges of elements.

We first present some auxiliary technical tools that we need to prove our results. Some of them will be used in the next section.

Let $\tilde{V}^{h/2}(\Omega_k)$ be the conforming space of piecewise linear continuous functions on the triangulation $T_{\frac{h}{2}}(\Omega_k)$ which is constructed by dividing each triangle from $T_h(\Omega_k)$ into four ones by joining the midpoints of the edges of this element and let $\tilde{V}_0^{h/2}(\Omega_k)$ be the subspace of $\tilde{V}^{h/2}(\Omega_k)$ of functions with zero traces on $\partial\Omega_k$. We first introduce two *local equivalence maps (isomorphisms)*, as in [100], to obtain some properties of local nonconforming spaces $X_h(\Omega_k)$. We now define a *local equivalence map* $\mathcal{M}_k$: $X_h(\Omega_k) \to \tilde{V}^{h/2}(\Omega_k)$, see [100].

Definition 3.3.1 *Given $u \in X_h(\Omega_k)$, we define $\mathcal{M}_k u \in \tilde{V}^{h/2}(\Omega_k)$ by the values of $\mathcal{M}_k u$ at the vertices of the triangulation $T_{\frac{h}{2}}(\Omega_k)$. The vertices are divided into three sets of points:*

- *If $p \in \Omega_{k,h}^{CR}$, then*

$$(\mathcal{M}_k u)(p) = u(p).$$

- *If $p \in \Omega_{k,h} \setminus \partial\Omega_{k,h}$ and p is a vertex of an element of $T_h(\Omega_k)$, then*

$$(\mathcal{M}_k u)(p) = 1/(N(p)) \sum_{\tau_j^h} u_{|\tau_j^h}(p),$$

  where the sum is taken over all triangles τ_j^h with the common vertex p and $N(p)$ is the number of these triangles.

- *If $q \in \partial\Omega_{k,h}$, then*

$$(\mathcal{M}_k u)(q) = \frac{|q_l q|}{|q_l q_r|} u(q_l) + \frac{|q q_r|}{|q_l q_r|} u(q_r),$$

  where q_l, q_r are the left and right neighboring CR nodal points of q.

Lemma 3.3.2 *Let $\mathcal{M}_k u$ be defined as above, then for all $u \in X_h(\Omega_k)$, we have*

$$|u|_{H_h^1(\Omega_k)} \asymp |\mathcal{M}_k u|_{H^1(\Omega_k)},$$

$$\|u\|_{L^2(\Omega_k)} \asymp \|\mathcal{M}_k u\|_{L^2(\Omega_k)},$$

$$\int_{\partial\Omega_k} (\mathcal{M}_k u)(s)\, ds = \int_{\partial\Omega_k} u(s)\, ds,$$

$$\|\mathcal{M}_k u - u\|_{L^2(\Omega_k)} \preceq\ h_k\, |u|_{H_h^1(\Omega_k)},$$

$$\|\mathcal{M}_k u - u\|_{L^2(\mathcal{E})} \preceq\ h_k^{1/2}\, |u|_{H_h^1(\Omega_k)}.$$

Here $\mathcal{E}$ is an edge of Ω_k.

Proof. The proof of the first four inequalities can be found in [99], see Lemma 5.3, p.102 there, cf. also Lemma 3, p.390 in [100]. We now prove the last inequality. Using the standard trace theorem, see Theorem 1.5.2.1 p.42 in [71], on each subsegment of $\mathcal{E}$ that is an edge of a triangle in $T_h(\Omega_k)$ combined with a scaling argument, we get

$$\|\mathcal{M}_k u - u\|^2_{L^2(\mathcal{E})} = \sum_{e \in T_h^k(\mathcal{E})} \|\mathcal{M}_k u - u\|^2_{L^2(e)} \preceq \sum_{s=0}^{1} \sum_{\tau : \partial\tau \cap \mathcal{E} \neq \emptyset} h_k^{2s-1} |\mathcal{M}_k u - u|^2_{H^s(\tau)} \leq$$

$$\leq \frac{1}{h_k}\, \|\mathcal{M}_k u - u\|^2_{L^2(\Omega_k)} + h_k |\mathcal{M}_k u - u|^2_{H_h^1(\Omega_k)}.$$

The first and the fourth inequalities of the lemma yield the desired bound. □

We now define, for each edge $\mathcal{E}$ of Ω_k ($\mathcal{E}$ can be either a master or a slave) $X_h^{\mathcal{E}}(\Omega_k)$ as a subspace of $X_h(\Omega_k)$ of functions that are equal to zero at all nodes of $\partial\Omega^{CR}_{k,h} \setminus \mathcal{E}_h^{CR}$.

Let $\mathcal{M}_k^{\mathcal{E}} : X_h^{\mathcal{E}}(\Omega_k) \to \tilde{V}^{h/2}(\Omega_k)$ be a *local equivalence map* defined as follows, cf. [100]:

Definition 3.3.2 *Given $u \in X_h^{\mathcal{E}}(\Omega_k)$, we define $\mathcal{M}_k^{\mathcal{E}} u \in \tilde{V}^{h/2}(\Omega_k)$ by the values of $\mathcal{M}_k^{\mathcal{E}} u$ at the vertices of the triangulation $T_{\frac{h}{2}}(\Omega_k)$.*

- *If $p \in \Omega^{CR}_{k,h}$ or $p \in \Omega_{k,h} \setminus \partial\Omega_{k,h}$, then the value $(\mathcal{M}_k^{\mathcal{E}} u)(p)$ is defined as for $\mathcal{M}_k u$, see Definition 3.3.1.*
- *If $q \in \partial\Omega_{k,h} \setminus \mathcal{E}_h$, then $(\mathcal{M}_k^{\mathcal{E}} u)(q) = 0$.*

- *If* $q \in \mathcal{E}_h \setminus \mathcal{E}_h^{CR}$, *then*

$$(\mathcal{M}_k^{\mathcal{E}} u)(q) = \frac{|qq_r|}{|q_l q_r|} u(q_l) + \frac{|q_l q|}{|q_l q_r|} u(q_r),$$

where q_l, q_r *are the left and right neighboring CR nodal points of* q.

Note that $\mathcal{M}^{\mathcal{E}}$ is piecewise linear between nodes of $\mathcal{E}_h^{CR}$. The next lemma states the properties of $\mathcal{M}_k^{\mathcal{E}}$.

Lemma 3.3.3 *Let* $\mathcal{M}_k^{\mathcal{E}} u$ *be defined as in Definition 3.3.2, then for all* $u \in X_h^{\mathcal{E}}(\Omega_k)$, *we have*

$$|u|_{H_h^1(\Omega_k)} \asymp |\mathcal{M}_k^{\mathcal{E}} u|_{H^1(\Omega_k)}, \qquad \|u\|_{L^2(\Omega_k)} \asymp \|\mathcal{M}_k^{\mathcal{E}} u\|_{L^2(\Omega_k)},$$
$$\|\mathcal{M}_k^{\mathcal{E}} u - u\|_{L^2(\Omega_k)} \preceq \ h_k \, |u|_{H_h^1(\Omega_k)}, \qquad \|\mathcal{M}_k^{\mathcal{E}} u - u\|_{L^2(\mathcal{E})} \preceq \ h_k^{1/2} \, |u|_{H_h^1(\Omega_k)}.$$

The proof is analogous to the proof of the previous lemma.

We define a pseudo-inverse map $(\mathcal{M}_k)^{\dagger} : \tilde{V}^{h/2}(\Omega_k) \to X_h(\Omega_k)$, as in [100], by

$$(\mathcal{M}_k)^{\dagger} u(p) = u(p)$$

for all $p \in \Omega_{k,h}^{CR}$.

It follows from the definition that for all $u \in X_h(\Omega_k)$, we have

$$(\mathcal{M}_k)^{\dagger} \mathcal{M}_k u = u$$

and for $u \in \tilde{V}^{h/2}(\Omega_k)$

$$|(\mathcal{M}_k)^{\dagger} u|_{H_h^1(\Omega_k)} \preceq |u|_{H_h^1(\Omega_k)} \qquad \|(\mathcal{M}_k)^{\dagger} u\|_{L_h^2(\Omega_k)} \preceq \|u\|_{L_h^2(\Omega_k)}, \tag{3.8}$$

as $\Omega_{k,h}^{CR}$ are contained in the set of nodes of $\tilde{V}^{h/2}(\Omega_k)$, see [100]. Note that for $u \in X_h^{\mathcal{E}}(\Omega_k)$, we also obtain

$$(\mathcal{M}_k)^{\dagger} \mathcal{M}_k^{\mathcal{E}} u = u.$$

We now split of any local function $u_k \in X_h(\Omega_k)$ into two parts that are orthogonal to each other in terms of the form $a_{h,k}(\cdot,\cdot)$, i.e.

$$u_k = H_k u_k + P_k u_k. \tag{3.9}$$

Here $H_k u_k$ is a discrete harmonic part of u_k and $P_k u_k$ is the orthogonal projection onto the special subspace $\tilde{X}_h(\Omega_k) \subset X_h(\Omega_k)$ defined as follows. Let $\mathcal{A}_k$ be the special set of CR nodes containing all nodes that belong to $\partial\Omega_{k,h}^{CR}$ and those that are midpoints of the edges of triangles which have one (or possibly two) sides on a master $\gamma_{m,k}$ (or possibly two masters). Then we define $\tilde{X}_h(\Omega_k)$ as a space of functions of $X_h(\Omega_k)$ which are equal to zero at all nodes of $\mathcal{A}_k$. In other words, a function is in this subspace if it is equal to zero in $\partial\Omega_{k,h}^{CR}$ and over all triangles $\tau \in T_h(\Omega_k)$ such that $\partial\tau \cap \gamma_{m,k}$ is an edge of τ for any master $\gamma_{m,k} \subset \partial\Omega_k$.

We now define the discrete harmonic part of $u \in X_h(\Omega_k)$ as

$$\begin{cases} a_{h,k}(H_k u_k, v_k) = 0 & \forall v_k \in \tilde{X}_h(\Omega_k), \\ H_k u_k(p) = u_k(p) \quad for & p \in \mathcal{A}_k \end{cases}$$

and $P_k u_k$ is the orthogonal projection (in the sense of $a_{h,k}(\cdot,\cdot)$) of $u \in X_h(\Omega_k)$ onto $\tilde{X}_h(\Omega_k)$ defined by

$$a_{h,k}(P_k u_k, v_k) = a_{h,k}(u_k, v_k) \quad \forall v_k \in \tilde{X}_h(\Omega_k).$$

Note that $H_k = I - P_k$, and hence, $H_k u_k$ is orthogonal (in the sense of $a_{h,k}(\cdot,\cdot)$) to $P_k u_k$.

We now define an auxiliary operator $\Pi_m : L^2(\delta_{m,j}) \rightarrow W_0^{h_j}(\delta_{m,j})$, where $W_0^{h_j}(\delta_{m,j})$ is the space formed by all continuous functions which are equal to zero at the ends of $\delta_{m,j}$ and are piecewise linear over all segments which have their ends in $\delta_{m,j,h}^{CR}$. $\Pi_m u$ is a function in $W_0^{h_j}(\delta_{m,j})$ which interpolates $Q_m u$ at the nodes of $\delta_{m,j,h}^{CR}$. Thus it is sufficient to define values of $\Pi_m u$ in this set of points. For $p \in \delta_{m,j,h}^{CR}$ set

$$\Pi_m u(p) = Q_m u(p). \tag{3.10}$$

Here $Q_m u$ is the L^2 orthogonal projection onto $M_{-1,0}^{h_j}(\delta_{m,j})$ defined in (3.2). Thus $Q_m u$ is constant over each element of $T_h^j(\delta_{m,j})$ and the value of $Q_m u$ is properly defined at any point $p \in \delta_{m,j,h}^{CR}$ (which is a midpoint of an element $e \in T_h^j(\delta_{m,j})$).

The next lemma states the L^2 and $H_{00}^{1/2}$ stability of Π_m.

Lemma 3.3.4 *For $\Pi_m : L^2(\delta_{m,j}) \rightarrow W_0^{h_j}(\delta_{m,j})$ defined above, we have*

$$\|\Pi_m u\|_{L^2(\delta_{m,j})} \preceq \|u\|_{L^2(\delta_{m,j})} \quad \forall u \in L^2(\delta_{m,j}),$$

$$\|\Pi_m u\|_{H_{00}^{1/2}(\delta_{m,j})} \preceq \|u\|_{H_{00}^{1/2}(\delta_{m,j})} \quad \forall u \in H_{00}^{1/2}(\delta_{m,j}).$$

Proof. The L^2 stability is fairly obvious as

$$\|\Pi_m u\|^2_{L^2(\delta_{m,j})} \asymp h_j \sum_{p\in\delta^{CR}_{m,j,h}} |\Pi_m u(p)|^2 =$$

$$= h_j \sum_{p\in\delta^{CR}_{m,j,h}} |Q_m u(p)|^2 \asymp \|Q_m u\|^2_{L^2(\delta_{m,j})} \leq \|u\|^2_{L^2(\delta_{m,j})}.$$

In order to prove the H_0^1 stability, we define an additional operator $I_{W^{h_j}} : H_0^1(\delta_{m,j}) \to W_0^{h_j}(\delta_{m,j})$ as pointwise linear interpolant defined by the values of function at $\delta^{CR}_{m,j,h}$. It follows from the continuous embedding $H^1 \subset C^0$ (in 1-D), see e.g. Theorem 1.4.4.1, p.27 in [71], that for all $u \in H_0^1(\delta_{m,j})$, we have

$$|I_{W^{h_j}} u|_{H^1(\delta_{m,j})} \preceq |u|_{H^1(\delta_{m,j})}, \tag{3.11}$$

e.g. see Corollary 4.4.24, p.109 in [38]. Then for $u \in H_0^1(\delta_{m,j})$, we can derive that

$$|\Pi_m u|_{H^1(\delta_{m,j})} \leq |\Pi_m u - I_{W^{h_j}} u|_{H^1(\delta_{m,j})} + |I_{W^{h_j}} u|_{H^1(\delta_{m,j})}.$$

The second term is estimated by utilizing (3.11). For the first one we use an inverse inequality and obtain

$$|\Pi_m u - I_{W^{h_j}} u|^2_{H^1(\delta_{m,j})} \preceq \frac{1}{h_j^2} \|\Pi_m u - I_{W^{h_j}} u\|^2_{L^2(\delta_{m,j})} \asymp$$

$$\asymp \frac{1}{h_j} \sum_{p\in\delta^{CR}_{m,j,h}} |Q_m u(p) - u(p)|^2 \preceq |u|^2_{H^1(\delta_{m,j})}.$$

Here we have used the observation that $Q_m u(p) = \frac{1}{|e|}\int_e u(s)\,ds$ where e is an element of $T_h^j(\delta_{m,j})$ with the midpoint p, and then the local embedding $H^1(e) \subset C^0(e)$, Poincaré's inequality and a scaling argument.

Thus using the interpolation technique, e.g. see Proposition 12.1.5, p.279 in [38], we end the proof. □

The next lemma states that the seminorm over a subregion of discrete harmonic (in the sense of $a_{h,j}(\cdot,\cdot)$) function u_j which is nonzero only at CR nodes of a slave $\delta_{m,j} \subset \mathcal{A}_j$ can be estimated by the $H_{00}^{1/2}$-norm of $\mathcal{M}_j^{\mathcal{E}} u_j$ over this slave.

Lemma 3.3.5 *Let u be discrete harmonic in the terms of $a_{h,j}(\cdot,\cdot)$ in Ω_j and $u = 0$ at CR nodes of $\mathcal{A}_j \setminus \delta_{m,j}$. Then we have*

$$|u|_{H_h^1(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}} u_j\|_{H_{00}^{1/2}(\delta_{m,j})}.$$

Here $\mathcal{M}_j^{\mathcal{E}}$ is the local equivalence map defined in Definition 3.3.2 for $\mathcal{E} = \delta_{m,j}$.

Proof. We need another auxiliary operator: $\mathcal{H} : \tilde{V}^{\frac{h}{2}}(\partial\Omega_j) \to \tilde{V}^{\frac{h}{2}}(\Omega_j)$ - the discrete harmonic extension (in the conforming sense) defined by

$$\begin{cases} (\nabla \mathcal{H}v, \nabla\psi)_{L^2(\Omega_j)} = 0 & \forall \psi \in \tilde{V}_0^{h/2}(\Omega_j), \\ \mathcal{H}v = v & on \quad \partial\Omega_j. \end{cases}$$

We now estimate $|u|_{H^1_h(\Omega_j)}$ as follows, cf. the proof of Lemma 5.3, p.127 in [99]. We first note that

$$|(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)} \preceq |\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}}u\|_{H^{1/2}_{00}(\delta_{m,j})}. \tag{3.12}$$

The first inequality follows from (3.8) and the second one from the extension property of conforming discrete harmonic functions (in the sense as in the definition of $\mathcal{H}$), see Lemma 5.1, p.1112 in [21]. The function $(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u$ can differ from u at all CR nodes of $\mathcal{A}_j$ that are not in $\partial\Omega^{CR}_{j,h}$. We then modify the function $(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u$ by setting its values in $\mathcal{A}_j \setminus \partial\Omega^{CR}_{j,h}$ to zero. The new function denoted by $\tilde{u}$ is equal to u at nodes of $\mathcal{A}_j$ and we have

$$|\tilde{u}|_{H^1_h(\Omega_j)} \leq |(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)} + |\tilde{u} - (\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)} \preceq |(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)},$$

as $\tilde{u}$ and $(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u$ may differ only in the set of points that are associated with the triangles that have one edge on $\partial\Omega_j \setminus \delta_{m,j}$, and the second function is zero in $\partial\Omega^{CR}_{j,h} \setminus \delta^{CR}_{m,j,h}$. Thus we have

$$|u|_{H^1_h(\Omega_j)} \preceq |\tilde{u}|_{H^1_h(\Omega_j)} \preceq |(\mathcal{M}_j)^{\dagger}\mathcal{H}\mathcal{M}_j^{\mathcal{E}}u|_{H^1_h(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}}u\|_{H^{1/2}_{00}(\delta_{m,j})}.$$

The first inequality follows from the fact that u has the smallest energy norm on Ω_j among all functions $w \in X_h(\Omega_j)$ such that $w(p) = u(p)$ for $p \in \mathcal{A}_j$, since u is discrete harmonic (in the sense of $a_{h,j}(\cdot,\cdot)$). This can be proved in the same way as in the conforming case. The last estimate is the result of (3.12). $\square$

The next result is the approximation property of Q_m the L^2 orthogonal projection onto $M^{h_j}_{-1,0}(\delta_{m,j})$ defined in (3.2). The proof utilizes the Hilbertian interpolation and standard arguments, cf. the proof of Lemma 3.3.4 above and e.g. [47] or [32].

Lemma 3.3.6 *If $g \in H^s(\delta_{m,j})$, then holds*

$$\|g - Q_m g\|_{L^2(\delta_{m,j})} \preceq h_j^s |g|_{H^s(\delta_{m,j})} \quad s \in \{0, \frac{1}{2}, 1\},$$

where Q_m is the L^2 orthogonal projection onto $M^{h_j}_{-1,0}(\delta_{m,j})$ defined in (3.2).

3.3.1 Analysis of the consistency error

Let us turn to the consistency term. Our main result concerning this error we give in the following lemma:

Lemma 3.3.7 *Under the assumptions of Theorem 3.3.1, we have*

$$\sum_{k=1}^{N} \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau} \frac{\partial u^*}{\partial n} w \, ds \preceq \left(\sum_{k=1}^{N} h_k^2 |u^*|^2_{H^2(\Omega_k)} \right)^{1/2} |w|_{H^1_H(\Omega)}.$$

Proof.

First the consistency term can be rewritten as the sum of two terms: the first one being the sum over edges of fine elements contained in the subdomains and in $\partial\Omega$, and the second one being the sum of edges contained in the interface Γ, i.e.

$$\sum_{k=1}^{N} \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau} \frac{\partial u^*}{\partial n} w_k \, ds = \sum_{k=1}^{N} \sum_{\tau \in T_h(\Omega_k)} \sum_{e \subset (\partial\tau \setminus \Gamma)} \int_e \frac{\partial u^*}{\partial n} [w] \, ds + \sum_{\Gamma_{ij} \subset \Gamma} \int_{\Gamma_{ij}} \frac{\partial u^*}{\partial n} [w] \, ds,$$

where e is an edge of element τ, $[w]$ - jump across the edge e or Γ_{ij}, and for $e \subset \partial\Omega$ we have $[w] = w$.

The first term is estimated by $c\,(h_k^2 \sum_{k=1}^{N} |u^*|^2_{H^2(\Omega_k)})^{1/2} |w|_{H^1_H(\Omega)}$ by Lemma 8.3.7 and Lemma 8.3.9 of [38] for $e \subset \Omega \setminus \Gamma$ and $e \subset \partial\Omega$, respectively.

We now estimate the second term. Let Q_m be the standard L^2- orthogonal projection onto $M^{h_j}_{-1,0}(\delta_{m,j})$, see (3.2). By (3.3), we have

$$\int_{\delta_{m,j}} \frac{\partial u^*}{\partial n} [w] \, ds = \int_{\delta_{m,j}} (\frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n}) \, (w_i - w_j) \, ds =$$

$$= \int_{\delta_{m,j}} (\frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n})(w_i - \mathcal{M}_i w_i + \mathcal{M}_i w_i - Q_m \mathcal{M}_i w_i) \, ds -$$

$$- \int_{\delta_{m,j}} (\frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n})(w_j - Q_m w_j) \, ds,$$

where $\mathcal{M}_i u_i, \mathcal{M}_j u_j$ are defined in Definition 3.3.1. We have also used the fact that for all $u, v \in L^2(\delta_{m,j})$, we have $(Q_m u, (I - Q_m) v)_{L^2(\delta_{m,j})} = 0$ what follows from the properties of Q_m - the L^2-orthogonal projection. Now using Schwarz inequality, we have

$$\int_{\delta_{m,j}} \frac{\partial u^*}{\partial n} [w] \, ds \le \| \frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n} \|_{L^2(\delta_{m,j})} (\| w_i - \mathcal{M}_i w_i \|_{L^2(\delta_{m,j})} +$$

$$+\|\mathcal{M}_i w_i - Q_m \mathcal{M}_i w_i\|_{L^2(\delta_{m,j})} + \|w_j - Q_m w_j\|_{L^2(\delta_{m,j})}).$$

The first and third terms, see Lemma 3.3.6, can be estimated by

$$\|\frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n}\|_{L^2(\delta_{m,j})} \preceq h_j^{1/2} \, |\frac{\partial u^*}{\partial n}|_{H^{1/2}(\delta_{m,j})} \preceq h_j^{1/2} \, |u^*|_{H^2(\Omega_j)}$$

and

$$\|\mathcal{M}_i w_i - Q_m \mathcal{M}_i w_i\|_{L^2(\delta_{m,j})} \preceq h_j^{1/2} \, |\mathcal{M}_i w_i|_{H^{1/2}(\delta_{m,j})} \preceq h_j^{1/2} \, |\mathcal{M}_i w_i|_{H^1(\Omega_i)}.$$

We have also used the standard trace bound, see Theorem 1.5.2.1, p.42 in [71]. Next Lemma 3.3.2 yields that

$$\|\mathcal{M}_i w_i - Q_m \mathcal{M}_i w_i\|_{L^2(\delta_{m,j})} \preceq h_j^{1/2} \, |w_i|_{H_h^1(\Omega_i)}$$

and

$$\|w_i - \mathcal{M}_i w_i\|_{L^2(\delta_{m,j})} \preceq h_i^{1/2} |w_i|_{H_h^1(\Omega_i)} \leq h_j^{1/2} |w_i|_{H_h^1(\Omega_i)}.$$

Here we have considered the case when $h_i \leq h_j$. The case $h_i > h_j$ is discussed below. The last term, $\|w_j - Q_m w_j\|^2_{L^2(\delta_{m,j})}$, is estimated as follows:

$$\int_{\delta_{m,j}} (w_j - Q_m w_j)^2 \, ds = \sum_{e \in T_h^j(\delta_{m,j})} \int_e (w_j - Q_e w_j)^2 \, ds,$$

where Q_e is the L^2- orthogonal projection onto the one-dimensional space of constant functions on an element e. We have used the fact that functions in $M_{-1,0}^{h_j}(\delta_{m,j})$ being constant on one element and equal to zero on others form the L^2 orthogonal basis. Now using the reference element $\hat{e} \subset \partial\hat{\tau}$, we have for any constant d

$$\int_e (w_j - Q_e w_j)^2 \, ds \leq \int_e (w_j - d)^2 \, ds \asymp h_j \int_{\hat{e}} (\hat{w}_j - d)^2 \, d\hat{s}$$

$$\preceq h_j \, \|\hat{w}_j - d\|^2_{H^1(\hat{\tau})} \preceq h_j \, |\hat{w}_j|^2_{H^1(\hat{\tau})} \preceq h_j \, |w_j|^2_{H^1(\tau)}.$$

We used the trace bound, see [6], and Poincaré's inequality for $\hat{\tau}$.

Finally, summing over all elements of the slave triangulation of $\delta_{m,j}$, we obtain

$$\|w_j - Q_m w_j\|_{L^2(\delta_{m,j})} \preceq h_j^{1/2} |w_j|_{H_h^1(\Omega_j)}$$

Summing over all slaves $\delta_m \subset \Gamma$ ends the proof of the lemma.

We now consider the case when $h_i \geq h_j$. Then as before, we have

$$\int_{\delta_{m,j}} \frac{\partial u^*}{\partial n} [w] \, ds \leq \| \frac{\partial u^*}{\partial n} - Q_m \frac{\partial u^*}{\partial n} \|_{L^2(\delta_{m,j})} \cdot$$

$$\cdot \left(\|w_i - Q_m w_i\|_{L^2(\delta_{m,j})} + \|w_j - Q_m w_j\|_{L^2(\delta_{m,j})} \right),$$

since $Q_m w_i = Q_m w_j$, see (3.3). The first and the third term are estimated as above. We only have to deal with the second term. As before we obtain

$$\|w_i - Q_m w_i\|^2_{L^2(\delta_{m,j})} = \sum_{e \in T_h^j(\delta_{m,j})} \int_e (w_i - Q_e w_i)^2 \, ds,$$

where Q_e is the same as in the first part of the proof. The sum over elements e such that e is contained in one of element of h_i-triangulation $T_h^i(\gamma_{m,i})$ can be estimated in the same manner as above. Note that then $w_{i|e}$ is continuous. The problem is with the remaining elements. Let $e = [a, b]$ be a such element with the left end denoted by a, the right end by b and let $c \in [a, b]$ be an end of two joint elements of h_i-triangulation $T_h^i(\gamma_{m,i})$. We denote them as $[d, c]$ and $[c, p]$. Thus we have $a \in [d, c]$, $b \in [c, p]$ and there is a jump of function w_i at c. We denote the left value of w_i at x as $w_i^l(x)$ and the right one as $w_i^r(x)$, where x can be any of d, c, p. Then we define a constant $\beta = 0.5\,(w_i^l(c) + w_i^r(c))$ and we have

$$\|w_i - Q_e w_i\|^2_{L^2(e)} \leq \|w_i - \beta\|^2_{L^2(e)} = \int_a^c (w_i - \beta)^2 \, dt + \int_c^b (w_i - \beta)^2 \, dt.$$

We now straightforwardly calculate

$$\int_c^b (w_i - \beta)^2 \, dt \preceq h_j \, |w_i^r(c) - w_i^l(c)|^2 + (h_j^3/h_i^2) \, |w_i^l(p) - w_i^r(c)|^2 \preceq$$

$$\preceq h_j \sum_{c \in \partial\tau} |w_i|^2_{H^1(\tau)}.$$

The last sum is taken over all triangles τ with the common vertex c. The norm over ac is estimated in the same way. Summing over all elements e of the $T_h^j(\delta_{m,j})$, we obtain

$$\|w_i - Q_m w_i\|^2_{L^2(\delta_{m,j})} \preceq h_j^{1/2} |w_i|_{H_h^1(\Omega_i)},$$

what ends the proof of this case. □

3.3.2 Analysis of the approximation error

Let us turn to the best approximation error. The estimate of it follows from the following result.

Lemma 3.3.8 *For any function $u \in H_0^1(\Omega)$ with $u_{|\Omega_k} \in H^2(\Omega_k)$, we have*

$$\inf_{v \in V^h} |u - v|_{H^1_H(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 \, |u|^2_{H^2(\Omega_k)} \right)^{1/2}.$$

Proof. Let $\tilde{v}_k$ be a continuous piecewise linear interpolant of u define by the values of u at the vertices of all triangles of $T_h(\Omega_k)$ - the h_k triangulation of Ω_k. We have $\tilde{v}_k \in X_h(\Omega_k)$ and, see e.g. Corollary 4.4.24, p.109 in [38],

$$h_k^{-2} \, \|u_{|\Omega_k} - \tilde{v}_k\|^2_{L^2(\Omega_k)} + |u_{|\Omega_k} - \tilde{v}_k|^2_{H^1(\Omega_k)} \preceq h_k^2 |u_{|\Omega_k}|^2_{H^2(\Omega_k)}. \qquad (3.13)$$

That function $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_N) \in X_h(\Omega)$ may not satisfy the matching condition across the interfaces. To deal with this, we define a function w such that $v = w + \tilde{v}$ satisfies the mortar condition. To do it, we first determine w at nodal points of $\mathcal{A}_k, k = 1, \ldots, N$. Let w be equal to zero at all nodes associated with masters, i.e. in $\mathcal{A}_k \setminus \sum_s \delta^{CR}_{s,k,h}$, and on the slave side of an interface Γ_{kl} (in $\delta^{CR}_{m,k,h}$) be defined by

$$\int_{\delta_{m,k}} w \, \psi \, ds = \int_{\delta_{m,k}} (\tilde{v}_l - \tilde{v}_k) \psi \, ds \quad \forall \psi \in M^{h_k}_{-1,0}(\delta_{m,k}),$$

where $M^{h_k}_{-1,0}(\delta_{m,k})$ is the test space defined on the slave triangulation of $\delta_{m,k}$. We next define w as discrete harmonic in the sense of (3.9) in all subdomains. Then it is obvious that $v = w + \tilde{v}$ satisfies the matching condition (3.3). Thus we can deduce that

$$|u - v|_{H^1_H(\Omega)} \leq |u - \tilde{v}|_{H^1_H(\Omega)} + |w|_{H^1_H(\Omega)}.$$

The first term is estimated by (3.13), hence we only need to estimate $|w|_{H^1_H(\Omega)}$.

Let us consider one substructure and decompose $w_{|\Omega_k}$ as

$$w_{|\Omega_k} = \sum_{\delta_{m,k} \subset \partial\Omega_k} w_{m,k},$$

where the sum is taken over all slaves of Ω_k and $w_{m,k}$ is the function which is equal to w in $\delta^{CR}_{m,k,h}$, to zero at the remaining nodes of $\mathcal{A}_k$ and is discrete harmonic in Ω_k as in (3.9).

Thus we deduce that

$$|w|^2_{H^1_H(\Omega)} \leq \sum_{k=1}^{N} |w_k|^2_{H^1_h(\Omega_k)} \leq \sum_{k=1}^{N} \sum_{\delta_{m,k}\subset\partial\Omega_k} |w_{m,k}|^2_{H^1_h(\Omega_k)}.$$

We consider $w_{m,k}$ for one slave $\delta_{m,k} \subset \partial\Omega_k$ with the corresponding master $\gamma_{m,l} \subset \partial\Omega_l$. From Definition 3.3.2 and (3.10) follows that $\mathcal{M}^{\mathcal{E}}_j w_{m,k} = \Pi_m(\tilde{v}_l - \tilde{v}_k)$ on $\delta_{m,k}$. Then by Lemmas 3.3.4 and 3.3.5, we have

$$|w_{m,k}|_{H^1_h(\Omega_k)} \preceq \|\mathcal{M}^{\mathcal{E}}_j w_{m,k}\|_{H^{1/2}_{00}(\Gamma_{kl})} = \|\Pi_m(\tilde{v}_l - \tilde{v}_k)\|_{H^{1/2}_{00}(\Gamma_{kl})} \preceq$$

$$\preceq \|\tilde{v}_l - \tilde{v}_k\|_{H^{1/2}_{00}(\Gamma_{kl})} \leq \|\tilde{v}_l - u\|_{H^{1/2}_{00}(\Gamma_{kl})} + \|u - \tilde{v}_k\|_{H^{1/2}_{00}(\Gamma_{kl})}.$$

Using the trace theorem, see e.g. Theorem 1.5.2.1, p.42 in [71], combined with a scaling argument on each element of the h_l triangulation of Γ_{kl} and (3.13), we obtain

$$\|\tilde{v}_l - u\|_{L^2(\Gamma_{kl})} + h_l|\tilde{v}_l - u|_{H^1(\Gamma_{kl})} \preceq h_l^{3/2}|u|_{H^2(\Omega_l)}.$$

We remind that $\tilde{v}_l$ is the local continuous piecewise linear interpolant of u defined by the values of u at nodal points of $T_h(\Omega_l)$. Then an interpolation argument, see e.g. Proposition 12.1.5, p.279 in [38], (as $H^{1/2}_{00}(\Gamma_{kl}) = [L^2(\Gamma_{kl}), H^1_0(\Gamma_{kl})]_{1/2}$) yields the following estimate

$$\|\tilde{v}_l - u\|_{H^{1/2}_{00}(\Gamma_{kl})} \preceq h_l|u|_{H^2(\Omega_l)}.$$

The $H^{1/2}_{00}$ norm of $\tilde{v}_k - u$ can be estimated in the same way. Finally, we get

$$|w_{m,k}|_{H^1_h(\Omega_k)} \preceq h_l|u|_{H^2(\Omega_l)} + h_k|u|_{H^2(\Omega_k)}.$$

Summing over the slaves $\delta_{m,k} \subset \partial\Omega_k$ and afterwards over the subdomains ends the proof of the lemma. □

3.4 Additive Schwarz method

In this section, we propose a parallel method for solving the problem (3.5) arising from discretization of the boundary value problem (3.1) by the method described in Section 3.2. The method is designed and analyzed using the general ASM framework, see Section 1.4.

Using this framework, the method is defined in terms of a decomposition of V^h into subspaces and projections on these subspaces in terms of certain bilinear forms. We want to remind that we can straightforwardly develop another algorithm for solving the problem (3.5) - a multiplicative Schwarz method (MSM) based on the same decomposition of the discrete space and the same local bilinear forms, cf. Section 1.4.

3.4.1 Description of ASM

For simplicity of presentation, we describe the method with the additional assumption that the subdomains Ω_i are triangles and form a quasiuniform triangulation with a parameter H, cf. [47]. We remind that the master sides of an interface $\Gamma_{ij} = \gamma_{m,i} = \delta_{m,j}$ is chosen according to the rule $h_i \le h_j$. This assumption is here necessary.

We first have to define a decomposition of the discrete space V^h. We now introduce nodal basis functions of the subspace V^h. We divide nodes that are not on slaves into two sets, first of nodes of $\mathcal{A}_k$ for any Ω_k, cf. (3.9), and the second set of remaining nodes. With each *CR nodal point* that is not on a slave side, we associate a basis function which is equal to one at this point and zero at all *CR nodal points* that are not in any $\delta^{CR}_{s,h}$. A basis function associated with a node of the second set is a standard nonconforming nodal basis function, i.e. equal to one at the respective node and zero at all remaining nodes. A basis function associated with a node of the first set is one at this node, zero at remaining nodes except ones in any $\delta^{CR}_{s,h}$, where is determined by the mortar condition, see (3.3), at nodes of each slave. In this way, a basis function of the first set associated with a node of any $\mathcal{A}_k \setminus \bigcup \delta^{CR}_{s,k,h}$ can be nonzero not only at this node but also at some CR nodes on one or two slave sides of substructures that have a common side to Ω_k.

We first define a conforming coarse space denoted by $V^H = V_0$, the space of continuous functions which are piecewise linear on the coarse triangulation and equal to zero on $\partial\Omega$, obviously $V_0 \subset V^h$. We next define local spaces V_i^s associated with subregions which have slaves as all their edges contained in Γ. Let S denote the set of indices of those subdomains. Then let $u \in V_i^s$ for $i \in S$ if u be locally in $X_m(\Omega_i)$ and zero in all remaining subdomains. Here $X_m(\Omega_i)$ is a subspace of $X_h(\Omega_i)$ formed by the functions which are equal to zero at all CR nodes of $\mathcal{A}_i$.

To define local subspaces V_m associated with masters, we introduce for each master $\gamma_{m,k}$ a set of CR nodal points $\mathcal{A}_{\gamma_{m,k}}$ which is a subset of $\mathcal{A}_k$ defined as follows. Let

$$\mathcal{A}_{\gamma_{m,k}} = \gamma^{CR}_{m,k,h} \cup \mathcal{A}^{(0)}_{\gamma_{m,k}},$$

where

$$\mathcal{A}^{(0)}_{\gamma_{m,k}} = \{p \in \mathcal{A}_k \setminus \partial\Omega^{CR}_{k,h} : p \in \partial\tau \ \ and \ \ \partial\tau \cap \gamma_{m,k} = e \ an \ edge \ for \ \tau \in T_h(\Omega_k)\}.$$

We see that $\mathcal{A}_{\gamma_{m,k}}$ contains CR nodal points which are either in $\gamma^{CR}_{m,k,h}$ or in $\Omega^{CR}_{k,h}\setminus\partial\Omega^{CR}_{k,h}$ as midpoints of edges of triangles that have an edge on $\gamma_{m,k}$, (cf. definition of $\mathcal{A}_k$). Note that those sets for two masters of Ω_k which have the common end can have common points (near the common end of those masters). Then we can define a local subspaces V_m associated with that master $\gamma_{m,k} \subset \partial\Omega_k$ as follows: let $u \in V_m$ if

- is zero in $\mathcal{A}_k \setminus \mathcal{A}_{\gamma_{m,k}}$ for Ω_k,
- is discrete harmonic in all substructures Ω_j that have a slave which is the common edge to Ω_k and is zero in $\mathcal{A}_j \setminus \delta^{CR}_{m,j}$ for those Ω_j,
- is zero in all remaining substructures.

Thus we introduce the decomposition:

$$V^h = V_0 + \sum_{i\in S} V_i^s + \sum_{m=1}^{K} V_m,$$

where K is the number of all masters $\gamma_{m,k} \subset \Gamma$. It is easy to see that this decomposition is properly defined. We now introduce bilinear forms defined on the these subspaces. We set that all local bilinear forms for V_0, V_i^s for $i \in S$ are equal to the original form $a_H(\cdot,\cdot)$ and for the master local subspaces we set that $b_m(\cdot,\cdot) = a_{h,i}(\cdot,\cdot)$ where Ω_i is the substructure such that γ_m is its master, i.e. $\gamma_m \subset \partial\Omega_i$.

Let us introduce operators T_0, $P_k, k \in S$, and $T_m, m = 1,\ldots,K$, as

$$a_H(T_0u, v) = a_H(u, v) \quad \forall v \in V_0,$$

$$a_H(P_ku, v) = a_H(u, v) \quad \forall v \in V_k^s,$$

and

$$b_m(T_mu, v) = a_H(u, v) \quad \forall v \in V_m.$$

Let

$$T = T_0 + \sum_{i\in S} P_i + \sum_{m=1}^{K} T_m.$$

The problem (3.5) is replaced by

$$Tu_h^* = g_h, \tag{3.14}$$

where

$$g_h = g_0 + \sum_{i\in S} g_i^s + \sum_{m=1}^{K} g_m,$$

$$g_0 = T_0 u_h^*, \quad g_i^s = P_i u_h^*, \quad g_m = T_m u_h^*$$

and u_h^* is the solution of (3.5). We ought to point out that these functions can be computed without knowing u_h^*.

The main result of this section is given in the following theorem.

Theorem 3.4.1 *For all $u \in V^h$ we have*

$$(1 + \log(H/\underline{h}))^{-2} a_H(u,u) \preceq a_H(Tu,u) \preceq a_H(u,u),$$

where $\underline{h} = \inf_k h_k$.

3.4.2 Technical tools

We first prove some technical lemmas which are used in the proof of Theorem 3.4.1.

Lemma 3.4.1 *For a master $\gamma_{m,i} \subset \partial\Omega_i$ with its associated slave $\delta_{m,j} \subset \partial\Omega_j$ holds*

$$|u_j|_{H_h^1(\Omega_j)} \preceq \ |u_i|_{H_h^1(\Omega_i)} \quad \forall u \in V_m,$$

where V_m is the local subspace associated with the master $\gamma_{m,i}$.

Proof. We first note that for $u \in V_m$ and $p \in \delta_{m,j,h}^{CR}$, we have $(\mathcal{E} = \delta_{m,j})$

$$\left(\mathcal{M}_j^{\mathcal{E}} u_j\right)(p) = u_j(p) = (Q_m u_j)(p) = (Q_m u_i)(p) = (\Pi_m u_i)(p), \tag{3.15}$$

what follows from Definition 3.3.2, (3.10) and (3.3).

By Lemma 3.3.5, we have

$$|u_j|_{H_h^1(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}} u_j\|_{H_{00}^{1/2}(\delta_{m,j})}$$

and then from Lemma 3.3.4, we get with the help of the operator Π_m

$$|u_j|_{H_h^1(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}} u_j - \Pi_m \mathcal{M}_i^{\mathcal{E}} u_i\|_{H_{00}^{1/2}(\delta_{m,j})} + \|\Pi_m \mathcal{M}_i^{\mathcal{E}} u_i\|_{H_{00}^{1/2}(\delta_{m,j})} \preceq$$

$$\preceq \frac{1}{h_j^{1/2}} \|\mathcal{M}_j^{\mathcal{E}} u_j - \Pi_m \mathcal{M}_i^{\mathcal{E}} u_i\|_{L^2(\delta_{m,j})} + \|\mathcal{M}_i^{\mathcal{E}} u_i\|_{H_{00}^{1/2}(\delta_{m,j})}.$$

We have used an inverse inequality and Lemma 3.3.4. The second term is estimated using the standard trace theorem, cf. Theorem 1.5.1.2, p.37 in [71], and Lemma 3.3.3. The first term is estimated using (3.15) and we have

$$\frac{1}{h_j}\|\mathcal{M}_j^{\mathcal{E}} u_j - \Pi_m \mathcal{M}_i^{\mathcal{E}} u_i\|_{L^2(\delta_{m,j})}^2 \asymp \sum_{p \in \delta_{m,j,h}^{CR}} |Q_m u_i(p) - Q_m \mathcal{M}_i^{\mathcal{E}} u_i(p)|^2 \asymp$$

$$\asymp \frac{1}{h_j}\|Q_m(u_i - \mathcal{M}_i^{\mathcal{E}} u_i)\|_{L^2(\delta_{m,j})}^2 \leq \frac{1}{h_j}\|u_i - \mathcal{M}_i^{\mathcal{E}} u_i\|_{L^2(\delta_{m,j})}^2 \preceq \frac{h_i}{h_j}\, |u_i|_{H^1(\Omega_i)}^2.$$

The last inequality follows from Lemma 3.3.3. Using the assumption $h_i \leq h_j$ ends the proof of this lemma. □

In the next lemma, we state a Sobolev like inequality for nonconforming finite element functions. It directly follows from a Sobolev like inequality for conforming P_1 functions, e.g. see Lemma 7, p.170 in [20], and the properties of $\mathcal{M}_i$, cf. Lemma 3.3.2.

Lemma 3.4.2 *For a function $u \in X_h(\Omega_k)$ holds*

$$\|u\|_{L^\infty(\Omega_k)}^2 \preceq (1 + \log(H_k/h_k)) \left(\frac{1}{H_k^2}\|u\|_{L^2(\Omega_k)}^2 + |u|_{H_h^1(\Omega_k)}^2 \right),$$

where H_k is the diameter of Ω_k.

We now introduce an auxiliary function associated with each master.

Definition 3.4.1 *Let $\theta_{m,k}$ be discrete harmonic in the sense of (3.9) in all substructures and equal to*

- $\frac{1}{2}$ *at points of $\mathcal{A}_{\gamma_{m,k}} \cap (\bigcup_{s \neq m} \mathcal{A}_{\gamma_{s,k}})$,*
- *one in $\mathcal{A}_{\gamma_{m,k}} \setminus (\bigcup_{s \neq m} \mathcal{A}_{\gamma_{s,k}})$,*

- *zero at all other CR nodes of $\mathcal{A}_k$ and of $\mathcal{A}_j \setminus \delta_{l,j,h}^{CR}$ for $j \neq k$, where $\delta_{l,j}$ is a slave that is the common edge to Ω_k and Ω_j.*

Note that $\theta_{m,k}$ can be nonzero only in Ω_k and in Ω_j that have the common edge to Ω_k.

The next lemma states one property of $\theta_{m,k}$.

Lemma 3.4.3 *Let for $u \in V^h$ define $u^{m,k}$ as a function equal to $I_h^{CR}(\theta_{m,k}u)$ in $\mathcal{A}_j$, for $j = 1, \ldots, N$, and discrete harmonic in the sense of (3.9) in all subdomains. Then*

$$a_{h,k}(u^{m,k}, u^{m,k}) \preceq (1 + \log(H_k/h_k))^2 \left\{ \frac{1}{H_k^2} \|u\|_{L^2(\Omega_k)}^2 + |u|_{H_h^1(\Omega_k)}^2 \right\},$$

where H_k is the diameter of Ω_k and I_h^{CR} is the pointwise interpolant at CR nodes.

In the proof of this lemma, similar ideas to those of the proof of Lemma 4.5, p.1676 in [60], are used. The seminorm $|u^{m,k}|_{H_h^1(\Omega_k)}$ can be estimated by constructing a special function which is equal to u^m in $\mathcal{A}_k$ and for which our estimate is done and by noting that the discrete harmonic function has minimal energy.

3.4.3 Proof of the main theorem

We now give the proof of Theorem 3.4.1.

Proof. Using the general ASM framework, we have to check three key assumptions specified in Theorem 1.4.1 in Section 1.4.

Assumption (i)
We want to prove that there is a positive constant c independent of h_i and H such that for all $u \in V^h$ there exist functions $u_0 \in V_0$, $u_i \in V_i^s$ and $u^m \in V_m$ such that $u = u_0 + \sum_{i \in S} u_i + \sum_{m=1}^{K} u^m$ and

$$a_H(u_0, u_0) + \sum_{i \in S} a_H(u_i, u_i) + \sum_{m=1}^{K} b_m(u^m, u^m) \leq c\,(1 + \log(H/\underline{h}))^2 a_H(u, u). \quad (3.16)$$

We first select $u_0 \in V_0 = V^H$ by making $u_0(x_r) = (1/N_{x_r}) \sum \overline{u}_i$, where $x_r \in \Gamma$ is a crosspoint. Here the sum is taken over the subdomains that have x_r as a vertex, N_{x_r} is the number of such subdomains and $\overline{u}_i$ is the average value of u over Ω_i. Let $w = u - u_0$. We now decompose w as

$$w = Hw + Pw,$$

where

$$Hw = \sum_{k=1}^{N} H_i w, \quad Pw = \sum_{k=1}^{N} P_i w.$$

$H_i w, P_i w$ are defined as in (3.9) and extended as zero onto other substructures. We simply define for $i \in S$

$$u_i = P_i w.$$

We next decompose Hw as

$$Hw = \sum_{m=1}^{K} v^m,$$

where $v^m \in V_m$ is defined as equal to $I_h^{CR}(\theta_{m,k} u)$ in $\mathcal{A}_j$, for $j = 1, \ldots, N$ and discrete harmonic in all subdomains. Then we can define $u^m \in V_m$ as

$$u^m = v^m + (1/N(k))\, P_k w,$$

where $N(k)$ is the number of masters $\gamma_{l,k} \subset \partial\Omega_k$. Note that

$$u = u_0 + w = u_0 + \sum_{i \in S} u_i + \sum_{m=1}^{K} u^m.$$

We first estimate $a_H(u_0, u_0)$ as

$$a_H(u_0, u_0) = \sum_{k=1}^{N} |u_0 - \overline{u}_k|^2_{H_h^1(\Omega_k)} \preceq \sum_{k=1}^{N} \sum_{x_r \in \partial\Omega_k} |u_0(x_r) - \overline{u}_k|^2,$$

where the second sum is taken over vertices of Ω_k. Then we consider $|u_0(x_r) - \overline{u}_k|^2$ and have

$$|u_0(x_r) - \overline{u}_k|^2 \preceq \sum_{i \neq k} |\overline{u}_k - \overline{u}_i|^2,$$

where the sum is taken over all subdomains that have x_r as the common vertex.

Using the fact that the average values of $u \in V^h$ over a master $\gamma_{m,i}$ and a slave $\delta_{m,j}$ that occupies geometrically the same place, are equal to each other, since $Q_m u_i = Q_m u_j$, see (3.3), and Poincaré's inequality for nonconforming elements, see Lemma 5, p.392 in [100], we obtain

$$a_H(u_0, u_0) \preceq \sum_{i=1}^{N} |u_i|^2_{H_h^1(\Omega_i)} \leq a_H(u, u). \tag{3.17}$$

We now note that for a master $\gamma_m \subset \partial\Omega_k$

$$b_m(u^m, u^m) \preceq a_H(P_k w, P_k w) \; + \; a_{h,k}(v^m, v^m).$$

Thus

$$\sum_{m=1}^{K} b_m(u^m, u^m) + \sum_{i \in S} a_H(u_i, u_i) \preceq \sum_{k=1}^{N} a_H(P_k w, P_k w) \; + \; \sum_{m=1}^{K} a_{h,k}(v^{m,k}, v^{m,k}).$$

The first sum can estimated as

$$\sum_{k=1}^{N} a_H(P_k w, P_k w) \preceq \sum_{k=1}^{N} \{a_H(P_k u_0, P_k u_0) + a_H(P_k u, P_k u)\} \leq$$

$$\leq a_H(u_0, u_0) + a_H(u, u) \preceq a_H(u, u).$$

We have used (3.17) and the fact that P_k is the orthogonal projection in terms of $a_{h,k}(\cdot, \cdot)$. We next estimate the seminorm of $v^{m,k}$ over Ω_k. At nodes of $\mathcal{A}_k$ we have

$$v^{m,k} = I_h^{CR}(\theta_{m,k}(u_0 - \overline{u}_k)) + I_h^{CR}(\theta_{m,k}(\overline{u}_k - u)).$$

Those functions are extended onto slave sides by the mortar condition and further as discrete harmonic. The seminorm of the second function we estimate by Lemma 3.4.3 and Poincaré's inequality for nonconforming elements. The seminorm of the first one can be estimated using Lemma 3.4.3 and (3.17). Thus we have

$$a_{h,k}(v^{m,k}, v^{m,k}) \preceq (1 + \log(H/h))^2 |u|^2_{H_h^1(\Omega_k)}.$$

Summing over all masters yields the desired estimate.

Assumption (iii)

We notice that $\omega = 1$ for V_0 and V_i^s, $i \in S$, since we have set $a_H(\cdot, \cdot)$ as our local bilinear forms for these subspaces. We now estimate ω for $V_{m,k}$. We want to prove that

$$a_H(u, u) \preceq b_m(u, u) \quad \forall\, u \in V_{m,k}.$$

Let $u \in V_{m,k}$ for a master $\gamma_{m,k} \subset \partial\Omega_k$. This function is nonzero only in the substructures which have a slave that is the common edge to Ω_k. We first estimate the seminorm over Ω_j one of subdomains such that $\partial\Omega_j \cap \partial\Omega_k = \overline{\delta}_{s,j} = \overline{\gamma}_{s,k} \neq \overline{\gamma}_{m,k}$. By Lemma 3.3.5 and an inverse inequality, we have

$$|u_j|_{H_h^1(\Omega_j)} \preceq \|\mathcal{M}_j^{\mathcal{E}} u_j\|_{H_{00}^{1/2}(\delta_{m,j})} \preceq \frac{1}{h_j^{1/2}} \|\mathcal{M}_j^{\mathcal{E}} u_j\|_{L^2(\delta_{m,j})} \preceq (\sum_{p \in \delta_{s,j,h}^{CR}} |u_j(p)|^2)^{\frac{1}{2}}.$$

Here $\mathcal{E} = \delta_{m,j}$. Then by (3.15), we have

$$|u_j|^2_{H^1_h(\Omega_j)} \preceq \sum_{p \in \delta^{CR}_{s,j,h}} |Q_m u_j(p)|^2 = \sum_{p \in \delta^{CR}_{s,j,h}} |Q_m u_k(p)|^2 \asymp$$

$$\asymp \frac{1}{h_j} \|Q_m u_k\|^2_{L^2(\Gamma_{kj})} \preceq \frac{1}{h_j} \|u_k\|^2_{L^2(\Gamma_{kj})} = \frac{1}{h_j} \sum_{e \in T^k_h(\gamma_{s,k})} \|u_k\|^2_{L^2(e)}.$$

Note that u_k can be nonzero only in few elements of the h_k triangulation of $\gamma_{s,k}$, (near the common end to $\gamma_{m,k}$), and that

$$u_k(p) = 0, \quad p \in \gamma^{CR}_{s,k,h}, \tag{3.18}$$

what follows from the definition of $V_{m,k}$. Since u_k is linear over each segment e that is the edge of τ an element of $T_h(\Omega_k)$ the h_k - triangulation of Ω_k, we have $\frac{1}{|e|}\int_e u_k\,ds = 0$ what follows from (3.18). Then using the trace theorem, the Poincaré's inequality for each e and a scaling argument, we have

$$|u_j|^2_{H^1_h(\Omega_j)} \preceq (h_i/h_j)\,|u_k|^2_{H^1_h(\Omega_k)} \le |u_k|^2_{H^1_h(\Omega_k)}.$$

We have used the assumption that $h_i \le h_j$.

Finally, the seminorm over the subdomain Ω_l such that $\partial\Omega_l \cap \partial\Omega_k = \overline{\delta}_{m,l} = \overline{\gamma}_{m,k}$ can be estimated from Lemma 3.4.1 by the seminorm of u over Ω_k.

Assumption (ii)
It is satisfied as functions from V_m and V_i^s have local supports. □

3.4.4 Implementation

In this subsection, we briefly describe an implementation of our method. For simplicity, we present our method in terms of Richardson iteration, while in practice a CG method is used.

$$u^{n+1} = u^n - \tau\ \{T(u^n) - g\} = u^n - \tau\,\{r^n_0 + \sum_{i \in S} r^n_i + \sum_{m=1}^{K} r^n_m\},$$

where τ is a properly chosen parameter, $r^n_0 = T_0(u^n - u^*_h)$, $r^n_i = P_i(u^n - u^*_h)$ for $i \in S$ and $r^n_m = T_m(u^n - u^*_h)$ for each master $\gamma_m \subset \Gamma$, cf. (3.14).

Algorithm 3.4.1 • *Let $u^0 \in V^h$ be arbitrary.*

- *For $n = 0$ until convergence,*
  - *Compute* $r_0^n = T_0(u^n - u_h^*)$, *solving*

$$a_H(r_0^n, v) = a_H(u^n, v) - (f, v) \quad \forall v \in V_0$$

  - *Compute* $r_i^n = P_i(u^n - u_h^*)$ *for* $i \in S$, *solving*

$$a_H(r_i^n, v) = a_H(u^n, v) - (f, v) \quad \forall v \in V_i$$

  - *Compute* $r_m^n = T_m(u^n - u_h^*)$ *for all masters* $\gamma_m \subset \Gamma$, *solving*

$$b_m(r_m^n, v) = a_H(u^n, v) - (f, v) \quad \forall v \in V_m \tag{3.19}$$

  - $u^{n+1} = u^n - \tau\left(r_0^n + \sum_{i \in S} r_i^n + \sum_{m=1}^{K} r_m^n\right)$

- *End* n.

The r_0^n and $r_i^n, i \in S$ can be computed in the standard way. We now briefly discuss the case of r_m^n for a master $\gamma_{m,k} \subset \partial\Omega_k$. The right-hand side of (3.19) is a sum of integrals over Ω_k and over the substructures which have slaves that are common edges to Ω_k. From the definition follows that $v \in V_m$ is discrete harmonic over those subdomains and therefore the right-hand side of (3.19) has to be computed in a special way, see p.101-110 in Chapter 4 from [20]. After solving (3.19), we obtain the function r_m^n locally over Ω_k, then (3.3) sets its values over each slave $\delta_{l,j} = \gamma_{l,k}$ and further its values at the nodes of $\Omega_{j,h}^{CR} \setminus \mathcal{A}_j$ are determined by the values in $\delta_{l,j,h}^{CR}$ as $r_m^n \in V_m$ is discrete harmonic in Ω_j, in the sense of (3.9), and zero in $\mathcal{A}_j \setminus \delta_{l,j,h}^{CR}$, cf. [20].

3.5 Numerical Experiments

In this section, we present some preliminary results of numerical experiments. We carry out a few numerical experiments to test the error estimates and then some to test the convergence our method. Our algorithm has been implemented in PETSCs 2.0 (the Portable, Extensible Toolkit for Scientific Computation) in C on Sun Sparc Workstation. The region Ω is the rectangle $(0, 2) \times (0, 1)$ divided into two adjacent unit squares. Each substructure Ω_k is divide into a grid of smaller squares. These small squares are then divided into two triangles by drawing the lines from bottom left to top right. The meshes do not match on the interface. We assign the right side of the interface as a master one and in our tables we denote the number of unknowns of the subspace of the right subdomain as N_m and the one of the left one as N_s. h_m is a diameter of the small triangles in the right (master) subdomain and h_s is the parameter of the mesh of the left (slave) substructure.

Table 3.1: Accuracy tests.

l=0	h_s	h_m	H^1	L^2
l=0	1/5	1/6	0.1962	0.0257
l=1	1/10	1/12	9.7585D-02(2.01)	6.13813D-03(4.19)
l=2	1/20	1/24	4.8734D-02(2.00)	1.50103D-03(4.09)
l=3	1/40	1/48	2.4360D-02(2.00)	3.71052D-04(4.05)
l=4	1/80	1/96	1.2179D-02(2.00)	9.22363D-05(4.02)
l=5	1/160	1/192	6.0893D-03(2.00)	2.29932D-05(4.01)

We first compute the unknowns of the solution of the inner nodes of the left subdomain and then compute the values of the remaining unknowns (of the right subdomain) by CG method using as a preconditioner the exact solver over the right (master) substructure, i.e. we use a preconditioner of Neumann-Dirichlet type, see p.112-116 in [20]. We first perform a few numerical experiments to support the ac-

Table 3.2: $h_s = 0.1$ constant

N_m	N_s	h_m/hs	No. of iteration
1180	280	1/2	6
10740	280	1/6	6
29900	280	1/10	6
76640	280	1/16	6
119800	280	1/20	6
287990	280	1/30	6

curacy theory developed in the Section 3. In this accuracy tests, we assume that the exact solution u^* has the form

$$u^*(x, y) = sin(\frac{\pi}{2}x) * \sin(\pi y).$$

We denoted by $u = (u_1, u_2) \in V^h$ the computed solution. Let $I_{h_i}^{CR}$ be the pointwise piecewise linear interpolation operator in $X_h(\Omega_i)$. The error we report in Table 3.1 is

Table 3.3: h_m/h_s constant

N_m	N_s	No. of iteration
1180	280	6
4760	1160	8
10740	2640	8
19120	4720	8
76640	19040	8
119800	29800	8

defined by $e = (e_1, e_2) = (I_{h_1}^{CR}u^* - u_1, I_{h_2}^{CR}u^* - u_2)$. The norm L^2 and the seminorm H_h^1 are used to measure the error. In the first initial test, we take $h_s = h_1 = 1/5$ and $h_m = h_2 = 1/6$. Then in the following tests the refinement is done by cutting each triangle into four equal triangles. We use l to denote the level of refinement. The

Table 3.4: $h_m = 0.1$ constant

N_m	N_s	h_m/hs	No. of iteration
290	2640	3	10
290	7400	5	10
290	14560	7	11
290	29800	10	11
290	67200	15	12

results are summarized in Table 3.1. In row l the number in ( ) is the ratio of the error in row l to the one in row $l-1$. One can see that the error in H_h^1 seminorm is of first order.

In the next tables, we present the results of the experiments in which we test the convergence of our Neumann-Dirichlet method and its dependence on the ratio of parameters. In Table 3.2, we set the value of N_s and increase N_m, thus the ratio of h_m to h_s diminishes. In Table 3.3, we present the experiments in which we set the ratio of h_m to h_s to $1/2$ and increase the number of unknowns.

The results show that the method for two subdomain is independent of the number of unknowns and the ratio of h_m to h_s if $h_m \leq h_s$. In Table 3.4, we presents the results of the experiments in those we check if the assumption $h_m \leq h_s$ is really necessary (cf. Lemma 3.4.1). We set the value of h_m and decrease h_s. The results show that the number of iterations may be independent or weakly dependent on the ratio of h_m to h_s what implies that the choice of the mortar side of an edge may be arbitrary.

Chapter 4

Mortar methods for discretizations of a plate problem

Contents

4.1 Introduction.

In this chapter, we study certain mortar element methods for the clamped plate problem. For our knowledge, there are no results concerning such topics. Belhachmi [8] discussed the mortar method for the biharmonic problem, but his results concern only the case of local spectral discretizations. He carried out the error analysis for that case only.

We consider locally the conforming bicubic element, the reduced Hsieh-Clough-Tocher (HCT) and the Hsieh-Clough-Tocher macro elements, cf. [48], and the nonconforming Adini and Morley finite element methods. We present the error analysis for all these discretizations and discuss some methods for solving the discrete problems. We restrict ourselves to the geometrically conforming version of mortar method, i.e. the polygonal domain Ω is divided into polygonal subdomains Ω_i which form a coarse triangulation: the intersection of two subregions is either the empty set, an edge or a vertex.

We first introduce independent local discretizations of one of the five types mentioned above in all subdomains. The 2-D meshes of two neighboring subregions do not necessarily match on their common interface. Then the mortar technique for plate problems which we present here requires the continuity of the solution at the vertices of subdomains and that the solution on the two neighboring subdomain satisfies two mortar conditions of the L^2 type. Those conditions depend on the local discretization methods.

For the locally conforming methods (i.e. bicubic, HCT and reduced HCT), the mortar conditions on the common edge of two subdomains are equivalent to the equality of the L^2 projections on two mortar spaces of the solutions and of the normal derivatives of the solutions on these two subdomains. The mortar spaces defined on the common interface depend on the local discretization methods.

For the two nonconforming methods (i.e. Adini and Morley elements), the mortar conditions are of the same type, but additionally involve some interpolants which are defined locally on each interface. One of the reason of the introduction of these interpolants into mortar conditions, for Adini and Morley mortar methods, is the fact that the respective traces of local functions depend also on the values of degrees of freedom at interior nodal points.

We have for the cases of conforming local discretizations that there are degrees of freedom at each vertex which are associated with some derivatives of the first or even second order. Because we do not assume continuity of partial derivatives at the vertices of the substructures, the mortar element functions will not, generally, be C^1 continuous over Ω and therefore they will not be in the space $H^2(\Omega)$, even if the meshes match across the interface between two adjacent subdomains.

We propose four parallel methods for solving some of discrete problems. These algorithms are described as additive Schwarz methods (ASM), see Section 1.4, (cf. also [20], [64] or [60]). All these methods, except one, are of iterative substructuring type. They are applied to the Schur complement of respective discrete problems, i.e. interior variables are first eliminated using some direct methods.

There are many iterative methods for mortar finite elements for second order elliptic problems, e.g. see [1], [2], [4], [3], [5], [43], [44], [58], [57], [83], [80]. For iterative substructuring methods for plate problems with globally conforming or nonconforming discretizations defined on one global triangulation of Ω, we refer e.g. to [39], [82].

The first two methods, described in Sections 4.5.1 and 4.5.2, are designed for mortar methods with local HCT or reduced HCT discretizations and are based on analogous decomposition of the discrete space. The first method is of iterative substructuring type and the second one is not. In the definition of both methods, we represent a discrete space as a sum which consists of a coarse space, local one dimensional spaces associated with degrees of freedom of order one at vertices of subdomains and of certain local spaces associated with interfaces. The difference lies in the fact that the first method is of iterative substructuring type, but the second one is not. Additionally, the second method uses a nonstandard outer coarse grid. Therefore, we have to introduce a special interpolation operator which maps the coarse grid onto the mortar discrete space.

The next method is of Neumann-Neumann type, cf. [82], [64]. The origin of the Neumann-Neumann algorithms can be traced back to the work of Dihn, Glowinski and Périaux [54]. These algorithms have been developed further by a number of French scientists in particular Bourgat, Le Tallec and Vidrascu [23], De Roeck and Le Tallec

[53] and Le Tallec, De Roeck and Vidrascu [81].

Our Neumann-Neumann method is designed for mortar methods built on the decomposition which satisfies one additional condition: we assume that it is possible to choose master edges of substructures in such way that each subdomain has all its edges either as masters or as slaves. This assumption is due to the property of functions in discrete spaces built by the mortar method. Namely, if we have the common interface of two subdomains with master $\gamma_{m,k} \subset \partial\Omega_k$ and slave $\delta_{m,l} \subset \partial\Omega_l$, then some norms over this interface of trace $Tr\ u_{l|\delta_{m,l}}$ can be bounded by the respective norms of $Tr\ u_{k|\gamma_{m,k}}$, but not vice versa.

This Neumann-Neumann method is based on the modified abstract scheme of Le Tallec, Mandel and Vidrascu [82] and can be applied for mortar methods with all conforming local discretizations.

The last method, presented in Section 4.5.4, is the adaptation of the first method to the case of the mortar method with locally nonconforming Adini discretizations. We distinguish this case because the analysis requires special coarse grid and technical tools necessary to overcome some technical difficulties which are due to the local nonconformity of the solution.

All methods presented in this chapter are almost optimal, i.e. the number of iterations required to decrease the energy norm of the error by a conjugate gradient method is proportional in each case to $(1+\log(\frac{H}{\underline{h}}))$. Here H and h_i are the parameters of the coarse decomposition and the fine triangulation on Ω_i, respectively, and $\underline{h} = \inf_i h_i$.

The outline of this chapter is as follows. In Section 4.2, we formulate the differential problem and discuss the mortar element methods for different locally conforming and nonconforming discretizations. We also consider the problems of the existence and uniqueness of the arising discrete problems. In Section 4.3, we prove the ellipticity of discrete bilinear forms over mortar discrete spaces and in Section 4.4, we state and prove the error estimates of the mortar elements methods introduced in Section 4.2. Finally, Section 4.5 is devoted to parallel algorithms of solving discrete problems.

4.2 Discrete problem

4.2.1 Clamped plate problem

Let Ω be a polygonal domain in $\Re^2$. The differential problem is to find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = \int_\Omega fv \, dx \quad \forall v \in H_0^2(\Omega), \tag{4.1}$$

where u^* is the displacement, $f \in L^2(\Omega)$ is the body force,

$$a(u, v) = \int_\Omega \left[\triangle u \triangle v + (1-\nu)\left(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1}\right)\right] dx.$$

Here

$$H_0^2(\Omega) = \{v \in H^2(\Omega) : \; v = \partial_n v = 0 \;\; on \;\; \partial\Omega\},$$

∂_n is the normal unit derivative outward to $\partial\Omega$, and $u_{x_ix_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i, j = 1, 2$. The Poisson ratio ν satisfies $0 < \nu < 1/2$. The Lax-Milgram theorem, utilizing the continuity and ellipticity of the bilinear form $a(\cdot,\cdot)$ yields the existence and the uniqueness of the solution, see e.g. [38] or [47].

Assumptions: Let Ω be a union of non-overlapping polygonal subdomains that are arbitrary for the Morley, reduced HCT and HCT elements and rectangles for the bicubic and Adini elements, i.e.

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k \;\; with \;\; \Omega_k \cap \Omega_l = \emptyset, \quad k \neq l.$$

We assume that the intersection of boundaries of two different subdomains $\partial\Omega_k \cap \partial\Omega_l, k \neq l$, is either the empty set, a vertex or a common edge. Thus $\{\Omega_k\}$ forms a decomposition of Ω that we call the coarse triangulation with a parameter $H = \max_k H_k$.

We assume the shape regularity of that decomposition, cf. Section 2.3.

We triangulate each subdomain Ω_k into nonoverlapping rectangles for bicubic and Adini elements and into triangles for Morley, reduced HCT and HCT methods. The rectangles (or triangles) of this triangulation are denoted by τ_i and called elements. We assume that the arising fine triangulation $T_h(\Omega_k)$ is quasiuniform with parameter $h_k = \max(\text{ diam } \tau)$ for $\tau \in T_h(\Omega_k)$, cf. [38]. We also introduce additional notation. Let the set of all vertices of elements of the triangulation of Ω_k, $\overline{\Omega}_k$ $\partial\Omega_k$ and $\mathcal{E}$ be denoted by $\Omega_{k,h}$, $\overline{\Omega}_{k,h}$, $\partial\Omega_{k,h}$ and $\mathcal{E}_h$, respectively. Here $\mathcal{E}$ is an edge of a subdomain.

4.2.2 Bicubic element

In this subsection, we present a mortar method for plate problem with locally bicubic element, known as the Bogner-Fox-Schmit rectangle, see [22] and Figure 4.1.

The local finite element space $X_h^B(\Omega_k) \subset H_C^2(\Omega_k)$ is defined by

$$X_h^B(\Omega_k) = \{v \in C^1(\Omega_k) : \ v_{|\tau_i} \in Q_3(\tau_i) \ \ and \ \ v = \partial_n v = 0 \ \ on \ \ \partial\Omega_k \cap \partial\Omega\},$$

where $H_C^2(\Omega_k) = \{v \in H^2(\Omega_k) : \ v = \partial_n v = 0 \ \ on \ \ \partial\Omega_k \cap \partial\Omega\}$.

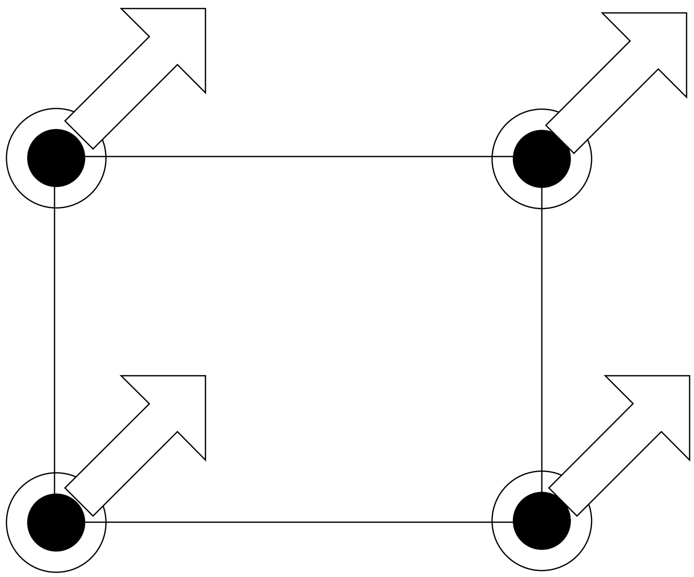

Figure 4.1: Bicubic element.

The degrees of freedom of the bicubic element are given by

$$\{v(p), v_{x_1}(p), v_{x_2}(p), v_{x_1x_2}(p)\},$$

where p is a vertex of an rectangular element, see Figure 4.1, and a bullet means the value, circle the gradient and an oblique arrow the mixed derivative of a finite element function.

To define a mortar finite element method, we introduce some notations and spaces. Define a global space

$$X_h^B(\Omega) = X_h^B(\Omega_1) \times \ldots \times X_h^B(\Omega_N) \subset \prod_{k=1}^{N} H_C^2(\Omega_k)$$

and

$$\Gamma = \bigcup_k \partial\Omega_k \setminus \partial\Omega.$$

Let Γ_{kl} denote a common edge to Ω_k and Ω_l. Each interface $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ has two sides corresponding to Ω_k and to Ω_l.

We now select an open side of Γ_{kl} and name it as master (mortar) and denote by $\gamma_{m,k}$ if it is a side of $\partial\Omega_k$. The side of Γ_{kl} belonging to $\partial\Omega_l$ is the slave (nonmortar) and is denoted by $\delta_{m,l}$. Thus we have $\overline{\Gamma} = \bigcup_{\gamma_{m,i}\subset\Gamma} \overline{\gamma}_{m,i}$. Let $W^{h_k}(\Gamma_{kl})$ be the subspace of C^1 continuous functions that are piecewise cubic on the 1-D h_k-triangulation of Γ_{kl} inherited from the 2-D h_k-triangulation of Ω_k. Note that on $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l}$ there are two 1-D triangulations and two different spaces $W^{h_k}(\gamma_{m,k})$ and $W^{h_l}(\delta_{m,l})$.

Let $M^{h_l}_{1,3}(\delta_{m,l})$ denote the subspace of $W^{h_l}(\delta_{m,l})$ formed by functions which are piecewise linear on two segments that touch the ends of Γ_{kl}, i.e.

$$M^{h_l}_{1,3}(\delta_{m,l}) = \{v \in C^1(\delta_{m,l}) : v_{|e} \in P_3(e) \;\; for \;\; \overline{e} \cap \partial\Gamma_{kl} = \emptyset \tag{4.2}$$

$$and \;\; v_{|e} \in P_1(e) \;\; for \;\; \overline{e} \cap \partial\Gamma_{kl} \neq \emptyset\},$$

where e is an edge of an element of $T_h(\Omega_l)$ belonging to $\delta_{m,l}$. Note that, actually, for interfaces with one end touching $\partial\Omega$, we have $M^{h_l}_{1,3}(\delta_{m,l}) \not\subset W^{h_l}(\delta_{m,l})$ since for any function $v \in W^{h_l}(\delta_{m,l})$, we have $v(p) = v'(p) = 0$ for $p = \partial\Omega \cap \delta_{m,l}$. We say that $u_k \in X^B_h(\Omega_k)$ and $u_l \in X^B_h(\Omega_l)$ on $\partial\Omega_k \cap \partial\Omega_l = \Gamma_{kl}$ satisfy the mortar conditions if

$$\int_{\delta_m} (u_k - u_l)_{|\Gamma_{kl}} \psi \, ds = 0 \quad \forall \psi \in M^{h_l}_{1,3}(\delta_{m,l}) \tag{4.3}$$

and

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)_{|\Gamma_{kl}} \psi \, ds = 0 \quad \forall \psi \in M^{h_l}_{1,3}(\delta_{m,l}). \tag{4.4}$$

We now define the discrete space V^B_h as the subspace of $X^B_h(\Omega)$ formed by functions which satisfy the mortar conditions (4.3) and (4.4) and are continuous at all crosspoints, where a crosspoint $c_r \in \Gamma$ is a common point of some substructures.

Remark 4.2.1 *For each interface Γ_{kl} with the master (mortar) $\gamma_{m,k}$ and slave $\delta_{m,l}$ and any $u \in V^B_h$, the trace $Tr_{|\delta_m} u_l$ is determined by $Tr_{|\gamma_m} u_k$ and the values of degrees of freedom of u_l at ends (which are also vertices of substructure Ω_l) of this slave $\delta_{m,l}$. Note that by the assumption, $u_k = u_l$ at the ends of Γ_{kl}. Here $Tr_{|C_k} v_k = (v_{|C_k}, \nabla v_{|C_k})$ and C_k is $\partial\Omega_k$ or Γ_{kj}, an edge of substructure Ω_k. $Tr_{|C_l} v_l$ is defined in a similar way.*

The discretization of (4.1) using V_h^B is of the form:
Find $u_h^B \in V_h^B$ such that

$$a_H(u_h^B, v) = \int_\Omega fv\,dx \quad \forall v \in V_h^B, \tag{4.5}$$

where $a_H(u,v) = \sum_{k=1}^N a_{h,k}(u,v)$ and

$$a_{h,k}(u,v) = \sum_{\tau \in T_h(\Omega_k)} \int_\tau [\triangle u \triangle v + (1-\nu)(2u_{x_1x_2}v_{x_1x_2}$$
$$-u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1})]\,dx.$$

By calculations, cf. e.g. (5.9.2), p.143 in [38], we get

$$|a_H(u,v)| \le (1+\nu)|u|_{H_H^2(\Omega)}|v|_{H_H^2(\Omega)} \quad \forall u,v \in X_h^B(\Omega) \tag{4.6}$$

and

$$a_H(u,u) \ge (1-\nu)|u|^2_{H_H^2(\Omega)} \quad \forall u \in X_h^B(\Omega), \tag{4.7}$$

where

$$|v|^2_{H_H^2(\Omega)} = \sum_{k=1}^N |v|^2_{H_h^2(\Omega_k)} = \sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} |v|^2_{H^2(\tau)}$$

and

$$|v|^2_{H^2(\tau)} = \int_\tau (v^2_{x_1x_1} + 2v^2_{x_1x_2} + v^2_{x_2x_2})\,dx.$$

The form $a_H(\cdot,\cdot)$ is positive definite over V_h^B since $a_H(u,u) = 0$ implies that u is linear in all Ω_k and from the mortar condition follows that u linear in Ω. Then the boundary conditions yield $u = 0$. Thus we have

Proposition 4.2.1 *The problem (4.5) has a unique solution.*

The nodal basis

Here we introduce a nodal basis, and present a matrix form of the discrete problem.

We divide sets of nodes into the following three groups:

- all nodes interior to the substructures,

- all nodes interior to the masters,
- all nodes of vertices of subregions except those on $\partial\Omega$.

We associate a basis function with each respective degree of freedom (nodal values) at each node of these sets, i.e. if $x \in \overline{\Omega}_{k,h}$ is a nodal point in one of these three sets and α is an admissible multi-index, (here for bicubic element the set of admissible multi-indices is the following one $\{(0,0),(1,0),(0,1),(1,1)\}$), then we define a basis function $\phi^{\alpha}_{k,x}$ as follows:

$$\begin{array}{l} \partial^{\beta}\phi^{\alpha}_{k,x}(y) = 1 \quad for \quad \beta = \alpha \quad and \quad y = x, \\ \partial^{\beta}\phi^{\alpha}_{k,x}(y) = 0 \quad for \quad \beta \neq \alpha \quad\ or \quad\ y \neq x, \end{array}$$

where y is an arbitrary nodal point of one of these three sets.

Note that $\phi^{\alpha}_{k,x}$ is properly defined, i.e. the values of degrees of freedom of $\phi^{\alpha}_{k,x}$ at all nodal points are uniquely determined: at nodal points which are vertices or are interior to subdomains as zero or one (the value of α degree of freedom at x) and at nodal points on a slave (nonmortar) δ_r by the values of degrees of freedom at ends of this slave and at nodal points of its associated master γ_r by the mortar conditions (4.3) and (4.4), cf. Remark 4.2.1.

We now describe these basis functions in a more detailed way.

The functions corresponding to degrees of freedom at nodes in the interiors of the substructures are standard nodal basis functions as in the conforming finite element discretization of plate problems, cf. e.g. p.77 in [47].

A basis nodal function associated with an α degree of freedom and a node x interior to the master $\gamma_{m,i}$, can be defined as follows. The value of its α degree of freedom at x is one, the values of remaining degrees of freedom at this node are zero and the values of all admissible degrees of freedom at the remaining nodes defined above are also zero. The values of respective degrees of freedom of this basis function at the interior nodes of slave $\delta_{m,j} = \gamma_{m,i}$ are determined by the mortar conditions (4.3) and (4.4), with zero values of respective degrees of freedom at the ends of $\delta_{m,j}$.

We now define basis functions associated with the degrees of freedom at vertices of the substructures. We first define basis functions that correspond to multi-indices of length greater than zero. Let c_r be a crosspoint, i.e. c_r is a common vertex of four substructures, i.e. $c_r \in \bigcap_{k=1}^{4} \partial\Omega_{i_k}$. The set of indices of substructures $\{\Omega_{i_k}\}_{k=1,2,3,4.}$ we denote by $\mathcal{N}(c_r)$. Note that ∇v and v_{xy} for $v \in V_h^B$ can be discontinuous at c_r (only the value of a function is continuous at a crosspoint by the definition of V_h^B).

Thus we can distinguish between the vertices of all subdomains Ω_k for $k \in \mathcal{N}(c_r)$ despite the fact that these vertices occupy the same geometrical position of c_r. The set of these vertices we denote by $\mathcal{V}(c_r)$. We also denote by $\mathcal{V}(\Omega_k) \subset \Gamma$ the set of vertices of $\Omega_k \cap \Gamma$ and introduce $\mathcal{V} := \bigcup_{k=1}^{N} \mathcal{V}(\Omega_k)$.

Let α be a multi-index of length greater than zero, i.e. $|\alpha| > 0$ and $v_i \in \mathcal{V}(c_r)$ be a vertex of Ω_i. Then ϕ^{α}_{i,v_i}, a basis function associated with α and $v_i \in \mathcal{V}(c_r)$, has $\partial^{\alpha}\phi^{\alpha}_{i,v_i}(v_i)$ equal to one. All remaining degrees of freedom are zero at v_i, at all other vertices of $\mathcal{V}$, and at all interior nodes of all substructures. We now define ϕ^{α}_{i,v_i} on Γ, i.e. on all masters and slaves. There are three possible situations: the vertex v_i can be the common end of two masters $\gamma_{n,i}$ and $\gamma_{m,i}$, the common end of two slaves $\delta_{l,i}$ and $\delta_{k,i}$, or the common end of a slave $\delta_{s,i}$ and a master $\gamma_{p,i}$.

In the first case, $Tr_{|\gamma_{n,i}}\phi^{\alpha}_{i,v_i}$ and $Tr_{|\gamma_{m,i}}\phi^{\alpha}_{i,v_i}$ are the traces (in the sense of $Tru = (u, \nabla u)$) of a standard nodal function corresponding to the multi-index α and to v_i, i.e. the one with α degree of freedom equal to one at v_i and all other admissible degrees of freedom at v_i and all degrees of freedom at the remaining nodes of the both masters equal to zero. On slaves $\delta_n = \gamma_{n,i}$ and $\delta_m = \gamma_{m,i}$, the traces (in the sense of the triple Tr) of this function are determined by the mortar conditions (4.3) and (4.4) with all degrees of freedom equal to zero at the ends of δ_n and δ_m, respectively.

In the second case, the traces $Tr_{|\gamma_l}\phi^{\alpha}_{i,v_i}$ and $Tr_{|\gamma_k}\phi^{\alpha}_{i,v_i}$ are zero and the traces on $\delta_{l,i}$ and $\delta_{k,i}$ are determined by the mortar conditions with the α degree of freedom equal to one at v_i and all remaining degrees of freedom at v_i and all ones at the other ends of $\delta_{l,i}$ and $\delta_{k,i}$ equal to zero.

In the last case, $Tr_{|\gamma_{p,i}}\phi^{\alpha}_{i,v_i}$ is equal to the trace of the respective standard nodal function and on slave $\delta_p = \gamma_p$ is defined as in the first case while on the slave $\delta_{s,i}$ (and $\gamma_s = \delta_{s,i}$) is defined analogously to the second case. In all cases, $Tr\,\phi^{\alpha}_{i,v_i}$ is zero on the remaining masters and slaves.

As functions in V_h^B are continuous at crosspoints, we have to consider only one basis function that corresponds to the multi-index $\alpha = (0,0)$ and a crosspoint c_r which is a common vertex of four subdomains. We denote this function by $\phi^{(0,0)}_{c_r}$. On each master $\gamma_{m,i}$ which have c_r as one end, $Tr\ \phi^{(0,0)}_{c_r}$ is equal to the trace of standard nodal function which corresponds to the h_i triangulation of this master and to the multi-index $(0,0)$ which denotes a degree of freedom at c_r. On each slave, its trace is determined by the trace on respective master, mortar conditions and the values of admissible degrees of freedom at ends of this slave in the same manner as above.

It is obvious that all those functions form a basis of the space V_h^B, i.e.

$$V_h^B = \sum_{k=1}^{N} \sum_x \sum_\alpha \operatorname{span}\{\phi_{k,x}^\alpha\},$$

where the sums are taken over all subdomains Ω_k, all respective nodal points of those three sets corresponding to Ω_k, and admissible multi-indices.

Let the solution of (4.5) be represented as $u_h^B = \sum_{k=1}^N \sum_x \sum_\alpha u_{h,k,x}^{B,\alpha} \phi_{k,x}^\alpha$ with $u_{h,k,x}^{B,\alpha} = \partial^\alpha u_{h,k}^B(x)$. We next introduce a symmetric, positive definite matrix A and a vector $\mathbf{f}$ by

$$A = \{a_{k,x,l,y}^{\alpha,\beta}\}_{l,y,\beta}^{k,x,\alpha} \quad \textit{and} \quad \mathbf{f} = \{f_{l,y}^\beta\}_{l,y,\beta},$$

where $a_{k,x,l,y}^{\alpha,\beta} = a_H(\phi_{k,x}^\alpha, \phi_{l,y}^\beta)$ and $f_{l,y}^\beta = (f, \phi_{l,y}^\beta)_{L^2(\Omega)}$. Here and below, if u is a function in V_h^B, then $\mathbf{u}$ denotes the vector representation of u in terms of the nodal basis, i.e. if $u = \sum_{k,x,\alpha} u_{k,x}^\alpha \phi_{k,x}^\alpha$, then $\mathbf{u} = \{u_{k,x}^\alpha\}$.

Utilizing this notations, we can rewrite the problem (4.5) as the following system of linear equations

$$A\mathbf{u}_h^B = \mathbf{f}. \tag{4.8}$$

Utilizing Corollary 4.3.1, see below, and an inverse inequality, we can obtain a bound of the condition number of the matrix A, i.e.

Proposition 4.2.2 *If we assume that $h_i \asymp h_j$, for an interface $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$, then*

$$\operatorname{cond}(A) \preceq \underline{h}^{-4},$$

where $\underline{h} = \inf_k h_k$.

4.2.3 Adini element

In this subsection, we introduce a mortar method that locally uses the Adini element, cf. [7] and Chapter 7, Section 49, p.298 in [48]. The local finite element space $X_h^A(\Omega_k)$ of Adini element is defined by

$$X_h^A(\Omega_k) = \{v \in L_2(\Omega_k) : \ v_{|\tau} \in P_3(\tau) \oplus \operatorname{span}\{x_1^3 x_2, x_1 x_2^3\} \ \textit{for} \ \tau \in T_h(\Omega_k),$$

$$v, v_{x_1}, v_{x_2} \ \textit{continuous at the vertices and}$$

$$v(a) = v_{x_1}(a) = v_{x_2}(a) = 0 \ \textit{for a vertex} \ a \in \partial\Omega_k \cap \partial\Omega\},$$

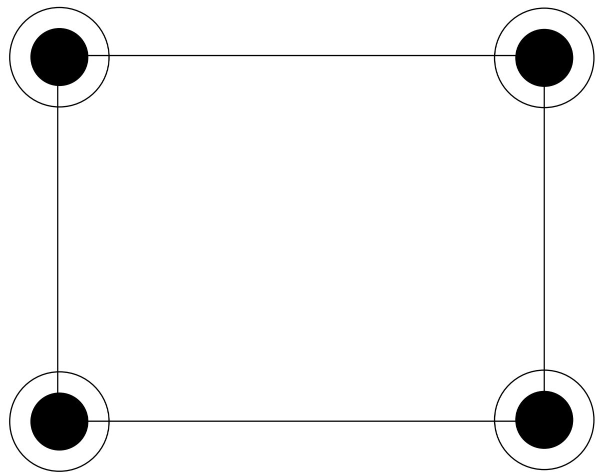

Figure 4.2: Adini element.

where $\tau \in T_h(\Omega_k)$ is an rectangular element, cf. [7] and Figure 4.2.

The degrees of freedom of the Adini element are given by

$$\{v(p), v_{x_1}(p), v_{x_2}(p)\},$$

where p is a vertex of an element, cf. Figure 4.2.

As in the previous subsection, we introduce the global space $X_h^A(\Omega)$ with the same local bilinear forms $a_{h,k}(\cdot,\cdot)$ and the same global form $a_H(\cdot,\cdot)$. For each interface $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$, we choose one side as a master denoted by $\gamma_{m,k} \subset \partial\Omega_k$ and the second one as a slave $\delta_{m,l} \subset \partial\Omega_l$ if $h_k \leq h_l$. This assumption is necessary for the proofs of some technical results and is due to the fact that any local finite element function is not sufficiently regular, cf. Section 4.4.2 and Section 4.5.5.

We introduce additional auxiliary spaces on each slave (nonmortar) δ_m. Let the first one denoted by $M_{1,3}^{h_l}(\delta_{m,l})$ be the space introduced in the previous subsection, i.e. C^1 smooth functions that are piecewise cubic except for two elements, that touch the ends of slave, where are piecewise linear, see (4.2). We now define a class of spaces for all positive integer, cf. [9]. In this subsection, we need only the space with index $s = 1$, but later we will also need the one with index $s = 2$. Let the space denoted by $M_{0,s}^{h_l}(\delta_{m,l})$ be the space formed by continuous functions that are polynomials of degree s over each element of the h_l triangulation of $\delta_{m,l}$ except of two elements which touch the ends of this slave and on these two elements functions from this space are polynomials of order $s-1$. Thus $M_{0,1}^{h_l}(\delta_{m,l})$ is the space of continuous piecewise linear functions which are constant on the two elements which touch the ends of the slave $\delta_{m,j}$.

We say that $u_k \in X_h^A(\Omega_k)$ and $u_l \in X_h^A(\Omega_l)$ for $\partial\Omega_l \cap \partial\Omega_k = \Gamma_{kl}$, satisfy the mortar conditions if

$$\int_{\delta_m} (u_k - u_l)\psi \, ds = 0 \ \ \forall \psi \in M_{1,3}^{h_l}(\delta_{m,l}), \tag{4.9}$$

$$\int_{\delta_m} (I_{h_k}\partial_n u_k - I_{h_l}\partial_n u_l)\psi \, ds = 0 \ \ \forall \psi \in M_{0,1}^{h_l}(\delta_{m,l}), \tag{4.10}$$

where I_{h_l}, I_{h_k} are the standard piecewise linear interpolants onto the h_l and h_k meshes of $\delta_{m,l}$ and $\gamma_{m,k}$, respectively. Note that $I_{h_i}\partial_n u_i$, for $i = k, l$, equals the trace of the piecewise bilinear interpolant defined over Ω_i by the values of $\partial_n u_i$ at the vertices of rectangular elements of $T_h(\Omega_i)$.

We now define the discrete space V_h^A as the subspace of $X_h^A(\Omega)$ formed by functions which satisfy the mortar conditions (4.9) and (4.10) and are continuous at all crosspoints.

The discretization of (4.1) using V_h^A is of the form:
Find $u_h^A \in V_h^A$ such that

$$a_H(u_h^A, v) = \int_\Omega fv \, dx \quad \forall v \in V_h^A. \tag{4.11}$$

The form $a_H(\cdot,\cdot)$ is positive definite over V_h^A what follows from the fact that $a_H(u,u) = 0$ implies that u is linear in all rectangles of Ω_k, then from the continuity of u, u_{x_1}, u_{x_2} at all vertices of $\Omega_{k,h}$ follows that u linear in Ω_k and from the mortar condition follows that u linear in Ω. Then the boundary conditions yield $u = 0$. Hence

Proposition 4.2.3 *The problem (4.11) has a unique solution.*

As in Section 4.2.2, we can introduce a nodal basis, i.e. with each degree of freedom at all nodal points which are not in $\partial\Omega$ or in any slave δ_m, we associate one nodal basis function. Then we can rewrite the discrete problem (4.11) into a system of linear equations. If we additionally assume that $h_i \asymp h_j$, for an interface $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$, we can see that the condition number of the resulting matrix is bounded by $C\underline{h}^{-4}$, where $\underline{h} = \inf_k h_k$ and C is a positive constant independent of any h_k and the number of subdomains.

4.2.4 HCT and reduced HCT methods

In this subsection, we introduce two mortar methods the first one that locally uses the reduced Hsieh-Clough-Tocher (HCT) macro element, cf. Chapter 7, Section 46, p.285 in [48], and the second one that uses the Hsieh-Clough-Tocher (HCT) macro element, cf. [49] and Chapter 7, Section 46, p.279 in [48]. The local finite element space $X_h^H(\Omega_k)$ for HCT element is defined by, cf. Figure 4.3,

$$X_h^H(\Omega_k) = \{v \in C^1(\Omega_k) : \;\; v_{|\tau} \in P_3(\tau_i), \;\; for \;\; triangles \;\; \tau_i, \;\; i = 1, 2, 3,$$
$$formed \;\; by \;\; connecting \;\; the \;\; vertices \;\; of \;\; \tau \in T_h(\Omega_k)$$
$$to \;\; its \;\; centroid, \;\; v = \partial_n v = 0 \;\; on \;\; \partial\Omega_k \cap \partial\Omega\}.$$

The local finite element space of reduced HCT element denoted by $X_h^{RH}(\Omega_k)$ is a subspace of $X_h^H(\Omega_k)$ defined by

$$X_h^{RH}(\Omega_k) = \{v \in X_h^H(\Omega_k) : \partial_n v_{|e} \in P_1(e)$$
$$for \;\; any \;\; side \;\; e \;\; of \;\; a \;\; triangle \;\; \tau \in T_h(\Omega_k)\},$$

cf. [49], [48] and see Figure 4.3.

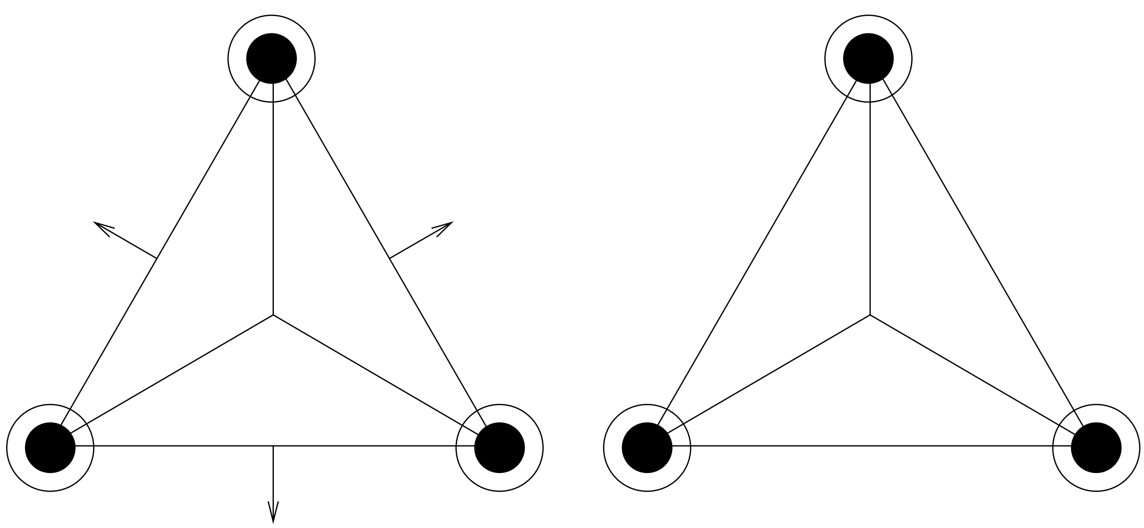

Figure 4.3: HCT and reduced HCT macro elements.

The degrees of freedom of the reduced HCT element are given by

$$\{v(p), v_{x_1}(p), v_{x_2}(p)\}$$

and of HCT element

$$\{v(p), v_{x_1}(p), v_{x_2}(p), \partial_n v(m)\},$$

where p is a vertex of an element and m the midpoint of an edge of an element, cf. Figure 4.3

As in the previous subsection, we introduce the global spaces $X_h^{RH}(\Omega)$ and $X_h^H(\Omega)$ with the same local bilinear forms $a_{h,k}(\cdot,\cdot)$ and the same global form $a_H(\cdot,\cdot)$. For each interface $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$, we choose one side as a master (mortar) denoted by $\gamma_{m,k} \subset \partial\Omega_k$ and the second one as a slave $\delta_{m,l} \subset \partial\Omega_l$. Here the choice of the master side is arbitrary. The test spaces $M_{1,3}^{h_l}(\delta_{m,l})$, $M_{0,1}^{h_l}(\delta_{m,l})$ and $M_{0,2}^{h_l}(\delta_{m,l})$ are the auxiliary spaces defined over a slave $\delta_{m,l}$ introduced in the previous subsection, i.e. e.g. $M_{0,2}^{h_l}(\delta_{m,l})$ is the space of continuous piecewise quadratic functions which are linear on the two elements which touch the ends of the slave $\delta_{m,j}$.

We say that $u_k \in X_h^{RH}(\Omega_k)$ and $u_l \in X_h^{RH}(\Omega_l)$ ($u_k \in X_h^H(\Omega_k)$ and $u_l \in X_h^H(\Omega_l)$) for $\partial\Omega_l \cap \partial\Omega_k = \Gamma_{kl}$, satisfy the mortar conditions if

$$\int_{\delta_m} (u_k - u_l)\psi \, ds = 0 \quad \forall \psi \in M_{1,3}^{h_l}(\delta_{m,l}) \tag{4.12}$$

for both reduced HCT and HCT methods, and

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)\psi \, ds = 0 \quad \forall \psi \in M_{0,1}^{h_l}(\delta_{m,l}) \tag{4.13}$$

in the case of reduced HCT method, and

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)\psi \, ds = 0 \quad \forall \psi \in M_{0,2}^{h_l}(\delta_{m,l}) \tag{4.14}$$

in the case of HCT method.

We now define the discrete space V_h^{RH} as the subspace of $X_h^{RH}(\Omega)$ formed by functions which satisfy the mortar conditions (4.12) and (4.13) and are continuous at all crosspoints, and analogously the discrete space V_h^H as the subspace of $X_h^H(\Omega)$ formed by functions which satisfy the mortar conditions (4.12) and (4.14) and are continuous at all crosspoints.

The discretization of (4.1) using V_h^{RH} (or V_h^H) is of the form
Find $u_h^{RH} \in V_h^{RH}$ and $u_h^H \in V_h^H$ such that

$$a_H(u_h^{RH}, v) = \int_\Omega f v \, dx \quad \forall v \in V_h^{RH}, \tag{4.15}$$

$$a_H(u_h^H, v) = \int_\Omega f v \, dx \quad \forall v \in V_h^H. \tag{4.16}$$

The form $a_H(\cdot,\cdot)$ is positive definite over V_h^{RH} and V_h^H what follows from the fact that $a_H(u,u) = 0$ implies that u is linear in all subdomains Ω_k and from the mortar conditions follows that u linear in Ω. Then the boundary conditions yield $u = 0$.

Proposition 4.2.4 *The problems (4.15) and (4.16) have unique solutions.*

As in Section 4.2.2, we can introduce nodal bases for the methods, i.e. in each case with each degree of freedom at each nodal point which is not in $\partial\Omega$ and a slave δ_m, we associate one nodal basis function. Then we can rewrite the discrete problems (4.15) and (4.16) as systems of linear algebraic equations. If we additionally assume that $h_i \asymp h_j$, for an interface $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$, we can get that the condition numbers of the resulting matrices are bounded by $C\underline{h}^{-4}$, where $\underline{h} = \inf_k h_k$ and C is a positive constant independent of any h_k and the number of subdomains.

4.2.5 Morley element

In this subsection, we introduce a mortar method that locally uses the Morley element, cf. [90] or [79]. The local finite element space $X_h^M(\Omega_k)$ is defined by, see Figure 4.4,

$$X_h^M(\Omega_k) = \{v \in L_2(\Omega_k) : \ v_{|\tau} \in P_2(\tau), \ v \ continuous \ at \ vertices \ of \ \tau \in T_h(\Omega_k)$$

$$and \ \partial_n v \ continuous \ at \ midpoints \ of \ edges \ \ of \ \tau \ and$$

$$v(p) = \partial_n v(m) = 0 \ for \ a \ vertex \ p \in \partial\Omega \ and \ a \ midpoint \ m \in \partial\Omega\}.$$

The degrees of freedom of the Morley element are given by

$$\{v(p), \partial_n v(m)\},$$

where p is a vertex of an element and m is the midpoint of an edge of an element, cf. Figure 4.4.

We also define a global space $X_h^M(\Omega) = \prod_{k=1}^N X_h^M(\Omega_k)$ as in the previous subsections.

We now select an open disjoint side of $\Gamma_{kl} \subset \Gamma \cap \partial\Omega_k$ denoted by $\gamma_{m,k}$ and name it master (mortar) if $h_k \leq h_l$. This assumption is necessary for the proofs of some

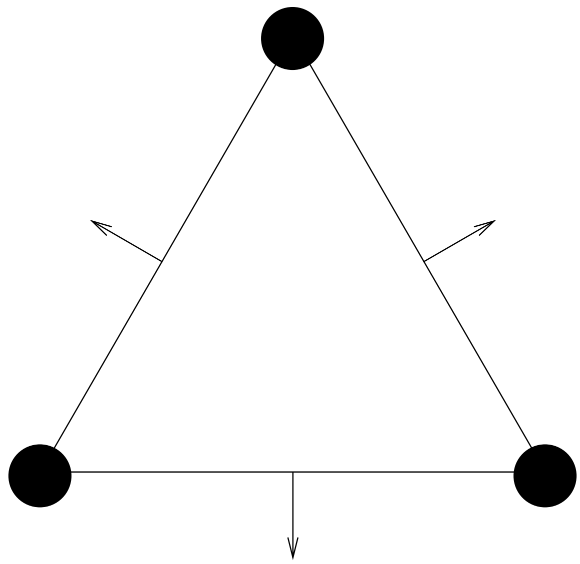

Figure 4.4: Morley element.

technical results and is due to the fact that any local finite element function is not sufficiently regular, cf. Section 4.2.3 and Section 4.4.4. The side of $\Gamma_{kl} \subset \partial\Omega_l$ is called slave (nonmortar) and is denoted by $\delta_{m,l}$. As $h_k \leq h_l$ and the both triangulations are quasiuniform, we can assume that the two side elements of the slave triangulation $T_h^l(\delta_{m,l})$, i.e. the ones that touch the ends of $\delta_{m,j}$, are longer than the respective elements of the master (mortar) triangulation $T_h^k(\gamma_{m,k})$.

We introduce additionally two auxiliary spaces on each slave (nonmortar) $\delta_{m,l}$. Let the first one denoted by $M_{-1,0}^{h_l}(\delta_{m,l})$, be the space of functions which are piecewise constant on the h_l-triangulation of this slave.

For the simplicity of presentation, we also assume that the both 1-D triangulations of the interface Γ_{kl}, the h_k one of its master $\gamma_{m,k}$ and the h_l one of its slave $\delta_{m,l}$, have even numbers of the elements. Let consider $\delta_{m,l}$ and let $\overline{\delta}_{m,l,h} = \{p_0, p_1, \ldots, p_{N_{m,l}}\}$ be a set of vertices of the h_l triangulation of this slave, ($N_{m,l}$ is even). Then we introduce an operator $I_{2h_l,2} : C^0(\Gamma_{kl}) \to L^2(\Gamma_{kl})$ as follows

Definition 4.2.1 *Let $I_{2h_l,2} : C^0(\Gamma_{kl}) \to L^2(\Gamma_{kl})$ and $I_{2h_l,2}u$ be defined by the values of u at all points of $\overline{\delta}_{m,l,h}$ as follows:*

- *$I_{2h_l,2}u \in P_2$ on each $[p_i, p_{i+2}]$ for even i,*

- $I_{2h_l,2}u(p_i) = u(p_i) \;\; p_i \in \overline{\delta}_{m,l,h}$.

In other words, $I_{2h_l,2}u$ is the piecewise quadratic interpolant defined over the $2h_l$ triangulation of $\delta_{m,l}$ that is made of elements $[p_i, p_{i+2}], i = 0, 2, \ldots, N_{m,l} - 2$. The operator $I_{2h_k,2}$ that corresponds to the h_k mesh of master $\overline{\gamma}_{m,k,h}$ we define in the same way. We next define an auxiliary space $M_{0,2}^{2h_l}(\delta_{m,l})$ as follows

$$M_{0,2}^{2h_l}(\delta_{m,l}) = \{v \in C^0(\delta_{m,l}) : \; v \in P_2([p_i, p_{i+2}]) \; for \; even \; i \neq 0, N_{m,l} - 2, \qquad (4.17)$$
$$and \; v \in \; P_1([p_i, p_{i+2}]) \; for \; i = 0, N_{m,l} - 2\}.$$

This space is defined in the same way as $M_{0,2}^h$ in Section 4.2.3, but over the $2h_l$ triangulation of $\delta_{m,l}$ which is made of segments $[p_i, p_{i+2}], i = 0, 2, \ldots, N_{m,l} - 2$. We now introduce two following mortar conditions on the interface $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l}$:

$$\int_{\delta_m} (I_{2h_k,2}u_k - I_{2h_l,2}u_l)\psi \, ds = 0 \;\; \forall \psi \in M_{0,2}^{2h_l}(\delta_{m,l}) \qquad (4.18)$$

and

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)\phi \, ds = 0 \;\; \forall \phi \in M_{-1,0}^{h_l}(\delta_{m,l}). \qquad (4.19)$$

We now define the discrete space V_h^M as the subspace of $X_h^M(\Omega)$ formed by functions which satisfy the mortar conditions (4.18) and (4.19), and are continuous at all crosspoints.

Remark 4.2.2 *The special interpolation operators $I_{2h_l,2}$ and $I_{2h_k,2}$ are introduced in the mortar condition (4.18) because of the non-conformity of Morley FE local spaces. If we consider an element $\tau \in T_h(\Omega_k)$ such that one of its edges e is contained in one of interfaces of Ω_k, i.e. $\partial\tau \cap \Gamma_{kl} = e$, then $Tr\, u_{k|e}$ is determined not only by the nodal points that belong to Γ_{kl}, but also by degrees of freedom at the remaining nodal points of this element τ (which are in interior of Ω_k). Therefore the choice of test spaces and the formulation of mortar conditions are not standard, as in the case of mortar method with locally conforming elements.*

The discretization of (4.1) using V_h^M is of the form:
Find $u_h^M \in V_h^M$ such that

$$a_H(u_h^M, v) = \int_\Omega fv \, dx \quad \forall v \in V_h^M, \qquad (4.20)$$

where $a_H(u,v) = \sum_{k=1}^{N} a_{h,k}(u,v)$ and $a_{h,k}(u,v)$ are defined as in Section 4.2.2. The form $a_H(\cdot,\cdot)$ is positive definite over V_h^M what follows from the fact that $a_H(u,u) = 0$ implies that u is linear in all triangles of Ω_k, then from the continuity of u at all vertices and of $\partial_n u$ at all midpoints of elements of $T_h(\Omega_k)$ follows that u is linear in Ω_k and from the mortar condition follows that u is linear in Ω. Then the boundary conditions yield $u = 0$.

Proposition 4.2.5 *The problem (4.20) has a unique solution.*

As in Section 4.2.2, we can introduce nodal basis, i.e. with each degree of freedom at all nodal points which are not in $\partial\Omega$ and a slave δ_m, we associate one nodal basis function. Then we can rewrite the discrete problem (4.20) as a system of linear algebraic equations. If we additionally assume that $h_i \asymp h_j$, for an interface $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$, we can see that the condition number of the resulting matrix is bounded by $C\underline{h}^{-4}$, where $\underline{h} = \inf_k h_k$ and C is a positive constant independent of any h_k and the number of subdomains.

4.3 Ellipticity of discrete problems

In this subsection, we prove that $a_H(\cdot,\cdot)$ is elliptic on the considered discrete spaces of the mortar methods with a constant independent of h_k and, what is also very important, the number of subdomains. These results are analogous to those for mortar method for second order elliptic problems proved in [16], The proofs of the results of this section utilize similar ideas to those of [16]. We distinguish the cases of mortar methods with conforming and nonconforming local discretizations. The proof of the first case is simpler due to the fact that in each subdomain we have properly smooth functions.

4.3.1 Ellipticity for locally conforming elements

In this subsection, we prove ellipticity for the discrete space of mortar method with local C^1 functions.

Lemma 4.3.1 *There exists a constant C independent of h_k and the number of sub-*

domains such that for $u \in V_h^B$

$$\sum_{k=1}^N \|u\|^2_{H^2(\Omega_k)} \le C \sum_{k=1}^N |u|^2_{H^2(\Omega_k)}.$$

This is also valid for $u \in V_h^{RH}$ and $u \in V_h^H$.

Proof. The proof is given for the case of locally bicubic elements, but after minor modifications it would be also valid for the mortar method with locally HCT or reduced HCT elements. We first prove that $\sum_{k=1}^N \|u_{x_1}\|^2_{L^2(\Omega_k)} \preceq \sum_{k=1}^N |u|^2_{H^2(\Omega_k)}$. Here and below, we consider $\partial^\alpha u$ for an admissible multi-index $|\alpha| \le 2$ and $u \in V_h^B$ as the L^2 function such that $\partial^\alpha u_{|\Omega_k} = \partial^\alpha u_k$.

We have for any $(x_1, x_2) \in \Omega$

$$u_{x_1}(x_1, x_2) = \int_a^{x_1} u_{x_1 x_1}(t, x_2)\, dt + \sum_{t_{kl} \in [a, x_1] \cap \Gamma_{kl}} [u_{x_1}](t_{kl}, x_2),$$

where $[\cdot]$ denotes the jump over Γ_{kl} at point t_{kl} and $(a, x_2) \in \partial\Omega$. Here t_{kl} is the point of intersection of a segment $[a, x_1]$ and an interface $\Gamma_{kl} \subset \partial\Omega_k \cap \partial\Omega_l$.

The first term can be estimated by

$$|\int_a^{x_1} u_{x_1 x_1}(t, x_2)\, dt| \le \int_a^b |u_{x_1 x_1}(t, x_2)|\, dt \le (b-a)^{1/2} (\int_a^b |u_{x_1 x_1}(t, x_2)|^2\, dt)^{1/2},$$

where b satisfies $(b, x_2) \in \partial\Omega$. The second one by

$$| \sum_{t_{kl} \in [a, x_1] \cap \Gamma_{kl}} [u_{x_1}](t_{kl}, x_2)| \le \sum_{t_{kl} \in [a, b] \cap \Gamma_{kl}} |[u_{x_1}](t_{kl}, x_2)| \le$$

$$\le \left(\sum_{t_{kl} \in [a, b] \cap \Gamma_{kl}} |\Gamma_{kl}| \right)^{1/2} \left(\sum_{t_{kl} \in [a, b] \cap \Gamma_{kl}} |\Gamma_{kl}|^{-1} |[u_{x_1}](t_{kl}, x_2)|^2 \right)^{1/2}.$$

We have $\sum_{t_{kl} \in [a,b] \cap \Gamma_{kl}} |\Gamma_{kl}| \preceq |b-a|$. It is obvious for V_h^B since all subdomains are rectangles, for V_h^{RH}, V_h^H we could use the result of [16], see Lemma 2.2 there. Note that u_{x_1} equals $\partial_n u$ on Γ_{kl} that are included in the above sum. Thus we have

$$(u_{x_1}(x_1, x_2))^2 \preceq \int_a^b |u_{x_1 x_1}(t, x_2)|^2\, dt + \sum_{t_{kl} \in [a, b] \cap \Gamma_{kl}} |\Gamma_{kl}|^{-1} |[u_{x_1}](t_{kl}, x_2)|^2.$$

We now integrate over dx_1 and dx_2 and get

$$\int_\Omega (u_{x_1})^2\, dx \preceq \int_\Omega |u_{x_1x_1}|^2\, dx + \sum_{\Gamma_{kl}\subset\Gamma} \int_{\Gamma_{kl}} |\Gamma_{kl}|^{-1} |\partial_n u_k - \partial_n u_l|^2\, ds.$$

We now have to estimate this sum over all interfaces. Each term of that sum is estimated separately. By (4.4), the average values of $\partial_n u_k$ and $\partial_n u_l$ over interface Γ_{kl} are equal to each other since $\int_{\Gamma_{kl}}(\partial_n u_k - \partial_n u_l)\, ds = 0$. Thus the standard trace theorem and Poincare's inequality yield that

$$\int_{\Gamma_{kl}} |\partial_n u_k - \partial_n u_l|^2\, ds \preceq H_k |u_k|^2_{H^2(\Omega_k)} + H_l |u_l|^2_{H^2(\Omega_l)}.$$

Thus summing over all interfaces ends the proof of estimate of the L^2 norm of u_{x_1}. The estimate of L^2 norm of u by $\sum_{k=1}^N |\nabla u_k|^2_{L^2(\Omega_k)}$ can be proved in similar way, cf. also [16]. □

Remark 4.3.1 *This proof was done for mortar discretizations of a clamped plate problem. But for a simply supported plate problem, cf. Chapter 5, Section 5.9 p.144 in [38], the proof of the same ellipticity property proceeds in a similar way since we have that on the part of boundary that is parallel to axis OX_k $u_{x_k} = 0$ for $k = 1, 2$; cf. also the proof of Lemma 4.3.2 below.*

We obtain also a straightforward corollary from the previous lemma i.e.

Corollary 4.3.1 *We have*

$$(u, u)_{L^2(\Omega)} \preceq a_H(u, u) \preceq \underline{h}^{-4}(u, u)_{L^2(\Omega)} \quad \forall u \in V_h^B \quad (V_h^{RH}, V_h^H),$$

where $\underline{h} = \inf_i h_i$.

The proof directly follows from an inverse inequality and Lemma 4.3.1.

4.3.2 Ellipticity for locally nonconforming elements

In this subsection, we prove ellipticity for the discrete space of mortar method with locally nonconforming functions of Adini element and Morley elements.

Lemma 4.3.2 *There exists a constant C independent of h_k and the number of subdomains such that for $u \in V_h^A$*

$$\sum_{k=1}^{N} \|u\|^2_{H_h^2(\Omega_k)} \leq C \sum_{k=1}^{N} |u|^2_{H_h^2(\Omega_k)}.$$

This is also valid for V_h^M.

For the proof of this lemma, we need a local equivalence mapping $\mathcal{M}_k^A : X_h^A(\Omega_k) \to X_h^B(\Omega_k)$, see [35], where $X_h^B(\Omega_k)$ is a space of C^1 smooth functions that are bicubic in each rectangular element of $T_h(\Omega_k)$. We also introduce $(\mathcal{M}_k^A)^\dagger : X_h^B(\Omega_k) \to X_h^A(\Omega_k)$ a quasi-inverse mapping of $\mathcal{M}_k^A$ which we use below, cf. Section 4.5.5.

Definition 4.3.1 *We define $\mathcal{M}_k^A : X_h^A(\Omega_k) \to X_h^B(\Omega_k)$ and $(\mathcal{M}_k^A)^\dagger : X_h^B(\Omega_k) \to X_h^A(\Omega_k)$ by setting their values of all respective degrees of freedom at all nodal points of $\overline{\Omega}_{k,h}$, as follows, let p be a nodal point of $\overline{\Omega}_{k,h}$, $u \in X_h^A(\Omega_k)$ and $v \in X_h^B(\Omega_k)$, then*

$$\begin{aligned} \mathcal{M}_k^A u\,(p) &= u(p), \\ \partial_{x_i}\mathcal{M}_k^A u\,(p) &= u_{x_i}(p) \quad i = 1, 2, \\ \partial_{x_1 x_2}\mathcal{M}_k^A u\,(p) &= 0, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{M}_k^A)^\dagger v\,(p) &= v(p), \\ \partial_{x_i}(\mathcal{M}_k^A)^\dagger v\,(p) &= v_{x_i}(p) \quad i = 1, 2. \end{aligned}$$

In the following lemma, we state some properties of these mappings. The proof of the first and second inequalities can be found in [35], see Lemma 5.1 there, the remaining equality and inequality follows from the definition of $(\mathcal{M}_k^A)^\dagger$.

Lemma 4.3.3 *Suppose $u \in X_h^A(\Omega_k)$ and $v \in X_h^B(\Omega_k)$. Then*

$$|\mathcal{M}_k^A u|_{H^s(\Omega_k)} \asymp |u|_{H_h^s(\Omega_k)} \quad s = 0, 1, 2,$$

$$\|u - \mathcal{M}_k^A u\|_{L^2(\Omega_k)} + h_k |u - \mathcal{M}_k^A u|_{H^1(\Omega_k)} \preceq h_k^2 |u|_{H_h^2(\Omega_k)},$$

$$(\mathcal{M}_k^A)^\dagger \mathcal{M}_k^A u = u,$$

$$|(\mathcal{M}_k^A)^\dagger v|_{H^s(\Omega_k)} \preceq |v|_{H_h^s(\Omega_k)} \quad s = 0, 1, 2.$$

Proof. (Lemma 4.3.2) The proof is similar to the proof of the previous lemma, but instead of the mortar conditions we want to use the continuity of functions of V_h^A (V_h^M) at crosspoints. There is another difficulty: the local functions are not contained in the space H^2. The proof is the same for both elements. Therefore we restrict ourselves to the case of the Adini element. The proof of the case of the Morley element follows the same lines, we only have to utilize local operator $\mathcal{M}_k^M$ and its properties, see Definition 4.4.1 and Lemma 4.4.13, instead of $\mathcal{M}_k^A$ in each substructure Ω_k.

Let $u = (u_1, \ldots, u_N) \in V_h^A \subset X_h^A(\Omega)$. First, for each subdomain Ω_k, we define $\tilde{u}_k = \mathcal{M}_k^A u_k$, see Definition 4.3.1. Next we define global function $\tilde{u} \in X_h^B(\Omega) \subset \prod_{k=1}^N H^2(\Omega_k)$ as equal to $\tilde{u}_k$ in Ω_k. By Lemma 4.3.3, we have

$$|u_k|^2_{H_h^s(\Omega_k)} \asymp |\tilde{u}_k|^2_{H^s(\Omega_k)} \quad s = 0, 1, 2. \tag{4.21}$$

We have for any $(x_1, x_2) \in \Omega$

$$\tilde{u}_{x_2}(x_1, x_2) = \int_a^{x_1} \tilde{u}_{x_1 x_2}(t, x_2)\, dt + \sum_{t_{kl} \in [a, x_1] \cap \Gamma_{kl}} [\tilde{u}_{x_2}](t_{kl}, x_2).$$

And then, as in the proof of the previous lemma, we conclude that

$$\int_\Omega (\tilde{u}_{x_2})^2\, dx \preceq \int_\Omega |\tilde{u}_{x_1 x_2}|^2\, dx + \sum_{\Gamma_{kl} \subset \Gamma} \int_{\Gamma_{kl}} H_k^{-1} |\partial_s \tilde{u}_k - \partial_s \tilde{u}_l|^2\, ds.$$

Here ∂_s is the tangential derivative over an interface Γ_{kl}. We estimate the first term by $\sum_{k=1}^N |u_k|^2_{H_h^2(\Omega_k)}$. We have used (4.21). We now have to estimate the second sum. Each term of the second sum is estimated separately.

From the continuity of $u \in V_h^A$ at the crosspoints, we see that $\tilde{u}_k(a) = u_k(a) = u_l(a) = \tilde{u}_l(a)$ if a is an end of Γ_{kl}. Then the linear interpolation polynomials of $\tilde{u}_k$ and $\tilde{u}_l$ over Γ_{kl} are equal and thus the standard trace theorem, a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], and (4.21) yield that

$$\int_{\Gamma_{kl}} |\partial_s \tilde{u}_k - \partial_s \tilde{u}_l|^2\, ds \preceq H_k |u_k|^2_{H^2(\Omega_k)} + H_l |u_l|^2_{H^2(\Omega_l)}.$$

Thus summing over all interfaces ends the proof of estimate of L^2 norm of u_{x_2}. The estimate of L^2 norm of u by $\sum_{k=1}^N |u_k|^2_{H_h^2(\Omega_k)}$ can be proved in a very similar way. □

4.4 Error estimates

In this section, we prove the error estimate for the mortar methods presented in the previous section.

The error estimates in the H_h^2 norm, we obtain from the second Strang lemma, cf. e.g. [12] or Lemma 8.1.9, p.198 in [38] which here is formulated as

$$\|u^* - u_h\|_h \leq \inf_{v \in V^h} \|u^* - v\|_h + \sup_{w \in V^h \setminus \{0\}} \frac{|a_H(u^* - u_h, w)|}{\|w\|_h}, \tag{4.22}$$

where u^* is the solution of (4.1), u_h is the respective solution of (4.5), (4.11), (4.15), (4.16) or (4.20), V^h is V_h^B, V_h^A, V_h^{RH}, V_h^H or V_h^M, respectively, and $\|\cdot\|_h = a_H(\cdot,\cdot)^{1/2}$.

The first term is called the approximation error while the second one is the consistency error.

We obtain the L^2 estimates of errors under the following regularity assumption for Ω:

Regularity assumption
For any $g \in L^2(\Omega)$ the problem

$$a(\phi, v) = \int_\Omega gv\, dx \quad \forall v \in H_0^2(\Omega)$$

has a unique solution $\phi \in H^4(\Omega) \cap H_0^2(\Omega)$ and

$$\|\phi\|_{H^4(\Omega)} \preceq \|g\|_{L^2(\Omega)}. \tag{4.23}$$

This regularity assumption holds for example for a polygonal domain which has the largest angle equal to $\pi/2$, cf. Chapter 7, Section 7.3.2, Remark 7.3.2.3 and Lemma 7.3.2.4, p.338-339 in [71], or for domains with sufficiently smooth boundary, cf. [84]. Then utilizing the standard duality technique for nonconforming elements, we obtain an L^2 estimate for mortar methods with locally conforming elements stated in the following proposition, for the proof see e.g. the proof of Theorem 2.2, p.17-18 in [79], cf. also [47] or [24].

Proposition 4.4.1 *Assume that the regularity assumption (4.23) holds and u^* is the solution of (4.1). Then*

$$\|u^* - u_h\|_{L^2(\Omega)} \preceq \sup_{\phi \in H^4(\Omega)} \left\{ \inf_{\phi_h \in V^h} \frac{|E(u^*, u_h, \phi, \phi_h)|}{\|\phi\|_{H^4(\Omega)}} \right\},$$

where

$$E(u^*, u_h, \phi, \phi_h) = a_H(u^* - u_h, \phi - \phi_h) - E_h(\phi, u^* - u_h) - E_h(u^*, \phi - \phi_h)$$

and

$$E_h(\psi, v) = a_H(\psi, v) - (\triangle^2\psi, v)_{L^2(\Omega)} \quad \forall\psi \in H^4(\Omega) \cap H_0^2(\Omega), \ \ \forall v \in \prod_{k=1}^{N} H_C^2(\Omega_k).$$

Here u_h is the respective solution of (4.5), (4.15) or (4.16) and V^h is V_h^B, V_h^{RH} or V_h^H, respectively.

Proposition 4.4.1 is also valid for the mortar methods that locally use nonconforming elements, i.e. Adini and Morley elements, but our proof technique fails to get proper error bounds in these two cases. This is due to the fact that in the proofs of consistency errors of Lemma 4.4.7 and Lemma 4.4.12, we use special local equivalence mappings ($\mathcal{M}_k^A$, $\mathcal{M}_i^M$) that are not defined for any locally H^2 function what in turn is necessary to obtain L^2 bounds.

4.4.1 Bicubic element

We now state our main result for the mortar method in V_h^B, i.e. for the local bicubic discretizations presented in Section 4.2.2.

Theorem 4.4.1 *Assume that u^* the solution of (4.1), is in the space $H_0^2(\Omega) \cap H^4(\Omega)$. Then holds*

$$|u^* - u_h^B|_{H_H^2(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2}. \tag{4.24}$$

If also the regularity assumption (4.23) holds, then we have

$$\|u^* - u_h^B\|_{L^2(\Omega)} \preceq \overline{h}^2 \left(\sum_{k=1}^{N} h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2},$$

where u_h^B is the solution of (4.5) and $\overline{h} = \max_k h_k$.

Proof. The proof of the H_h^2 estimate follows directly from (4.22) by using Lemmas 4.4.1 and 4.4.3, see below.

The L^2 estimate is obtained from Proposition 4.4.1. Each term of $E(u^*, u_h, \phi, \phi_h)$ is estimated separately. The first one can be bounded by

$$a_H(u^* - u_h^B, \phi - \phi_h) \preceq |u^* - u_h^B|_{H_H^2(\Omega)} |\phi - \phi_h|_{H_H^2(\Omega)} \preceq |\phi - \phi_h|_{H_H^2(\Omega)} \left(\sum_{k=1}^{N} h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2}.$$

We have used (4.24).

Note that Lemma 4.4.1 is valid not only for $w \in V_h^B$, but also for $w + v$ with $w \in V_h^B$ and $v \in H_0^2(\Omega)$ what is easy to observe in the proof of this lemma. Thus we get

$$E_h(\phi, u^* - u_h^B) \preceq \left(\sum_{k=1}^N h_k^4 |\phi|^2_{H^4(\Omega_k)}\right) |u^* - u_h^B|_{H_H^2(\Omega)} \preceq \overline{h}^2 |\phi|_{H^4(\Omega)} \left(\sum_{k=1}^N h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2}.$$

We have used Lemma 4.4.1 and (4.24). Next again by Lemma 4.4.1, we obtain

$$E_h(u^*, \phi - \phi_h) \preceq |\phi - \phi_h|_{H_H^2(\Omega)} \left(\sum_{k=1}^N h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2}.$$

Summing these three estimates and utilizing Lemma 4.4.3, we have

$$\inf_{\phi_h \in V_h^B} |E(u^*, u_h, \phi, \phi_h)| \preceq \overline{h}^2 |\phi|_{H^4(\Omega)} \left(\sum_{k=1}^N h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2}.$$

And this ends the proof of the L^2 estimate. □

The consistency error

Here we prove the bound for the consistency error. We state it in the following lemma.

Lemma 4.4.1 *Under assumptions of Theorem 4.4.1, holds*

$$|a_H(u^* - u_h^B, w)| = |E_h(u^*, w)| \preceq |w|_{H_H^2(\Omega)} \left(\sum_{k=1}^N h_k^4 |u^*|^2_{H^4(\Omega_k)}\right)^{1/2} \quad \forall w \in V_h^B.$$

Before we prove this lemma, we state an additional result, namely, the approximation property of the L^2 projection onto $M_{1,3}^{h_j}(\delta_{m,j})$.

Lemma 4.4.2 *If $g \in H^s(\delta_{m,j})$ for $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ then*

$$\|g - Q_1 g\|_{L^2(\delta_{m,j})} \preceq h_j^s |g|_{H^s(\delta_{m,j})}.$$

Here Q_1 is the L^2 orthogonal projection onto the space $M_{1,3}^{h_j}(\delta_{m,j})$.

The space $M_{1,3}^{h_j}(\delta_{m,j})$ is nonstandard, but the proof follows standard arguments, e.g. cf. Section 3 and the proof of Proposition 4.1 in [32].

Proof. (Lemma 4.4.1) Using the Green's formulas, e.g. see (1.2.5) and (1.2.9), p.14-15 in [47], and (4.5), we have

$$a_H(u^* - u_h^B, w) = a_H(u^*, w) - f(w) = E_h(u^*, w) = E_1(u^*, w) + E_2(u^*, w) + E_3(u^*, w)$$

with

$$E_1(u^*, w) = \int_\Gamma -\partial_n(\triangle u^*)[w]\, ds, \qquad E_2(u^*, w) = \int_\Gamma (1-\nu)\partial_n\partial_s u^*[\partial_s w]\, ds,$$

and

$$E_3(u^*, w) = \int_\Gamma (\triangle u^* - (1-\nu)\partial_s^2 u^*)[\partial_n w]\, ds.$$

Here ∂_n, ∂_s are normal and tangential derivatives, $[\cdot]$ is the jump over interface Γ.

Utilizing the fact that $[w]$ is equal to zero at the ends of any interface Γ_{kl}, we obtain

$$E_2(u^*, w) = -\sum_{\Gamma_{ij}} (1-\nu) \int_{\Gamma_{ij}} \partial_s\partial_n\partial_s u^*[w]\, ds.$$

Let $E_0(u^*, w) = E_1(u^*, w) + E_2(u^*, w)$ and we have $E_0(u^*, w) = \int_\Gamma G_3 u^*[w]\, ds$ with $G_3 u^* = -\partial_n \triangle u^* - (1-\nu)\partial_s\partial_n\partial_s u^*$.

We now consider one interface Γ_{ij} with the master $\gamma_{m,i}$ and slave $\delta_{m,j}$. The mortar condition (4.3) yields that $(Q_1 w_i = Q_1 w_j)$

$$\int_{\Gamma_{ij}} G_3 u^*[w]\, ds = \int_{\Gamma_{ij}} ((I - Q_1)G_3 u^*)\,[w]\, ds =$$

$$= \int_{\Gamma_{ij}} ((I - Q_1)G_3 u^*)\,(I - Q_1)w_i\, ds - \int_{\Gamma_{ij}} ((I - Q_1)G_3 u^*)\,(I - Q_1)w_j\, ds.$$

Then using Schwarz inequality, Lemma 4.4.2, and the trace theorem, see Theorem 1.5.2.1, p.42 in [71], we have

$$\int_{\Gamma_{ij}} G_3 u^*[w]\, ds \le \|(I - Q_1)G_3 u^*\|_{L^2(\delta_{m,j})} \cdot$$

$$\cdot \left\{ \|(I - Q_1)w_i\|_{L^2(\delta_{m,j})} + \|(I - Q_1)w_j\|_{L^2(\delta_{m,j})} \right\} \preceq h_j^{1/2} |G_3 u^*|_{H^{1/2}(\delta_{m,j})} \cdot$$

$$\cdot h_j^{3/2} (|w_i|_{H^{3/2}(\delta_{m,j})} + |w_j|_{H^{3/2}(\delta_{m,j})}) \preceq h_j^2 |u^*|_{H^4(\Omega_j)} (|w_i|_{H^2(\Omega_i)} + |w_j|_{H^2(\Omega_j)}).$$

Summing over all interfaces yields the estimate of $E_0(u^*, w)$.

Let $G_2 u^* = \triangle u^* - (1-\nu)(\partial_s^2 u^*)$, then in the same manner by (4.4), we obtain

$$\int_{\Gamma_{ij}} G_2 u^* [\partial_n w]\, ds \le \|(I-Q_1)G_2 u^*\|_{L^2(\delta_{m,j})}\{\|(I-Q_1)\partial_n w_i\|_{L^2(\delta_{m,j})}+$$
$$+\|(I-Q_1)\partial_n w_j\|_{L^2(\delta_{m,j})}\} \preceq h_j^2 |G_2 u^*|_{H^{3/2}((\delta_m)}(|\partial_n w_i|_{H^{1/2}(\delta_m)}+$$
$$+|\partial_n w_j|_{H^{1/2}(\delta_m)}) \preceq h_j^2 |u^*|_{H^4(\Omega_j)}(|w_i|_{H^2(\Omega_i)} + |w_j|_{H^2(\Omega_j)}).$$

Summing over all interfaces yields the estimate of $E_3(u^*, w)$. □

The approximation error

The approximation error is stated in the following lemma

Lemma 4.4.3 *If* $u \in H^4(\Omega) \cap H_0^2(\Omega)$, *then we have*

$$\inf_{v \in V_h^B} |u - v|^2_{H_H^2(\Omega)} \preceq \sum_{k=1}^N h_k^4 |u|^2_{H^4(\Omega_k)}.$$

Before we prove this lemma, we introduce some technical tools. First we define a new operator associated with a slave $\delta_{m,j}$ and state its stability property. The stability of this operator is crucial and further will be used to obtain approximation error bounds and in the analysis of algorithms.

Let $\Pi^1_{m,j} : L^2(\delta_{m,j}) \to H_0^2(\delta_{m,j}) \cap W^{h_j}(\delta_{m,j})$ be defined by

$$\int_{\delta_{m,j}} \left(I - \Pi^1_{m,j}\right) u\, \psi\, ds = 0 \quad \forall \psi \in M^{h_j}_{1,3}(\delta_{m,j}). \tag{4.25}$$

Here $W^{h_j}(\delta_{m,j})$ is a subspace of $C^1(\delta_{m,j})$ formed by functions which are piecewise cubic over elements of the h_j-triangulation of this slave.

Lemma 4.4.4 *Let* u *be a cubic polynomial and* $\tilde{u}$ *be a linear polynomial defined on the unit segment* $[0,1]$ *such that* $u(0) = u'(0) = 0$, $u(1) = \tilde{u}(1)$ *and* $u'(1) = \tilde{u}'(1)$. *Then holds*

$$\|u\|^2_{L^2([0,1])} \asymp \|\tilde{u}\|^2_{L^2([0,1])} \asymp \int_{[0,1]} u\tilde{u}\, ds.$$

Proof. We can interpret the norms of u and $\tilde{u}$ as norms of vector $(u(1), u'(1))^T = (A, B)^T$ therefore the first equivalence is obvious. To prove the second estimate it is sufficient to show that function $g(A,B) = (\int_{[0,1]} u\tilde{u}\, ds)^{1/2}$ is a norm over $\Re^2$. First we get that $u(t) = (3A - B)t^2 + (B - 2A)t^3$ and $\tilde{u}(t) = A - B + Bt$. Then

$$\int_{[0,1]} u\tilde{u}\, ds = (1/3)(A-B)(3A-B) + (1/4)[(A-B)(B-2A) + B(3A-B)]+$$

$$+(1/5)B(B-2A) = (1/2)A^2 - (7/30)AB + (1/30)B^2 = (M\mathbf{p}, \mathbf{p})_{\Re^2},$$

where $(\cdot,\cdot)_{\Re^2}$ is the standard inner product in $\Re^2$, $\mathbf{p}$ is a vector and M is a 2×2 symmetric matrix, i.e.

$$\mathbf{p} = \begin{pmatrix} A \\ B \end{pmatrix} \quad M = \begin{pmatrix} 1/2 & -7/60 \\ -7/60 & 1/30 \end{pmatrix}.$$

It easy to check that M is positive definite, what ends the proof of the lemma. □

Next lemma states the H^s stability of $\Pi^1_{m,j}$, $s \in [0,2]$.

Lemma 4.4.5 *For all $u \in [L^2(\delta_{m,j}), H^2_0(\delta_{m,j})]_s$ holds*

$$\|\Pi^1_{m,j} u\|_{[L^2(\delta_{m,j}),H^2_0(\delta_{m,j})]_s} \preceq \|u\|_{[L^2(\delta_{m,j}),H^2_0(\delta_{m,j})]_s} \quad s \in [0,2],$$

where $[L^2(\delta_{m,j}), H^2_0(\delta_{m,j})]_s$ is a Hilbertian interpolation space between $L^2(\delta_{m,j})$ and $H^2_0(\delta_{m,j})$.

Proof. We first prove the L^2 stability. Let $w = \Pi^1_m u$ and $\tilde{w}$ be function in $M^{h_j}_{1,3}(\delta_{m,j})$ such that at all nodal points $\tilde{w}^{(s)}(p) = w^{(s)}(p)$, $s = 0, 1$. Note that this function is properly defined and it differs from w only on two segments that touch the ends of the slave. Let one of those segment be denoted by e. Mapping to the reference element $[0,1]$ and utilizing Lemma 4.4.4, we obtain

$$\|w\|^2_{L^2(e)} \asymp \|\tilde{w}\|^2_{L^2(e)} \asymp \int_e \tilde{w} w\, ds.$$

Thus we have

$$\|w\|^2_{L^2(\delta_{m,j})} \preceq \sum_{e \subset \delta_{m,j}} \int_e \tilde{w} w\, ds \quad = \int_{\delta_{m,j}} \tilde{w} u\, ds \leq$$

$$\leq \|\tilde{w}\|_{L^2(\delta_{m,j})} \|u\|_{L^2(\delta_{m,j})} \preceq \|w\|_{L^2(\delta_{m,j})} \|u\|_{L^2(\delta_{m,j})}.$$

We have used the definition of Π^1_m, see (4.25), and Schwarz inequality. Dividing both sides by $\|w\|_{L^2(\delta_{m,j})}$ ends the proof of the L^2- stability.

Next we prove the H_0^2 stability. Let Q be the L^2 projection onto $H_0^2(\delta_{m,j}) \cap W^{h_j}(\delta_{m,j})$. It can be proved by using similar ideas to those in [32], cf. Section 3 and the proof of Proposition 4.1 there, that the projection Q is stable in H^s norm for $s = 0, 1, 2$, and satisfies

$$|u - Qu|_{H^s(\delta_{m,j})} \preceq h_j^{2-s} |u|_{H^2(\delta_{m,j})} \quad \forall u \in H_0^s(\delta_{m,j}) \quad s = 0, 1, 2. \tag{4.26}$$

Since $\Pi^1_{m,j}$ is also an L^2 projection (but not orthogonal) onto $H_0^2(\delta_{m,j}) \cap W^{h_j}(\delta_{m,j})$, we have $\Pi^1_{m,j} Q = Q$. Thus we can conclude that

$$|\Pi^1_{m,j} u|_{H^2(\delta_{m,j})} \leq |\Pi^1_{m,j} u - Qu|_{H^2(\delta_{m,j})} + |Qu|_{H^2(\delta_{m,j})} \preceq$$
$$\preceq h_j^{-2} \|\Pi^1_{m,j}(u - Qu)\|_{L^2(\delta_{m,j})} + |Qu|_{H^2(\delta_{m,j})} \preceq$$
$$\preceq h_j^{-2} \|u - Qu\|_{L^2(\delta_{m,j})} + |u|_{H^2(\delta_{m,j})} \preceq |u|_{H^2(\delta_{m,j})}.$$

We have used an inverse inequality, the L^2 stability of $\Pi^1_{m,j}$, the H^2 stability of Q and (4.26). A Hilbertian interpolation argument, e.g. see Proposition 12.1.5, p.279 in [38], ends the proof. □

The next lemma is a discrete analog of the extension theorem for Sobolev spaces, cf. [71]. The proof for HCT element can be found in [39], see Lemma 4.6 there, and for reduced HCT element in [82], see Theorem 4.4 there, and the proof for bicubic element can be done in similar way to that of results of [39] or [82].

Lemma 4.4.6 *Consider a local subdomain Ω_k and let $X_h^B(\Omega_k)$ ($X_h^{RH}(\Omega_k)$ or $X_h^H(\Omega_k)$) be the bicubic (reduced HCT or HCT) finite element space constructed on a quasiuniform triangulation made of rectangles (or triangles) of Ω_k. Let $v \in Tr\ X_h^B(\Omega_k)$ ($Tr\ X_h^{RH}(\Omega_k)$ or $Tr\ X_h^H(\Omega_k)$). Then there exists $Ext(v) \in X_h^B(\Omega_k)$ ($X_h^{RH}(\Omega_k)$ or $X_h^H(\Omega_k)$) such that*

$$Tr\ Ext(v)_{|\partial\Omega_k} = v,$$
$$|Ext(v)|_{H^2(\Omega_k)} \preceq |\nabla v|_{H^{1/2}(\partial\Omega_k)},$$

where $Tr\ v = (v_{|\partial\Omega_k}, \nabla v_{|\partial\Omega_k})$ for $v \in H^2(\Omega_k)$.

Proof. (Lemma 4.4.3) Let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)$ be in $X_h^B(\Omega)$ such that $\tilde{u}_k$ is the local piecewise bicubic interpolant of u defined by the values of all degrees of freedom of bicubic element at all nodal points of $\overline{\Omega}_{k,h}$. Then we have, cf. e.g. Theorem 48.1, p.296 in [48],

$$|u - \tilde{u}_k|_{H^s(\Omega_k)} \preceq h_k^{4-s} |u|_{H^4(\Omega_k)} \quad s = 0, 1, 2. \tag{4.27}$$

We next define for each interface Γ_{ij} with the master $\gamma_{m,i}$ and respective slave $\delta_{m,j}$ two functions $w_m, \partial_n w_m \in H_0^2(\delta_{m,j}) \cap W^{h_j}(\delta_{m,j})$ by conditions

$$\int_{\delta_m} w_m \psi \, ds = \int_{\delta_m} (\tilde{u}_i - \tilde{u}_j)_{|\Gamma_{ij}} \psi \, ds \quad \forall \psi \in M_{1,3}^{h_j}(\delta_{m,j})$$

and

$$\int_{\delta_m} \partial_n w_m \psi \, ds = \int_{\delta_m} (\partial_n \tilde{u}_i - \partial_n \tilde{u}_j)_{|\Gamma_{ij}} \psi \, ds \quad \forall \psi \in M_{1,3}^{h_j}(\delta_{m,j}),$$

where $\tilde{u}_i, \tilde{u}_j, \partial_n \tilde{u}_i, \partial_n \tilde{u}_j$ are respective traces onto the master $\gamma_{m,i}$ and the slave $\delta_{m,j}$. Then we define a global function $w = (w_1, \ldots, w_N) \in X_h^B(\Omega)$ as follows: if $\delta_{m,j}$ is parallel to axis OX_1 and $p \in \delta_{m,j,h}$, then we set $w_j(p) = w_m(p), w_{j,x_1}(p) = \partial_{x_1} w_m(p)$ and $w_{j,x_2}(p) = \partial_n w_m(p), w_{j,x_1x_2}(p) = \partial_{x_1}\partial_n w_m(p)$. On slaves parallel to axis OX_2, we define w analogously. Next we set w_j zero at all degrees of freedom of remaining nodes of $\partial\Omega_{j,h}$ i.e. at ones that are not on any slave $\delta_{m,j}$. Thus we have w defined at all degrees of freedom of all nodes of Γ. By Lemma 4.4.6, we know that for each subdomain exists $w_j = Ext(w_{|\partial\Omega_j}) \in X_h^B(\Omega_j)$. Then on each slave δ_m, we have $w_j = w_m$, $\partial_n w_j = \partial_n w_m$ and function $v = \tilde{u} + w$ is in V_h^B what follows from (4.3) and (4.4).

We can conclude that

$$|u - v|_{H_H^2(\Omega)} \le |u - \tilde{u}|_{H_H^2(\Omega)} + |w|_{H_H^2(\Omega)}.$$

The first term we estimate from (4.27).

We next estimate the seminorm of w. Note that any degrees of freedom of this function can be nonzero at nodal points of $\partial\Omega_{j,h}$ only at nodes of any $\delta_{m,j,h}$.

By Lemma 4.4.6, we have

$$|w|_{H_H^2(\Omega)}^2 \preceq \sum_{\delta_{m,j} \subset \Gamma} \|\nabla w_j\|_{H_{00}^{1/2}(\delta_{m,j})}^2 \preceq$$

$$\preceq \sum_{\delta_{m,j} \subset \Gamma} \{ \|\partial_s w_m\|_{H_{00}^{1/2}(\delta_{m,j})}^2 + \|\partial_n w_m\|_{H_{00}^{1/2}(\delta_{m,j})}^2 \}.$$

Note that on $\delta_{m,j}$, we have $w = w_m = \Pi_{m,j}^1(\tilde{u}_i - \tilde{u}_j)$ and $\partial_n w = \partial_n w_m = \Pi_{m,j}^1(\partial_n \tilde{u}_i - \partial_n \tilde{u}_j)$. Thus by Lemma 4.4.5, we have

$$|w|_{H_H^2(\Omega)}^2 \preceq \sum_{\delta_m \subset \Gamma} \left\{ \|\partial_s \tilde{u}_i - \partial_s \tilde{u}_j\|_{H_{00}^{1/2}(\delta_{m,j})}^2 + \|\partial_n \tilde{u}_i - \partial_n \tilde{u}_j\|_{H_{00}^{1/2}(\delta_{m,j})}^2 \right\} \preceq$$

$$\preceq \sum_{\gamma_m \subset \Gamma} \left\{ \|\partial_s \tilde{u}_i - \partial_s u\|^2_{H^{1/2}_{00}(\gamma_{m,i})} + \|\partial_n \tilde{u}_i - \partial_n u\|^2_{H^{1/2}_{00}(\gamma_{m,i})} \right\} +$$

$$+ \sum_{\delta_m \subset \Gamma} \left\{ \|\partial_s u - \partial_s \tilde{u}_j\|^2_{H^{1/2}_{00}(\delta_{m,j})} + \|\partial_n u - \partial_n \tilde{u}_j\|^2_{H^{1/2}_{00}(\delta_{m,j})} \right\}.$$

Utilizing (4.27) and ideas as in the proof of Lemma 3.3.8 in Section 3.3 and afterwards summing over all interfaces, we obtain

$$|w|^2_{H^2_H(\Omega)} \preceq \sum_{\Gamma_{ij} \subset \Gamma} [h_i^4 |u|^2_{H^4(\Omega_i)} + h_j^4 |u|^2_{H^4(\Omega_j)}] \preceq \sum_{k=1}^{N} h_k^4 |u|^2_{H^4(\Omega_k)},$$

what ends the proof. □

4.4.2 Adini element

In this subsection, we present error estimates for the mortar method in V_h^A, see Section 4.2.3, i.e. for the one that contains locally continuous, but not C^1- smooth functions.

Theorem 4.4.2 *Assume that u^*, the solution of (4.1), is in the space $H_0^2(\Omega) \cap H^4(\Omega)$. Then*

$$|u^* - u_h^A|^2_{H^2_H(\Omega)} \preceq \sum_{k=1}^{N} \left(h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)} \right),$$

where u_h^A is the solution of (4.11).

The proof follows from the second Strang lemma, see (4.22), and Lemmas 4.4.7 and 4.4.8, see below.

The consistency error

The main result of this paragraph is stated in the following lemma.

Lemma 4.4.7 *Under assumptions of Theorem 4.4.2, holds*

$$\sup_{w \in V_h^A \setminus \{0\}} \frac{|a_H(u^* - u_h^A, w)|}{|w|_{H^2_H(\Omega)}} \preceq \left(\sum_{k=1}^{N} (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right)^{1/2}.$$

Proof. Using the Green's integral formulas, cf. (1.2.5) and (1.2.9), p.14-15 in [47], and (4.11), we obtain for $w \in V_h^A$

$$a_H(u^* - u_h^A, w) = a_H(u^*, w) - f(w) = E_1(u^*, w) + E_2(u^*, w) + E_3(u^*, w)$$

with

$$E_1(u^*, w) = \int_\Gamma -\partial_n(\triangle u^*)[w]\, ds, \qquad E_2(u^*, w) = \int_\Gamma (1-\nu)\partial_n\partial_s u^*[\partial_s w]\, ds$$

and

$$E_3(u^*, w) = \sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau} (\triangle u^* - (1-\nu)\partial_s^2 u^*)\partial_n w\, ds.$$

Here ∂_n, ∂_s are normal and tangential derivative, $[\cdot]$ is the jump over interface Γ. We have used the fact that $X_h^A(\Omega_k) \subset H^1(\Omega_k)$.

Proceeding in the same way as in the case of bicubic element, see the proof of Lemma 4.4.1, we have

$$E_1(u^*, w) + E_2(u^*, w) = \sum_{\Gamma_{ij} \subset \Gamma} \int_{\Gamma_{ij}} G_3 u^*[w]\, ds,$$

where $G_3 u^* = -\partial_n \triangle u^* - (1-\nu)\partial_s\partial_n\partial_s u^*$.

We now estimate this term. Let consider one interface Γ_{ij} with the slave $\delta_{m,j}$ and master $\gamma_{m,i}$ and let $Q_{0,1}$ and Q_1 be L^2- orthogonal projections onto $M_{0,1}^{h_j}(\delta_{m,j})$ and $M_{1,3}^{h_j}(\delta_{m,j})$, respectively, cf. Sections 4.2.2 and 4.2.3. Then by the mortar condition (4.9), we obtain

$$\int_{\delta_{m,j}} G_3 u^*[w]\, ds = \int_{\delta_{m,j}} (I - Q_1) G_3 u^*\, [w]\, ds =$$

$$= \int_{\delta_{m,j}} (I - Q_1) G_3 u^* \left([w] + Q_1 \mathcal{M}_j^A w_j - Q_1 \mathcal{M}_i^A w_i\right) ds,$$

where $\mathcal{M}_j^A, \mathcal{M}_i^A$ are defined in Definition 4.3.1. We can write $[w] = (w_i - \mathcal{M}_i^A w_i) + \mathcal{M}_i^A w_i - (w_j - \mathcal{M}_j^A w_j) - \mathcal{M}_j^A w_j$ and by Schwarz and triangle inequalities, we obtain

$$\int_{\delta_m} G_3 u^*[w]\, ds = \|G_3 u^* - Q_1 G_3 u^*\|_{L^2(\delta_{m,j})} \{\|w_i - \mathcal{M}_i^A w_i\|_{L^2(\delta_{m,j})} +$$

$$+ \|w_j - \mathcal{M}_j^A w_j\|_{L^2(\delta_{m,j})} + \|\mathcal{M}_i^A w_i - Q_1 \mathcal{M}_i^A w_i\|_{L^2(\delta_{m,j})} + \|\mathcal{M}_j^A w_j - Q_1 \mathcal{M}_j^A w_j\|_{L^2(\delta_{m,j})}\}.$$

Note that $G_3u^* \in H^{1/2}(\delta_m)$ and $\mathcal{M}_i^A w_i, \mathcal{M}_j^A w_j$ restricted to $\Gamma_{ij} = \delta_{m,j}$ belong to $H^{3/2}(\delta_{m,j})$, thus the first, fourth and fifth terms we estimate using Lemma 4.4.2 and the trace bound, e.g. see Theorem 1.5.2.1, p.42 in [71]. The two remaining terms are estimated as follows, for $s = i$ and $s = j$

$$\|w_s - \mathcal{M}_s^A w_s\|^2_{L^2(\delta_m)} \preceq h_s^{-1}\|w_s - \mathcal{M}_s^A w_s\|^2_{L^2(\Omega_s)} +$$
$$+ h_s|w_s - \mathcal{M}_s^A w_s|^2_{H_h^1(\Omega_s)} \preceq h_s^3|w_s|^2_{H_h^2(\Omega_s)}.$$

We used the simple trace bound, see Theorem 1.5.2.1, p.42 in [71], on each element, a scaling argument and Lemma 4.3.3. The last term can be estimated in the same way.

We finally conclude that

$$\int_{\delta_m} G_3u^*[w]\,ds \preceq h_j^2|u^*|_{H^4(\Omega_j)}\left\{|w_j|_{H_h^2(\Omega_j)} + |w_i|_{H_h^2(\Omega_i)}\right\}.$$

We have also used the assumption $h_i \le h_j$.

Summing over all slaves, we get the estimate for $E_1(u^*,w) + E_2(u^*,w)$.

We now estimate the third remaining term $E_3(u^*,w)$. Let $G_2u^* = (\triangle u^* - (1-\nu)\partial_s^2 u^*)$. Adding and subtracting $I_{h_k}\partial_n w_k$ over all edges of elements belonging to $T_h(\Omega_k)$, we see that

$$E_3(u^*,w) = \sum_{\Gamma_{kl}\subset\Gamma}\int_{\Gamma_{kl}} G_2u^*(I_{h_k}\partial_n w_k - I_{h_l}\partial_n w_l)\,ds +$$
$$+ \sum_{k=1}^N \sum_{\tau\in T_h(\Omega_k)}\int_{\partial\tau} G_2u^*(\partial_n w_k - I_{h_k}\partial_n w_k)\,ds.$$

In [47], see the proof of Theorem 6.2.3, p.369-373 there, is proven that for each rectangle $\tau \in T_h(\Omega_k)$, we have

$$\int_{\partial\tau} G_2u^*(\partial_n w_k - I_{h_k}\partial_n w_k)\,ds \preceq h_k|u^*|_{H^3(\tau)}|w_k|_{H^2(\tau)}.$$

Thus summing over all elements of all $T_h(\Omega_k)$ and then the resulting estimate over all subdomains, we can conclude that the second term is estimated by the following $\left(\sum_{k=1}^N h_k^2|u^*|^2_{H^3(\Omega_k)}\right)^{1/2}|w|_{H_H^2(\Omega)}$.

We next estimate the remaining term of $E_3(u^*,w)$. We now consider a single interface Γ_{kl} with the slave $\delta_{m,k}$ and master $\gamma_{m,l}$. By the mortar condition (4.10), we have

$$\int_{\Gamma_{kl}} G_2u^*(I_{h_k}\partial_n w_k - I_{h_l}\partial_n w_l)\,ds = \int_{\Gamma_{kl}} (I - Q_{0,1})G_2u^*(I_{h_k}\partial_n w_k - I_{h_l}\partial_n w_l)\,ds.$$

Utilizing the properties of $Q_{0,1}$ an L^2 orthogonal projection, the fact that $I_{h_k}\partial_n w_k = I_{h_k}\partial_n \mathcal{M}_k^A w_k$ and $I_{h_l}\partial_n w_l = I_{h_l}\partial_n \mathcal{M}_l^A w_l$, and proceeding similar as above, we obtain

$$\int_{\Gamma_{kl}} G_2 u^*(I_{h_k}\partial_n w_k - I_{h_l}\partial_n w_l)\, ds \leq \|G_2 u^* - Q_{0,1}G_2 u^*\|_{L^2(\delta_m)}\cdot$$

$$\cdot\{\|\partial_n \mathcal{M}_l^A w_l - Q_{0,1}\partial_n \mathcal{M}_l^A w_l\|_{L^2(\delta_m)} + \|\partial_n \mathcal{M}_k^A w_k - Q_{0,1}\partial_n \mathcal{M}_k^A w_k\|_{L^2(\delta_m)} +$$
$$+ \|\partial_n \mathcal{M}_l^A w_l - I_{h_l}\partial_n \mathcal{M}_l^A w_l\|_{L^2(\delta_m)} + \|\partial_n \mathcal{M}_k^A w_k - I_{h_k}\partial_n \mathcal{M}_k^A w_k\|_{L^2(\delta_m)}\}.$$

We now use standard properties of $I_{h_s}, s = k, l$; see e.g. Corollary 4.4.24, p.109 in [38] and of $Q_{0,1}$, see e.g. Remark 3.4 in [25], and get

$$\int_{\Gamma_{kl}} G_2 u^*(I_{h_k}\partial_n w_k - I_{h_l}\partial_n w_l)\, ds \preceq h_k |G_2 u^*|_{H^{1/2}(\delta_m)}\{|\partial_n \mathcal{M}_k^A w_k|_{H^{1/2}(\delta_m)}$$

$$+ |\partial_n \mathcal{M}_l^A w_l|_{H^{1/2}(\delta_m)}\} \preceq h_k |u^*|_{H^3(\Omega_k)}\{|w_k|_{H_h^2(\Omega_k)} + |w_l|_{H_h^2(\Omega_l)}\}.$$

We have also used the trace bound, Lemma 4.3.3 and the fact that $h_l \leq h_k$. Summing over all interfaces ends the proof. □

The approximation error

In this paragraph, we estimate the approximation error of the mortar Adini method. The main result we state in the following lemma.

Lemma 4.4.8 *For $u \in H_0^2(\Omega) \cap H^3(\Omega)$, holds*

$$\inf_{v \in V_h^A} |u - v|^2_{H_H^2(\Omega)} \preceq \sum_{k=1}^N h_k^2 |u|^2_{H^3(\Omega)}.$$

For the proof we need to introduce $\Pi^0_{m,j,s} : L^2(\delta_{m,j}) \to H_0^1(\delta_{m,j})$, an operator corresponding to $\delta_{m,j}$, a slave, defined as follows, cf. [9]. For $u \in L^2(\delta_{m,j})$ let $\Pi^0_{m,j,s}u$ be a continuous function which is a polynomial of order s in all elements of the h_j triangulation of $\delta_{m,j}$ vanishing at ends of this slave and satisfying

$$\int_{\delta_{m,j}} \left(I - \Pi^0_{m,j,s}\right) u\, \psi\, ds = 0 \quad \forall \psi \in M_{0,s}^{h_j}(\delta_{m,j}). \tag{4.28}$$

Next lemma was proven in [9], see Lemma 1 there, and states the stability property of $\Pi^0_{m,j,s}$.

Lemma 4.4.9 *For the operator $\Pi^0_{m,j,s}$ for positive integer s holds*

$$\|\Pi^0_{m,j,s}u\|_{L^2(\delta_{m,j})} \preceq \|u\|_{L^2(\delta_{m,j})} \ \ \forall u \in L^2(\delta_{m,j})$$

and

$$\|\Pi^0_{m,j,s}u\|_{H^{1/2}_{00}(\delta_{m,j})} \preceq \|u\|_{H^{1/2}_{00}(\delta_{m,j})} \ \ \forall u \in H^{1/2}_{00}(\delta_{m,j}).$$

We will also use the operator $\Pi^1_{m,j}$, defined in (4.25).

Proof. (Lemma 4.4.8) Let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)$ be a function in $X^A_h(\Omega)$ such that for each subdomain

$$|u - \tilde{u}_k|_{H^s(\Omega_k)} \preceq h_k^{3-s}|u|_{H^3(\Omega_k)} \quad s = 0, 1, 2 \tag{4.29}$$

We set that $\tilde{u}_k$ is the interpolant of u defined at all degrees of freedom of Adini element at all nodal points of $\overline{\Omega}_{k,h}$, e.g. see Theorem 3.1.6, p.124 in [47].

Let $\Gamma_{ij} = \gamma_{m,i} = \delta_{m,j}$. We define two functions on $\delta_{m,j}$: the first one $w_m \in H^2_0(\delta_{m,j})$ - C^1 smooth, piecewise cubic and a piecewise linear one $\hat{w}_m \in H^1_0(\delta_{m,j})$ by conditions, cf. the mortar conditions (4.9) and (4.10),

$$\int_{\delta_m} w_m \psi \, ds = \int_{\delta_m} (\tilde{u}_i - \tilde{u}_j)_{|\Gamma_{ij}} \psi \, ds \ \ \forall \psi \in M^{h_j}_{1,3}(\delta_{m,j})$$

and

$$\int_{\delta_m} \hat{w}_m \psi \, ds = \int_{\delta_m} (I_{h_i}\partial_n \tilde{u}_i - I_{h_j}\partial_n \tilde{u}_j)_{|\Gamma_{ij}} \psi \, ds \ \ \forall \psi \in M^{h_j}_{0,1}(\delta_{m,j}),$$

where $\tilde{u}_i, \tilde{u}_j, \partial_n\tilde{u}_i, \partial_n\tilde{u}_j$ are respective traces onto the master $\gamma_{m,i}$ and the slave $\delta_{m,j}$. Note that $w_m = \Pi^1_{m,j}(\tilde{u}_i - \tilde{u}_j)$ and $\hat{w}_m = \Pi^0_{m,j,1}(I_{h_i}\partial_n\tilde{u}_i - I_{h_j}\partial_n\tilde{u}_j)$. We now define a global function $w \in X^A_h(\Omega)$ by setting the values of all degree of freedom at all nodal points of all subdomains. We first set to zero all degrees of freedom of w at all nodal points which are not in any $\delta_{m,k,h}$. Then we set values of degrees of freedom of w at nodal points of slaves as follows: For any $p \in \bigcup_{\delta_{m,k} \subset \Gamma} \delta_{m,k,h}$ let $w(p) = w_m(p)$, $\partial_s w(p) = \partial_s w_m(p)$ and $\partial_n w(p) = \hat{w}_m(p)$. We next define $v = \tilde{u} + w$. This function obviously satisfies mortar conditions (4.9) and (4.10) what follows from its definition.

We can further conclude that

$$|u - v|_{H^2_H(\Omega)} \le |u - \tilde{u}|_{H^2_H(\Omega)} + |w|_{H^2_H(\Omega)}.$$

The first term is estimated by (4.29).

It remains to estimate the seminorm of w. Note that this function in not zero only for an element $\tau \subset \Omega_j$ such that $\partial\tau \cap \overline{\delta}_{m,j} = \overline{e}$, an edge of the rectangular element τ.

Let now consider this element. Let the slave $\delta_{m,j}$ be parallel to the axis OX_1. Thus $\partial_n w_{|\delta_{m,j}} = w_{x_2|\delta_{m,j}}$ Using the reference rectangle and a scaling argument, we have

$$|w|^2_{H^2(\tau)} \preceq \sum_{p\in\partial\tau} \left\{ h_j^{-2} w^2(p) + w^2_{x_1}(p) + w^2_{x_2}(p) \right\},$$

where the sum is taken over all vertices of τ.

Note that $\|\hat{w}_m\|^2_{L^2(\delta_{m,j})} \asymp \sum_{p\in\delta_{m,j}} h_j \hat{w}^2_m(p) = \sum_{p\in\delta_{m,j}} h_j w^2_{x_2}(p)$ and $\|w_m\|^2_{L^2(\delta_{m,j})} \asymp \sum_{p\in\delta_{m,j}} (h_j w_m^2(p) + h_j^3 (w_m')^2(p)) = \sum_{p\in\delta_{m,j}} (h_j w^2_m(p) + h_j^3 w^2_{x_1}(p))$. Thus summing over all nodes of $\delta_{m,j,h}$ and then over all slaves, we have

$$|w|^2_{H^2_H(\Omega)} \preceq \sum_{\delta_m\subset\Gamma} \left\{ h_j^{-3} \|w_m\|^2_{L^2(\delta_{m,j})} + h_j^{-1} \|\hat{w}_m\|^2_{L^2(\delta_{m,j})} \right\}.$$

The sum of first terms, i.e. $\sum_{\delta_m\subset\Gamma} h_j^{-3}\|w_m\|^2_{L^2(\delta_{m,j})}$, we can estimate utilizing Lemma 4.4.5 and the sum of second terms using Lemma 4.4.9 and have

$$|w|^2_{H^2_H(\Omega)} \preceq \sum_{\Gamma_{ij}\subset\Gamma} \left\{ h_j^{-3} \|\tilde{u}_i - \tilde{u}_j\|^2_{L^2(\delta_{m,j})} + h_j^{-1} \|I_{h_i}\partial_n \tilde{u}_i - I_{h_j}\partial_n \tilde{u}_j\|^2_{L^2(\delta_{m,j})} \right\} \leq$$

$$\leq \sum_{\Gamma_{ij}\subset\Gamma} h_j^{-3} \left\{ \|\tilde{u}_i - u\|^2_{L^2(\delta_{m,j})} + \|u - \tilde{u}_j\|^2_{L^2(\delta_{m,j})} \right\} +$$

$$+ \sum_{\Gamma_{ij}\subset\Gamma} h_j^{-1} \left\{ \|I_{h_i}\partial_n \tilde{u}_i - \partial_n u\|^2_{L^2(\delta_{m,j})} + \|\partial_n u - I_{h_j}\partial_n \tilde{u}_j\|^2_{L^2(\delta_{m,j})} \right\}.$$

By (4.29), the trace theorem (utilized on each element of the h_i triangulation of $\gamma_{m,i}$), see Theorem 1.5.2.1, p.42 in [71], and a scaling argument, we conclude that

$$h_j^{-3}\|\tilde{u}_i - u\|^2_{L^2(\delta_{m,j})} \preceq h_j^{-3} h_i^{-1} \|\tilde{u}_i - u\|^2_{L^2(\Omega_i)} + h_j^{-3} h_i |\tilde{u}_i - u|^2_{H^1_h(\Omega_i)} \preceq h_i^2 |u|^2_{H^3(\Omega_i)}.$$

We have also used the assumption $h_i \leq h_j$. Analogously, we can obtain

$$h_j^{-3}\|\tilde{u}_j - u\|^2_{L^2(\delta_{m,j})} \preceq h_j^2 |u|^2_{H^3(\Omega_j)}.$$

We now estimate all remaining terms. We first note that at a nodal point p of any master or slave, e.g. let $p \in \gamma_{m,i,h}$, we have $I_{h_i}\partial_n \tilde{u}_i(p) = I_{h_i}\partial_n u(p)$, what follows from definition of $\tilde{u}$. Thus we get for any interface Γ_{ij} with the slave $\delta_{m,j}$ and master $\gamma_{m,i}$

$$h_j^{-1}\|\partial_n u - I_{h_i}\partial_n \tilde{u}_i\|^2_{L^2(\delta_{m,j})} = h_j^{-1}\|\partial_n u - I_{h_i}\partial_n u\|^2_{L^2(\delta_{m,j})} \preceq$$

$$h_j^{-1} h_i^{-1} \|\partial_n u - I_{h_i}\partial_n u\|^2_{L^2(\Omega_i)} + h_j^{-1} h_i |\partial_n u - I_{h_i}\partial_n u|^2_{H^1(\Omega_i)} \preceq h_i^2 |u|^2_{H^3(\Omega_i)}.$$

Here $I_{h_i}\partial_n u_i, I_{h_j}\partial_n u_i$ are the piecewise bilinear interpolants defined by the values of $\partial_n u_i, \partial_n u_j$ at the vertices of rectangular elements of $T_h(\Omega_i), T_h(\Omega_j)$, respectively. We have used the trace bound on each element of the h_i triangulation of $\gamma_{m,i}$, the standard finite element interpolation estimate, e.g. see Theorem 3.1.6, p.124 in [47], and the assumption $h_i \leq h_j$. Analogously, we get

$$h_j^{-1}\|\partial_n u - I_{h_j}\partial_n \tilde{u}_j\|^2_{L^2(\delta_{m,j})} \preceq h_j^2 |u|^2_{H^3(\Omega_j)}.$$

Summing over all interfaces and utilizing all above estimates ends the proof of this lemma. □

4.4.3 HCT elements

We now state our main result for the mortar methods presented in Section 4.2.4, i.e. the that locally contain smooth functions from reduced HCT finite element method or HCT method.

Theorem 4.4.3 (For reduced HCT method) *Assume that u^*, the solution of (4.1), is in the space $H_0^2(\Omega) \cap H^4(\Omega)$. Then holds*

$$|u^* - u_h^{RH}|_{H_H^2(\Omega)} \preceq \left(\sum_{k=1}^N (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right)^{1/2}.$$

If the regularity assumption (4.23) holds, then we have

$$\|u^* - u_h^{RH}\|_{L^2(\Omega)} \preceq \overline{h} \left(\sum_{k=1}^N (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right)^{1/2},$$

where $\overline{h} = \max_k h_k$ and u_h^{RH} is the solution of (4.15).

Theorem 4.4.4 (For HCT method) *Assume that u^*, the solution of (4.1), is in the space $H_0^2(\Omega) \cap H^4(\Omega)$. Then holds*

$$|u^* - u_h^{H}|_{H_H^2(\Omega)} \preceq \left(\sum_{k=1}^N h_k^4 |u|^2_{H^4(\Omega_k)} \right)^{1/2}.$$

If the regularity assumption (4.23) holds, then we have

$$\|u^* - u_h^{H}\|_{L^2(\Omega)} \preceq \overline{h}^2 \left(\sum_{k=1}^N h_k^4 |u^*|^2_{H^4(\Omega_k)} \right)^{1/2},$$

where $\overline{h} = \max_k h_k$ and u_h^H is the solution of (4.16).

The proofs are analogous to the one of Theorem 4.4.1 and follow from the second Strang lemma, see (4.22), Proposition 4.4.1 and Lemmas 4.4.10 and 4.4.11, see below. Therefore some details are omitted.

The consistency error

In the following lemma, we state estimates of the consistency errors.

Lemma 4.4.10 *Under the assumptions of Theorem 4.4.3, holds*

$$|a_H(u^* - u_h^{RH}, w)| \preceq |w|_{H_H^2(\Omega)} \left(\sum_{k=1}^{N} (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right)^{1/2} \quad \forall w \in V_h^{RH},$$

$$|a_H(u^* - u_h^{H}, w)| \preceq |w|_{H_H^2(\Omega)} \left(\sum_{k=1}^{N} (h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right)^{1/2} \quad \forall w \in V_h^{H}.$$

The proof of this lemma is very similar to the proof of Lemma 4.4.1 therefore we omit it.

The approximation error

The approximation error is stated in the following lemma.

Lemma 4.4.11 *If $u \in H^3(\Omega) \cap H_0^2(\Omega)$, then for reduced HCT element*

$$\inf_{v \in V_h^{RH}} |u - v|^2_{H_H^2(\Omega)} \preceq \sum_{k=1}^{N} h_k^2 |u|^2_{H^3(\Omega_k)}$$

and if $u \in H^4(\Omega) \cap H_0^2(\Omega)$, then for HCT element

$$\inf_{v \in V_h^{H}} |u - v|^2_{H_H^2(\Omega)} \preceq \sum_{k=1}^{N} h_k^4 |u|^2_{H^4(\Omega_k)}.$$

Proof. The proof uses similar arguments to that of Lemma 4.4.3. We first define $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N) \in X_h^{RH}(\Omega)$ or $(X_h^H(\Omega))$ as the local interpolant of u using all degrees of freedom of reduced HCT (or HCT) local discretizations. This function has desired approximation properties, but not necessarily belong to the discrete space V_h^{RH} (or V_h^H). Then we add a correction $w \in X_h^{RH}(\Omega)$ $(X_h^H(\Omega))$ such that $\tilde{u} + w \in V_h^{RH}$ (V_h^H) and have

$$|u - v|_{H_H^2(\Omega)} \leq |u - \tilde{u}|_{H_H^2(\Omega)} + |w|_{H_H^2(\Omega)}.$$

We define w locally, first only on boundaries of all subdomains: by zero on masters and on slaves by (4.12) and (4.13) or in case of HCT element by (4.12) and (4.14), respectively, as in the proof of Lemma 4.4.3. Then applying Lemma 4.4.6 from which follows that there exists an extension of w_k from $\partial\Omega_k$ onto Ω_k such that

$$|w|^2_{H_H^2(\Omega)} \preceq \sum_{\delta_m \subset \Gamma} \{|\partial_s w|^2_{H_{00}^{1/2}(\delta_m)} + |\partial_n w|^2_{H_{00}^{1/2}(\delta_m)}\},$$

and next proceeding as in the proof of Lemma 4.4.3, and utilizing Lemmas 4.4.5 and 4.4.9, we obtain the estimates of Lemma 4.4.11. □

4.4.4 Morley element

In this subsection, we prove the error estimate for the mortar methods with locally Morley nonconforming discretizations, see Section 4.2.5.

Theorem 4.4.5 *If $u^* \in H^4(\Omega) \cap H_0^2(\Omega)$, then*

$$|u^* - u_h^M|_{H_H^2(\Omega)} \preceq \left\{ \sum_{k=1}^{N} (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)}) \right\}^{1/2},$$

where u_h^M is the solution of (4.20).

The proof follows from the second Strang lemma, see (4.22), Lemmas 4.4.12 and 4.4.15, see below.

The consistency error

The main result of this paragraph is the following result.

Lemma 4.4.12 *Under the assumptions of Theorem 4.4.5, holds*

$$\sup_{w \in V_h^M \setminus \{0\}} \frac{|a_H(u^* - u_h^M, w)|}{\|w\|_h} \preceq \left\{ \sum_{k=1}^{N} (h_j^2 |u^*|^2_{H^3(\Omega_k)} + h_j^4 |u^*|^2_{H^4(\Omega_k)}) \right\}^{1/2}.$$

For the proof we introduce a local mapping, see Section 3, (3.2) in [39], $\mathcal{M}_k^M : X_h^M(\Omega_k) \to X_h^H(\Omega_k)$, where $X_h^H(\Omega_k)$ is a local subspace of C^1 smooth functions, the Hsieh-Clough-Tocher (HCT) macro element, cf. Section 4.2.4.

In the definition, the fact is used that for quadratic polynomial $q \in P_2([a,b])$ we have

$$q'((a+b)/2) = \frac{q(b) - q(a)}{b-a}.$$

Thus ∇u for $u \in X_h^M(\Omega_k)$ is well defined at all midpoints. Let m_p be an adjacent midpoint of the vertex p if both points belong to the same edge in $T_h(\Omega_k)$. The choice of the midpoint is not unique.

Definition 4.4.1 *We define $\mathcal{M}_k^M : X_h^M(\Omega_k) \to X_h^H(\Omega_k)$ by setting its degrees of freedom at all vertices and midpoints of Ω_k, i.e. let p be a vertex and m the midpoint of an edge of an element of $T_h(\Omega_k)$, then*

$$\mathcal{M}_k^M u\,(p) = u(p) \quad \forall \text{ vertices } p,$$

$$\partial_n \mathcal{M}_k^M u\,(m) = \partial_n u(m) \quad \forall \text{ midpoints } m,$$

$$\nabla \mathcal{M}_k^M u\,(p) = \nabla u(m_p) \;\; \forall \text{ vertices } p, \text{ where } m_p \text{ is an adjacent midpoint.}$$

In the following lemma, we state some properties of the local equivalence mapping defined above.

Lemma 4.4.13 *For all $u \in X_h^M(\Omega_k)$, holds*

$$|\mathcal{M}_k^M u|_{H^s(\Omega_k)} \asymp |u|_{H_h^s(\Omega_k)} \quad s = 0,1,2,$$

$$\|u - \mathcal{M}_k^M u\|_{L^2(\Omega_k)} + h_k |u - \mathcal{M}_k^M u|_{H_h^1(\Omega_k)} \preceq h_k^2 |u|_{H_h^2(\Omega_k)},$$

$$\|u - \mathcal{M}_k^M u\|_{L^2(\Gamma_{kl})} + h_k \|\partial_n u - \partial_n \mathcal{M}_k^M u\|_{L^2(\Gamma_{kl})} \preceq h_k^{3/2} |u|_{H_h^2(\Omega_k)}.$$

Here Γ_{kl} is an edge of Ω_k.

The proof of the first two statements can be found in [39], cf. Section 3, Corollary 3.3 and the proof of Lemma 3.1 there. The last inequality can be proved by the application of the trace bound on each h_k element of Γ_{kl}, see Theorem 1.5.2.1, p.42 in [71], and a scaling argument, and then by utilizing the second inequality of the lemma.

For the proof of Lemma 4.4.12, we need also the following result.

Lemma 4.4.14 *If $g \in H^s(\Gamma_{ij})$, $s \in [0,2]$, where $\Gamma_{ij} = \gamma_{m,i} = \delta_{m,j}$, then holds*

$$\|g - I_{2h_k,2} g\|_{L^2(\Gamma_{ij})} \preceq h_k^s |g|_{H^s(\Gamma_{ij})} \;\; s \in \{1, \frac{3}{2}, 2, \frac{5}{2}, 3\};\; k = i, j,$$

$$\|g - Q_2 g\|_{L^2(\Gamma_{ij})} \preceq h_j^s |g|_{H^s(\Gamma_{ij})} \;\; s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}.$$

Here Q_2 is the L^2 orthogonal projection onto the space $M_{0,2}^{2h_j}(\delta_{m,j})$, cf. (4.17), and $I_{2h_k,2}$, $k = i, j$, is defined in Definition 4.2.1.

Proof. The cases of $s = 1, 2$, for $I_{2h_k,2}$, $k = i, j$, are proved exactly as in [47], see Theorem 3.1.6, p.124 there, and of $s = 0, 1, 2$, for Q_2 are proved as in [32], see the proof of Proposition 4.1 there. For other s we use an interpolation argument, e.g. see Proposition 12.1.5, p.279 in [38]. □

Proof. (Lemma 4.4.12) As in the proof of Lemma 4.4.1, utilizing Green's integral formulas, e.g. see (1.2.5) and (1.2.9), p.14-15 in [47], we obtain

$$a_H(u^* - u_h^M, w) = a_H(u^*, w) - f(w) = E_1(u^*, w) + E_2(u^*, w) + E_3(u^*, w)$$

with

$$E_1(u^*, w) = -\sum_{k=1}^{N} \{ \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau \setminus \partial\Omega_k} \partial_n(\triangle u^*) w \, ds + \int_\Gamma \partial_n(\triangle u^*)[w] \, ds \},$$

$$E_2(u^*, w) = \sum_{k=1}^{N} \{ \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau \setminus \partial\Omega_k} (1-\nu)\partial_n \partial_s u^* \partial_s w \, ds + \int_\Gamma (1-\nu)\partial_n \partial_s u^* [\partial_s w] \, ds \},$$

and

$$E_3(u^*, w) = \sum_{k=1}^{N} \{ \sum_{\tau \in T_h(\Omega_k)} \int_{\partial\tau \setminus \partial\Omega_k} (\triangle u^* - (1-\nu)\partial_s^2 u^*)\partial_n w \, ds + \\ + \int_\Gamma (\triangle u^* - (1-\nu)\partial_s^2 u^*)[\partial_n w] \, ds \}.$$

Here ∂_n, ∂_s are normal and tangential derivatives and $[\cdot]$ denotes the jump. Each first term of $E_j(u^*, w), j = 1, 2, 3$, is estimated by $C \sum_{k=1}^N (h_k^2 |u^*|^2_{H^3(\Omega_k)} + h_k^4 |u^*|^2_{H^4(\Omega_k)})^{1/2} |w|_{H_h^2(\Omega_k)}$, where C is a positive constant independent of any h_k. The proof is given in [79], see Lemma 3.5, p.26 there.

Using the fact that $[w](p)=0$ for p, an end of the common edge of two adjacent substructures, we can conclude that for the second term of $E_2(u^*,w)$, holds

$$\int_\Gamma (1-\nu)\partial_n\partial_s u^*[\partial_s w]\,ds = \sum_{\Gamma_{ij}\subset\Gamma} -\int_{\Gamma_{ij}}(1-\nu)\partial_s\partial_n\partial_s u^*[w]\,ds.$$

Thus we have to estimate two terms, the first one: $\sum_{\Gamma_{ij}\subset\Gamma}\int_{\Gamma_{ij}} G_3u^*[w]\,ds$, where $G_3u^* = -\partial_n\triangle u^* - (1-\nu)\partial_s\partial_n\partial_s u^*$. And the second one $\int_\Gamma G_2u^*[\partial_n w]\,ds$, where $G_2u^* = (\triangle u^* - (1-\nu)\partial_s^2 u^*)$, as in the proof of Lemma 4.4.1.

Let consider one interface $\Gamma_{ij}=\delta_{m,j}=\gamma_{m,i}$ and let Q_2 and $Q_{-1,0}$ be the L^2 standard orthogonal projections onto $M_{0,2}^{2h_j}(\delta_{m,j})$ and $M_{-1,0}^{h_j}(\delta_{m,j})$, respectively, cf. Section 4.2.5.

We first estimate the second term. By the mortar condition (4.19), we obtain

$$\int_\Gamma G_2u^*[\partial_n w]\,ds = \int_\Gamma (I-Q_{-1,0})G_2u^*[\partial_n w]\,ds.$$

Using the following equality $[\partial_n w] = \partial_n w_i - \partial_n\mathcal{M}_i^M w_i + \partial_n\mathcal{M}_i^M w_i - \partial_n\mathcal{M}_j^M w_j + \partial_n\mathcal{M}_j^M w_j - \partial_n w_j$, we obtain

$$\int_\Gamma G_2u^*[\partial_n w]\,ds \le \|(I-Q_{-1,0})G_2u^*\|_{L^2(\Gamma_{ij})}\{\sum_{s=i,j}\|\partial_n w_s - \partial_n\mathcal{M}_s^M w_s\|_{L^2(\Gamma_{ij})}+$$

$$+\sum_{s=i,j}\|(I-Q_{-1,0})\partial_n\mathcal{M}_s^M w_s\|_{L^2(\Gamma_{ij})}\}.$$

Utilizing Lemma 3.3.6, the trace bound, e.g. see Theorem 1.5.2.1, p.42 in [71], a scaling argument, and Lemma 4.4.13, we have

$$\int_\Gamma G_2u^*[\partial_n w]\,ds \preceq (\sum_{k=1}^N h_k^2|u^*|^2_{H^3(\Omega_k)})^{1/2}|w|_{H^2_H(\Omega)}.$$

By the mortar condition (4.18), we obtain

$$\int_{\delta_m} G_3u^*[w]\,ds = \int_{\delta_m} G_3u^*(w_i - I_{2h_i,2}w_i)\,ds -$$

$$-\int_{\delta_m} G_3u^*(w_j - I_{2h_j,2}w_j)\,ds + \int_{\delta_m}(I-Q_2)G_3u^*\,(I_{2h_i,2}w_i - I_{2h_j,2}w_j)\,ds.$$

We represent $I_{2h_s,2}w_s$ as follows $I_{2h_s,2}w_s = I_{2h_s,2}\mathcal{M}_s^M w_s - \mathcal{M}_s^M w_s + \mathcal{M}_s^M w_s$ for $s=i,j$, and by Schwarz inequality we obtain

$$\int_{\delta_m} G_3u^*[w]\,ds \le \|G_3u^*\|_{L^2(\delta_m)}\,\{\sum_{s=i,j}(\|w_s - \mathcal{M}_s^M w_s\|_{L^2(\delta_m)}+$$

$$+2\,\|I_{2h_s,2}\mathcal{M}_s^M w_s - \mathcal{M}_s^M w_s\|_{L^2(\delta_m)} + \|\mathcal{M}_s^M w_s - Q_2\mathcal{M}_s^M w_s\|_{L^2(\delta_m)})\}.$$

We have used the facts that $I_{2h_s,2}w_s = I_{2h_s,2}\mathcal{M}_s^M w_s$, $s = i, j$, and that Q_2 is the L^2 orthogonal projection onto $M_{0,2}^{2h_j}(\delta_{m,j})$. We now estimate the terms for $s = i$. The case $s = j$ is estimated utilizing the same arguments.

We have $\mathcal{M}_i^M w_i \in H^{3/2}(\Gamma_{ij})$, thus all terms we estimate using Lemmas 4.4.14 and 4.4.13, the assumption $h_i \leq h_j$, and the trace bound, e.g. see Theorem 1.5.2.1, p.42 in [71], by $h_i^{3/2}\,|w_i|_{H_h^2(\Omega_i)}$.

Utilizing the trace bound on each element of the h_i triangulation of $\gamma_{m,i}$ and a scaling argument, we get $\|G_3u^*\|_{L^2(\delta_m)} \preceq (h_j^{-1}|u^*|^2_{H^3(\Omega_j)} + h_j|u^*|^2_{H^4(\Omega_j)})^{1/2}$. We finally conclude that

$$\int_{\delta_m} G_3u^*[w]\,ds \preceq (h_j^2|u^*|^2_{H^3(\Omega_j)} + h_j^4|u^*|^2_{H^4(\Omega_j)})^{1/2}\left\{|w_j|_{H_h^2(\Omega_j)} + |w_i|_{H_h^2(\Omega_i)}\right\}.$$

Summing over all interfaces ends the proof. □

The approximation error

In this paragraph, we estimate the approximation error of the mortar Morley method. The main result is stated in the following lemma.

Lemma 4.4.15 *For $u \in H_0^2(\Omega) \cap H^3(\Omega)$, holds*

$$\inf_{v \in V_h^M} |u - v|^2_{H_H^2(\Omega)} \preceq \sum_{k=1}^N h_k^2 |u|^2_{H^3(\Omega)}.$$

Proof. The scheme of this proof is similar to that of Lemma 4.4.8, therefore some details are omitted.

Let $\tilde{u} \in X_h^M(\Omega)$ be defined as the local interpolant of u using respective degrees of freedom at respective Morley nodal points (vertices and midpoints of edges of elements) in each subdomain. This function satisfies the estimate of the lemma, but may not be in V_h^M, in general.

Then we define two functions on each interface $\Gamma_{kl} = \gamma_{s,k} = \delta_{s,l}$. Let the first one $w_s \in H_0^1(\delta_{s,l})$ be piecewise quadratic on the $2h_l$ triangulation of the slave $\delta_{s,l}$, i.e. in the segments $[p_i, p_{i+2}], i = 0, 2, \ldots, N_{s,l} - 2$, where $\{p_i\}$ are the vertices of the elements of the h_l mesh of this slave. We remind the assumption that $N_{s,l}$, the number of elements of the h_l mesh of $\delta_{s,l}$, is even. And let w_s satisfy

$$\int_{\delta_s} w_s\psi\, ds = \int_{\delta_s} (I_{h_k,2}\tilde{u}_k - I_{h_l,2}\tilde{u}_l)\psi\, ds \;\; \forall \psi \in M_{0,2}^{2h_l}(\delta_{s,l}).$$

The second one $\hat{w}_s \in L^2(\delta_s)$, piecewise constant on the h_l mesh of $\delta_{s,l}$, is defined by

$$\int_{\delta_s} \hat{w}_s\psi\, ds = \int_{\delta_s} (\partial_n\tilde{u}_k - \partial_n\tilde{u}_l)\phi\, ds \;\; \forall \phi \in M_{-1,0}^{h_l}(\delta_{s,l}).$$

We now define a global function $w \in X_h^M(\Omega)$ by setting its values of all respective degrees of freedom at all nodal points of all subdomains. We first set all degrees of freedom of w to zero at all nodal points that are not in any slave $\delta_{s,l}$. Then we set values of degrees of freedom of w at nodal points of slaves as follows: For a vertex $p \in \delta_{s,l,h}$, let $w(p) = w_s(p)$ and for a midpoint m of an element of the h_l triangulation of $\delta_{s,l}$, let $\partial_n w(m) = \hat{w}_s(m)$. Thus w is properly defined and it is obvious that $v = \tilde{u} + w$ satisfies the mortar conditions (4.18) and (4.19). Further, we have $|u-v|_{H_H^2(\Omega)} \le |u-\tilde{u}|_{H_H^2(\Omega)} + |w|_{H_H^2(\Omega)}$ and because the first term satisfies the desired bound we must only estimate the second one.

Following the proof of Lemma 4.4.8, we obtain

$$|w|^2_{H_H^2(\Omega)} \preceq \sum_{\delta_{s,l}\subset\Gamma} \left\{ \sum_{p\in\delta_{s,l,h}} h_l^{-2}|w(p)|^2 + \sum_{m\in\delta_{s,l}} |\partial_n w(m)|^2 \right\} \asymp$$

$$\asymp \sum_{\delta_{s,l}\subset\Gamma} \left\{ h_l^{-3}\|w_s\|^2_{L^2(\delta_s)} + h_l^{-1}\|\hat{w}_s\|^2_{L^2(\delta_s)} \right\}.$$

We next consider one slave $\delta_{s,l}$ with its associated master $\gamma_{s,k}$. Note that $w_s = \Pi^0_{s,l,2}(I_{h_k,2}\tilde{u}_k - I_{h_l,2}\tilde{u}_l)$, where $\Pi^0_{s,l,2}$ is a special projection operator defined as in (4.28) for $M_{0,2}^{2h_l}(\delta_{s,l})$ and $\hat{w}_s = Q_{-1,0}(\partial_n\tilde{u}_k - \partial_n\tilde{u}_l)$, where $Q_{-1,0}$ is the L^2 orthogonal projection onto $M_{-1,0}^{h_l}(\delta_{s,l})$.

Following the lines of Lemma 4.4.8 and using the L^2 stability of $\Pi^0_{s,j,2}$ stated in Lemma 4.4.9 and the standard stability property of $Q_{-1,0}$, an L^2 orthogonal projection, we get the desired estimate. □

4.5 Additive Schwarz methods

In this section, some methods for solving discrete problems considered in Section 4.2, are described. We design and analyze these methods in terms of Additive Schwarz method, see Section 1.4, (or [20], [64], [70], [105]). We want to remind that for each ASM method, we have a corresponding multiplicative Schwarz method (MSM) based on the same decomposition of the discrete space and the same local bilinear forms, cf. Section 1.4.

4.5.1 First method

The method presented in this subsection is of iterative substructuring type, i.e. interior variables are first eliminated using direct methods. The method is for solving (4.15), i.e. for the discrete problem of the mortar method with locally reduced HCT discretizations, but it could be also applied for the mortar method with HCT local elements.

We now define $P_i u_i, H_i u_i \in X_h^{RH}(\Omega_i)$. Let $P_i u_i$ be the orthogonal projection (in the sense of local form $a_{h,i}(\cdot,\cdot)$) of a function $u_i \in X_h^{RH}(\Omega_i)$ onto $X_{0,h}^{RH}(\Omega_i) = H_0^2(\Omega_i) \cap X_h^{RH}(\Omega_i)$ and let $H_i u_i = u_i - P_i u_i$ be the discrete biharmonic part of u_i, i.e.

$$a_{h,i}(P_i u_i, v) = a_{h,i}(u_i, v) \quad \forall v \in X_{0,h}^{RH}(\Omega_i) \tag{4.30}$$

and

$$\begin{cases} a_{h,i}(H_i u_i, v) = 0 & \forall v \in X_{0,h}^{RH}(\Omega_i), \\ Tr\, H_i u_i = Tr\, u_i \quad on & \partial\Omega_i, \end{cases}$$

where $Tr\, u_i = (u_{i|\partial\Omega_i}, \nabla u_{i|\partial\Omega_i})$, e.g. cf. [60] or [82]. We next define a decomposition of any function $u \in V_h^{RH}$ into two parts, $u = Pu + Hu$, where $Pu = (P_1 u_1, \ldots, P_N u_N)$ and $Hu = (H_1 u_1, \ldots, H_N u_N)$. Since $P_i u_i$ is orthogonal to $H_i u_i$ (in terms of $a_{h,i}(\cdot,\cdot)$), Pu and Hu also are orthogonal in the terms of $a_H(\cdot,\cdot)$. Besides we have

$$a_{h,i}(H_i u_i, H_i u_i) = \inf\{a_{h,i}(v_i, v_i) :\ v_i \in X_h^{RH}(\Omega_i)\ \ such\ \ that\ \ Tr\, v_i = Tr\, u_i\}. \tag{4.31}$$

We next define $\tilde{V}_h^{RH} = \{Hu : u \in V_h^{RH}\}$, i.e. the subspace of V_h^{RH} of discrete biharmonic functions. We can decompose u_h^{RH}, the solution of (4.15), into $u_h^{RH} = Pu_h^{RH} + Hu_h^{RH}$. The first term we can compute solving N independent local problems

$$a_{h,i}(P_i u_h^{RH}, v) = f(v) \ \ \forall v \in X_{0,h}^{RH}(\Omega_i) \quad i = 1, \ldots, N.$$

The discrete biharmonic part of u_h^{RH}, further denoted by $\tilde{u}_h^{RH} = Hu_h^{RH}$, is a solution of a new variational discrete problem

$$a_H(\tilde{u}_h^{RH}, v) = f(v) \;\; \forall v \in \tilde{V}_h^{RH}. \tag{4.32}$$

We are now going to present a parallel algorithm for solving this problem. Note that a function $v = (v_1, \dots, v_N) \in \tilde{V}_h^{RH}$ is uniquely defined in Ω_k by $Tr\, v_{k|\partial\Omega_k}$.

Our method of ASM type is described in terms of decomposition of $\tilde{V}_h^{RH}$ into several subspaces and some bilinear forms defined on these subspaces, see Section 1.4.

We first define $V_0^{(1)} \subset \tilde{V}_h^{RH}$, a coarse space. A function $v = \{v_k\}$ is in $V_0^{(1)}$ if for each interface Γ_{kl} with the master $\gamma_{m,k}$ and the slave $\delta_{m,l}$ it satisfies the following conditions

- $Tr\, v_{k\,|\partial\Omega_k \cap \gamma_{m,k}} = Tr\, v_{l\,|\partial\Omega_l \cap \delta_{m,l}}$,
- $v_{|\Gamma_{kl}}$ is a cubic polynomial,
- $\partial_n v_{|\Gamma_{kl}}$ is a linear polynomial.

Note that $v \in V_0^{(1)}$ is a C^1 smooth function, discrete biharmonic in all subdomains, defined by its value and the value of its gradient at all crosspoints.

We next define local vertex spaces. Let c_r be a crosspoint, i.e. c_r is the common vertex of a few subregions. We remind that we can distinguish between the vertices of Ω_k, for $k \in \mathcal{N}(c_r)$, despite the fact that these vertices occupy the same geometrical position of c_r, cf. Section 4.2.2.

Let $\phi_{k,x}^{\alpha} \in \tilde{V}_h^{RH}$ be a discrete biharmonic nodal function associated with a nodal point $x \in \overline{\gamma}_{m,k,h} \subset \partial\Omega_{k,h}$ and a multi-index α such that

$$\partial^{\alpha}\phi_{k,x}^{\alpha}(x) = 1,$$

$$\partial^{\beta}\phi_{k,x}^{\alpha}(y) = 0 \;\; for \;\; \beta \neq \alpha \;\; or \;\; y \neq x$$

for $|\beta| \leq 1$ and any nodal point $y \in \overline{\gamma}_{s,h}$, where γ_s a master or y a vertex in $\mathcal{V}$. This definition is analogous to the one of nodal basis in Section 4.2.2. Note that $\phi_{k,x}^{\alpha}$ is properly defined, as the values of degrees of freedom at points in a slave (nonmortar) δ_r are uniquely determined by its values of degrees of freedom at the ends of this slave and at nodal points of its associated master γ_r by the mortar conditions (4.12) and (4.13).

Equivalently, we can say that $\phi^{\alpha}_{k,x}$ for a multi-index α and a nodal point x of any $\overline{\gamma}_m$ is the discrete biharmonic part of a nodal basis function corresponding to α and x defined analogously to Section 4.2.2. Both functions we denote by the same symbol, but it will follows from the context which one we consider at a moment.

Next for each vertex $x \in \mathcal{V}(\Omega_k)$ and a multi-index α such that $|\alpha| = 1$, we define $V^{(1)}_{x,\alpha} = \text{span}\{\phi^{\alpha}_{k,x}\}$, a one-dimensional subspace of $\tilde{V}^{RH}_h$. With each vertex $x \in \mathcal{V}(\Omega_k)$, we associate two spaces of this type, i.e. for $\alpha = (1,0), (0,1)$. Here $\mathcal{V}(\Omega_k) \subset \Gamma$ is the set of all vertices of $\Omega_k \cap \Gamma$.

We next define subspaces associated with masters. Let for an interface Γ_{kl}, $\gamma_{m,k}$ be the master and $\delta_{m,l}$ its associated slave. Then let $V^{(1)}_m = \tilde{V}^{RH}_h \cap H^2_0(\Omega_k \cup \Omega_l \cup \Gamma_{kl})$, i.e. let $V^{(1)}_m$ be a space formed by all functions $v \in \tilde{V}^{RH}_h$ for which

$$\partial^{\alpha} v(x) = 0 \;\; for \;\; x \in \bigcup_{s \neq m} \gamma_{s,h} \cup \bigcup_{i=1}^{N} \mathcal{V}(\Omega_i) \;\; and \;\; |\alpha| \leq 1.$$

Thus functions of $V^{(1)}_m$ can be nonzero only in $\Omega_k \cup \Omega_l \cup \gamma_{m,k} \cup \delta_{m,l}$. The values of degrees of freedom at nodal points of slave $\delta_{m,h}$ are determined by the mortar conditions (4.12) and (4.13).

We take the original form $a_H(\cdot,\cdot)$ for all local bilinear forms, cf. Section 1.4. Then we have the following decomposition

$$\tilde{V}^{RH}_h = V^{(1)}_0 + \sum_{\gamma_m \subset \Gamma} V^{(1)}_m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} V^{(1)}_{x,\alpha}.$$

Next we define operators $T^{(1)}_0 : \tilde{V}^{RH}_h \to V^{(1)}_0$, $T^{(1)}_m : \tilde{V}^{RH}_h \to V^{(1)}_m$ and $T^{(1)}_{x,\alpha} : \tilde{V}^{RH}_h \to V^{(1)}_{x,\alpha}$ as orthogonal projections onto respective subspaces, i.e.

$$a_H(T^{(1)}_0 u, v) = a_H(u, v) \;\; \forall v \in V^{(1)}_0,$$

$$a_H(T^{(1)}_m u, v) = a_H(u, v) \;\; \forall v \in V^{(1)}_m$$

for all masters γ_m, and

$$a_H(T^{(1)}_{x,\alpha} u, v) = a_H(u, v) \;\; \forall v \in V^{(1)}_{x,\alpha}$$

for all vertices $x \in \mathcal{V}(\Omega_k)$, $k = 1, \ldots, N$, and all multi-indices α of length one.

We define $T^{(1)} : \tilde{V}^{RH}_h \to \tilde{V}^{RH}_h$ as

$$T^{(1)} = T^{(1)}_0 + \sum_{\gamma_m \subset \Gamma} T^{(1)}_m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} T^{(1)}_{x,\alpha}.$$

Then we replace problem (4.32) by

$$T^{(1)}\tilde{u}_h^{RH} = g, \tag{4.33}$$

where $g = g_0 + \sum_{\gamma_m \subset \Gamma} g_m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} g_{x,\alpha}$ and $g_0 = T_0^{(1)}\tilde{u}_h^{RH}$, $g_m = T_m^{(1)}\tilde{u}_h^{RH}$ and $g_{x,\alpha} = T_{x,\alpha}^{(1)}\tilde{u}_h^{RH}$. The right-hand side g can be computed without knowing $\tilde{u}_h^{RH}$.

We now state the main result of this subsection.

Theorem 4.5.1 *For any $u \in \tilde{V}_h^{RH}$, holds*

$$(1 + \log(H/\underline{h}))^{-2} a_H(u,u) \preceq a_H(T^{(1)}u, u) \preceq a_H(u,u),$$

where $\underline{h} = \inf_k h_k$ and $H = \max_k H_k$.

The proof of this theorem can be found in Section 4.5.6.

4.5.2 Second method with outer coarse space

In this section, we present another ASM method for solving (4.15). We now decompose the discrete space V_h^{RH} instead of $\tilde{V}_h^{RH}$.

We additionally assume that subdomains are triangles that form a coarse, shape regular triangulation of Ω with the coarse parameter H, this assumption is not restrictive, see below Remark 4.5.1.

We first introduce an outer coarse space, i.e. one that is not contained in the discrete space V_h^{RH}. Let $\hat{V}_H^{RH} \subset H_0^2(\Omega)$ be a reduced HCT finite element space built on the coarse triangulation. Unfortunately, we have $\hat{V}_H^{RH} \not\subset V_h^{RH}$.

Therefore, we define an additional interpolation (grid transitional) operator $M_h^{RH} : \hat{V}_H^{RH} \to V_h^{RH}$ by setting the values of all respective degrees of freedom of $M_h^{RH}u$ at the nodal points of all subdomains as follows:

$$\partial^\alpha M_h^{RH} u(p) = \partial^\alpha u(p) \quad for \;\; p \in \overline{\Omega}_{k,h} \;\; \forall u \in \hat{V}_H^{RH}.$$

Note that $M_h^{RH}u$ is in V_h^{RH}, i.e. satisfies the mortar conditions because $u_{|\Gamma_{ij}}$ and $\partial_n u_{|\Gamma_{ij}}$ are cubic and linear polynomials, respectively. Hence

$$u_{|\Gamma_{ij}} = M_h^{RH} u_{|\gamma_{m,i}} = M_h^{RH} u_{|\delta_{m,j}}, \quad \partial_n u_{|\Gamma_{ij}} = \partial_n M_h^{RH} u_{|\gamma_{m,i}} = \partial_n M_h^{RH} u_{|\delta_{m,j}}.$$

We next define a coarse space $V_0^{(2)} = M_h^{RH} \hat{V}_H^{RH}$ and a non-exact bilinear form $b_0^{(2)}(\cdot,\cdot) : V_0^{(2)} \times V_0^{(2)} \to \Re$ by

$$b_0^{(2)}(u,v) = a(\hat{u},\hat{v}) \ \ \forall u,v \in V_0^{(2)},$$

where

$$u = M_h^{RH}\hat{u}, \ v = M_h^{RH}\hat{v} \ \ for \ \ \hat{u},\hat{v} \in \hat{V}_H^{RH}.$$

We also define one dimensional spaces corresponding to all degrees of freedom of order one at vertices of subdomains. Let $x \in \mathcal{V}(\Omega_k)$ be a vertex of $\Omega_k \cap \Gamma$ and α a multi-index of length one, then we define a special vertex function $\psi_{k,x}^\alpha$ as follows. Let $\psi_{k,x}^\alpha(p) = \phi_{k,x}^\alpha(p)$ for $p \in \partial\Omega_{j,h}$ for $j = 1,\ldots,N$. The function $\phi_{k,x}^\alpha$ was defined in the previous subsection. Thus $\psi_{k,x}^\alpha = \phi_{k,x}^\alpha$ on Γ. We extend $\psi_{k,x}^\alpha$ as discrete biharmonic in all subdomains $\Omega_j, j \neq k$ and we set its all degrees of freedom to zero at all nodes of $\Omega_{k,h}$. Thus both functions differs only at $\Omega_{k,h}$, where $\phi_{k,x}^\alpha$ is set as discrete biharmonic and $\psi_{k,x}^\alpha$ has the values of respective degrees of freedom equal to zero at the interior nodal points. Next let $V_{x,\alpha}^{(2)} = \text{span}\{\psi_{k,x}^\alpha\}$ and its associated bilinear form be equal to the original one, i.e. $a_H(\cdot,\cdot)$.

We next define the spaces corresponding to masters. Let $\gamma_{m,k} \subset \Omega_k$ be a master and $\delta_{m,l} \subset \Omega_l$ its associated slave. Then let $V_m^{(2)} \subset V_h^{RH}$ be defined as follows

$$V_m^{(2)} = \{v \in V_h^{RH} : \partial^\alpha v(p) = 0 \ \ for \ \ p \in \bigcup_{i=1}^N \overline{\Omega}_{i,h} \setminus (\gamma_{m,k,h} \cup \delta_{m,l,h} \cup \Omega_{k,h} \cup \Omega_{l,h})\}.$$

Here $|\alpha| \leq 1$. We take the exact bilinear form $a_H(\cdot,\cdot)$ for local bilinear forms associated with these local subspaces in ASM scheme, cf. Section 1.4.

Thus we have the following decomposition:

$$V_h^{RH} = V_0^{(2)} + \sum_{\gamma_m \subset \Gamma} V_m^{(2)} + \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} V_{x,\alpha}^{(2)}.$$

Next we define operators $T_0^{(2)} : V_h^{RH} \to V_0^{(2)}$, $T_m^{(2)} : V_h^{RH} \to V_m^{(2)}$ and $T_{x,\alpha}^{(2)} : V_h^{RH} \to V_{x,\alpha}^{(2)}$. Except of the first one, they are set as orthogonal projections onto respective subspaces, i.e.

$$b_0^{(2)}(T_0^{(2)}u,v) = a_H(u,v) \ \ \forall v \in V_0^{(2)},$$

$$a_H(T_m^{(2)}u,v) = a_H(u,v) \ \ \forall v \in V_m^{(2)}$$

for all masters γ_m, and

$$a_H(T^{(2)}_{x,\alpha}u, v) = a_H(u, v) \ \ \forall v \in V^{(2)}_{x,\alpha}$$

for all vertices $x \in \mathcal{V}(\Omega_k)$, $k = 1, \dots, N$, and all multi-indices α of length one.

Further, we follow the previous section and define $T^{(2)} : V_h^{RH} \to V_h^{RH}$ by

$$T^{(2)} = T_0^{(2)} + \sum_{\gamma_m \subset \Gamma} T_m^{(2)} + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} T^{(2)}_{x,\alpha}.$$

Then we replace problem (4.15) by

$$T^{(2)} u_h^{RH} = g, \tag{4.34}$$

where $g = g_0 + \sum_{\gamma_m \subset \Gamma} g_m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} g_{x,\alpha}$ and $g_0 = T_0^{(2)} u_h^{RH}$, $g_m = T_m^{(2)} u_h^{RH}$ and $g_{x,\alpha} = T^{(2)}_{x,\alpha} u_h^{RH}$. The right-hand side g can be computed without knowing u_h^{RH}.

We now state the main result of this subsection.

Theorem 4.5.2 *For any $u \in V_h^{RH}$, holds*

$$(1 + \log(H/\underline{h}))^{-2} a_H(u, u) \preceq a_H(T^{(2)}u, u) \preceq a_H(u, u),$$

where $\underline{h} = \inf_k h_k$ and $H = \max_k H_k$.

The proof is given in Section 4.5.6.

Remark 4.5.1 *We have described the ASM under the assumption that subdomains are triangles, but it is easy to see that if subdomains are polygonal, then we can divide each subdomain into coarse triangles which form a coarse triangulation of Ω. Further, we can define analogously an outer coarse space $\tilde{V}_H^{RH}$ as reduced HCT finite element space built over this coarse triangulation. All results and their proofs can be carried out in a very similar way.*

Remark 4.5.2 *If we assumed that $h_i \le h_j$, for an interface $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$, then in the definition of $V^{(2)}_{x,\alpha}$, we could replace a function $\psi^{\alpha}_{k,x}$ by the one which is equal to $\psi^{\alpha}_{k,x}$ on Γ and in Ω_k and has all degrees of freedom equal to zero at the interior nodal points of all substructures Ω_j. The resulting ASM algorithm would be simpler, since it would not utilize the discrete biharmonic extensions, and the statement of Theorem 4.5.2 would also be valid for the modified ASM.*

4.5.3 Algorithm of Neumann-Neumann type

In this subsection, we present a Neumann-Neumann algorithm following [82], where the conforming finite element was considered. Our algorithm is for the mortar methods. In this subsection, we assume that we can choose master sides of interfaces in such a way that we get two sets of subdomains: the first one of substructures that have all edges as masters and the second one of ones which have all their edges as slaves. The subdomains of the first set we call subdomains of Neumann type and the ones of the second set we call subdomains of Dirichlet type. Thus we have a "chess-board" ordering of subdomains. This assumption is due to the fact that in mortar methods the values of the degrees of freedom at nodes in a slave $\delta_{m,j}$ are determined by the values of the degrees of freedom at ends of the slave and at nodes of the associated master $\gamma_{m,i}$.

In the case of mortar method that uses locally piecewise bicubic element, this choice of master sides of the interfaces is always possible, but in the case of locally reduced HCT or HCT element this assumption can be not satisfied by some partitionings of the domain.

We define a decomposition of $u \in V^h$, where $V^h = V_h^B$ or V_h^H, V_h^{RH} into two parts $u = Pu + Hu$ orthogonal to each other in the sense of $a_H(\cdot,\cdot)$. In Section 4.5.1, we have considered the case of reduced HCT local discretizations. In the cases of locally piecewise bicubic or HCT mortar methods, we have to replace $X_{0,h}^{RH}(\Omega_i)$ by $X_{0,h}^B(\Omega_i) = H_0^2(\Omega_i) \cap X_h^B(\Omega_i)$ or $X_{0,h}^H(\Omega_i) = H_0^2(\Omega_i) \cap X_h^H(\Omega_i)$, respectively.

We further consider only case of locally piecewise bicubic mortar element method, but this method can be analogously defined for reduced HCT or HCT mortar methods.

We next decompose u_h^B, the solution of (4.5), as $u_h^B = Pu_h^B + Hu_h^B$. The first function can be computed by solving N independent problems, cf. Section 4.5.1, and the second one $\tilde{u}_h^B = Hu_h^B$ is the unique solution of the following problem

$$a_H(\tilde{u}_h^B, v) = f(v) \quad \forall v \in \tilde{V}_h^B, \tag{4.35}$$

where $\tilde{V}_h^B = \{Hu : u \in V_h^B\}$.

We rewrite the problem (4.35) in the operator form:

$$S\tilde{u}_h^B = f,$$

where

$$< Su, v >:= a_H(u,v) \quad \forall u, v \in \tilde{V}_h^B.$$

We use a modified version of abstract scheme of [82].

We first denote the set of indices of Neumann subdomain by $\mathcal{N}(\Omega)$, and the set of ones of Dirichlet ones by $\mathcal{D}(\Omega)$. We next introduce local subspaces formed by functions defined only in respective subdomains as follows:

$$V_i^N = H_i X_h^B(\Omega_i) \;\; for \;\; i \in \mathcal{N}(\Omega)$$

and

$$V_j^D = \sum_{x \in \mathcal{V}(\Omega_j)} \text{span}\{\phi_{j,x}^\alpha\} \;\; for \;\; j \in \mathcal{D}(\Omega) \;\; and \;\; \alpha = (1,0),(0,1),(1,1),$$

where $\mathcal{V}(\Omega_j)$ is the set of all vertices of $\Omega_j \cap \Gamma$ and $\phi_{j,x}^\alpha$ is a nodal discrete biharmonic function which corresponds to α degree of freedom at the vertex x defined analogously as in Section 4.5.1. Here for bicubic mortar method, we have the following set of possible multi-indices $\{(0,0),(1,0),(1,0),(1,1)\}$. Note that from the definition x, a vertex of a subdomain of Dirichlet type, is the common end of two slaves, thus we see that the support of $\phi_{j,x}^\alpha$ is contained in $\overline{\Omega}_j$.

We next define $B_i^N : V_i^N \to \tilde{V}_h^B$ for $i \in \mathcal{N}(\Omega)$ and $B_j^D : V_j^D \to \tilde{V}_h^B$ for $j \in \mathcal{D}(\Omega)$. prolongation operators for which

$$\tilde{V}_h^B = \sum_{i \in \mathcal{N}(\Omega)} B_i^N V_i^N + \sum_{j \in \mathcal{D}(\Omega)} B_j^D V_j^D.$$

Let B_j^D, for $j \in \mathcal{D}(\Omega)$, be defined by $B_j^D u_j = (0, \ldots, 0, u_j, 0, \ldots, 0)$ and B_i^N, for $i \in \mathcal{N}(\Omega)$, we define by setting the values of degrees of freedom of $B_i^N u_i$ at all nodes of the masters and the vertices as follows:

- If $p \in \mathcal{V}(\Omega_k)$ or $p \in \gamma_{s,k,h}$ for $k \neq i$, then let
$$\partial^\alpha B_i^N u_i(p) = 0.$$
- Next if $p \in \partial\Omega_{i,h} \setminus \mathcal{V}(\Omega_i)$, then we set
$$\partial^\alpha B_i^N u_i(p) = \partial^\alpha u_i(p).$$
- For a vertex $x \in \mathcal{V}(\Omega_i) \cap \mathcal{V}(c_r)$, we define
$$\partial^\alpha B_i^N u_i(x) = \partial^\alpha u_i(x) \;\; for \;\; \alpha = (1,0),(0,1),(1,1),$$
$$B_i^N u_i(x) = (2/N_{c_r}) u_i(x),$$
where $N_{c_r} = \text{card}(\mathcal{V}(c_r))$. Here $N_{c_r} = 4$.

Note that, for a function $v = \{v_k\} \in \tilde{V}_h^B$, the values $v_k(x)$ for $x \in \mathcal{V}(c_r)$ are equal to each other, but $\partial^\alpha v_k(x)$ for $x \in \mathcal{V}(c_r)$ may be not equal to each other.

We also need a decomposition of V_i^N for $i \in \mathcal{N}(\Omega)$ into

$$V_i^N = V_{0,i}^N \oplus Z_i^N,$$

where $V_{0,i}^N$ is a subspace of V_i^N over which S_i is nonsingular. Here $S_i : V_i^N \to V_i^N$ is defined by

$$< S_i u, v >= a_{i,h}(u, v) \ \ \forall u, v \in V_i^N$$

and

$$Z_i^N = \{v \in V_i^N :< S_i v, w >= 0 \ \ \forall w \in V_{0,i}^N\}.$$

In our case, we define $V_{0,i}^N, i \in \mathcal{N}(\Omega)$ as follows:

$$V_{0,i}^N = \{v \in V_i^N : \ \partial^\alpha v(x) = 0 \ \ \forall x \in \mathcal{V}(\Omega_i), \ \ \alpha = (0,0), (1,1)\}.$$

We now define our algorithm. This method is a variant of ASM and is presented in terms of decomposition of $\tilde{V}_h^B$ into several subspaces and special local bilinear forms defined on these subspaces, see Section 1.4. The coarse space we define by

$$V_0^N = \sum_{i \in \mathcal{N}(\Omega)} \text{span}\{B_i^N Z_i^N\} + \sum_{j \in \mathcal{D}(\Omega)} \text{span}\{B_j^D V_j^D\}.$$

Note that in our case Z_i^N for $i \in \mathcal{N}(\Omega)$ and V_j^D for $j \in \mathcal{D}(\Omega)$ are of small dimensions equal to eight or twelve, respectively.

We now define local subspaces

$$V_i^{N,\dagger} = (I - T_0^N) B_i^N V_i^N, \quad i \in \mathcal{N}(\Omega),$$

where T_0^N is the orthogonal projection (in terms of S) onto V_0^N, i.e.

$$< S T_0^N u, v >=< Su, v > \quad \forall v \in V_0^N$$

An associated bilinear form is defined as follows

$$b_i(u, v) = a_{i,h}(\tilde{u}_0, \tilde{v}_0) =< S_i \tilde{u}_0, \tilde{v}_0 >, \ \ u, v \in V_i^{N,\dagger},$$

where $\tilde{u}_0, \tilde{v}_0$ are the unique elements of $V_{00,i}^N$ for which

$$u = (I - T_0^N) B_i^N \tilde{u}_0, \quad v = (I - T_0^N) B_i^N \tilde{v}_0.$$

The space $V_{00,i}^N \subset V_{0,i}^N$ is defined as

$$V_{00,i}^N = \{v \in V_{0,i}^N : < S_i v, w >= 0 \ \ \forall w \in V_{0,i}^N \cap Ker(I - T_0^N)B_i^N\}.$$

We next define $T_i^N : \tilde{V}_h^B \to V_i^{N,\dagger}$ for $i \in \mathcal{N}(\Omega)$ by

$$b_i(T_i^N u, v) = a_H(u, v) =< Su, v > \quad \forall v \in V_i^{N,\dagger}$$

and $T^N : \tilde{V}_h^B \to \tilde{V}_h^B$ by

$$T^N = T_0^N + \sum_{i \in \mathcal{N}(\Omega)} T_i^N.$$

Then we replace problem (4.35) by

$$T^N \tilde{u}_h^B = g, \tag{4.36}$$

where $g = g_0 + \sum_{i \in \mathcal{N}(\Omega)} g_i$ and $g_0 = T_0^N \tilde{u}_h^B$, $g_i = T_i^N \tilde{u}_h^B$.

We now state the main theorem of this subsection.

Theorem 4.5.3 *For any $u \in \tilde{V}_h^B$, holds*

$$< Su, u > \leq < ST^N u, u > \preceq (1 + \log(H/\underline{h}))^2 < Su, u >,$$

where $\underline{h} = \min_k h_k$ and $H = \max_k H_k$.

The proof follows from Theorem 4.5.4 and Lemma 4.5.5, see below.

Theorem 4.5.4 *For any $u \in \tilde{V}_h^B$, holds*

$$< Su, u > \leq < ST^N u, u > \leq \ (1 + \max_{i \in \mathcal{N}(\Omega)} C(i))\, \omega < Su, u >,$$

where

$$\omega = \max_{i \in \mathcal{N}(\Omega)} \ \sup_{v \in V_{00,i}^N \setminus \{0\}} \frac{< SB_i^N v, B_i^N v >}{< S_i v, v >}$$

and

$$C(i) = \text{card}\left(\{j \in \mathcal{N}(\Omega) : < SB_i^N u_i, B_j^N u_j > \neq 0 \ \ \forall u_i \in V_i^N, \ \forall u_j \in V_j^N\}\right).$$

This theorem is a slightly changed version of Theorem 3.4 in [82], so we omit the proof which can be rewritten in a similar way.

Implementation

We now make some remarks on the implementation of this method following [82], see Sections 3.2 and 3.3 there.

The value of $T_i^N u$ for the operator $T_i^N : \tilde{V}_h^B \to V_i^{N,\dagger}$ can be computed as follows: $T_i^N u = (I - T_0^N) B_i^N \tilde{u}_0$, where $\tilde{u}_0 \in V_{0,i}^N$ is the solution of

$$b_0(T_i^N u, v) = a_{i,h}(\tilde{u}_0, \tilde{v}_0) =< Su, (I - T_0^N) B_i^N \tilde{v}_0 >=$$

$$=< (B_i^N)^T (I - T_0^N)^T Su, \tilde{v}_0 > \quad \forall \tilde{v}_0 \in V_{0,i}^N,$$

where $v = (I - T_0^N) B_i^N \tilde{v}_0$. Note that we take the space $V_{0,i}^N$ instead of $V_{00,i}^N$, it changes u_0, but not $(I - T_0^N) B_i^N u_0$, what follows from the orthogonal decomposition of $V_{0,i}^N$ into $V_{00,i}^N \oplus Ker(I - T_0^N) B_i^N \cap V_{0,i}^N$.

Thus we can represent operator $T_i^N : \tilde{V}_h^B \to V_i^{N,\dagger}$ as

$$T_i^N = (I - T_0^N) B_i^N S_{i,0}^{-1} (B_i^N)^T (I - T_0^N)^T S = (I - T_0^N) B_i^N S_{i,0}^{-1} (B_i^N)^T S (I - T_0^N),$$

where $S_{i,0}$ is the restriction of S_i to $V_{0,i}^N$.

And finally the operator T^N equals

$$T^N = T_0^N + \sum_{i \in \mathcal{N}(\Omega)} (I - T_0^N) B_i^N S_{i,0}^{-1} (B_i^N)^T S (I - T_0^N).$$

In application, to solve the problem (4.36), we use CG method, but for the simplicity of presentation, we restrict ourselves to the Richardson iterations.

Let u^0 be an arbitrary vector. Then in first step, we compute $u^1 = u^0 - T_0^N (u^0 - \tilde{u}_h^B)$ that satisfies

$$T_0^N (u^1 - \tilde{u}_h^B) = 0.$$

Then the Richardson iteration is defined as follows:

$$u^{n+1} = u^n - \tau\, T^N (u^n - \tilde{u}_h^B) \quad n = 1, 2, \ldots$$

where τ is a parameter chosen according to the spectral properties of T.

Note that by induction, we get

$$T_0^N (u^{n+1} - \tilde{u}_h^B) = T_0^N (u^n - \tilde{u}_h^B) - \tau\, T_0^N (u^n - \tilde{u}_h^B) +$$

$$+\tau\, T_0^N(I - T_0^N) \sum_{i\in\mathcal{N}(\Omega)} B_i^N S_{i,0}^{-1}(B_i^N)^T S(I - T_0^N)(u^n - \tilde{u}_h^B) = 0.$$

Hence our iterative method can be represented by

$$u^{n+1} = u^n - \tau\,(I - T_0^N) \sum_{i\in\mathcal{N}(\Omega)} B_i^N S_{i,0}^{-1}(B_i^N)^T S(u^n - \tilde{u}_h^B) \quad n = 1, 2, \ldots$$

Thus we have to apply T_0^N only once per iteration.

4.5.4 ASM method for the mortar method with locally non-conforming Adini discretization

In this subsection, we present an ASM for solving the discrete problem (4.11) of Section 4.2.3, i.e. the discrete problem arising from discretization of the plate problem (4.1) by a mortar finite element method that locally in each subdomain uses nonconforming Adini element. The ASM is analogous to the one described in Section 4.5.1. We keep the same notations of subspaces and respective projections.

We first decompose each function $u \in V_h^A$ into two parts orthogonal in terms of $a_H(\cdot,\cdot)$ as follows

$$u = Pu + Hu,$$

where $Pu = (P_1u, \ldots, P_iu, \ldots, P_Nu)$ and $Hu = (H_1u, \ldots, H_iu, \ldots, H_Nu)$. Here the orthogonal projection $P_iu \in X_{0,h}^A(\Omega_i)$ and the discrete biharmonic part $H_iu \in X_h^A(\Omega_i)$ are defined by

$$a_{h,i}(P_iu, v) = a_{h,i}(u, v) \quad \forall v \in X_{0,h}^A(\Omega_i)$$

and

$$\begin{cases} a_{i,h}(H_iu, v) &= 0 & \forall v \in X_{0,h}^A(\Omega_i), \\ \partial^\alpha H_iu(p) &= \partial^\alpha u(p) & \forall p \in \partial\Omega_{i,h}, \quad |\alpha| \le 1, \end{cases}$$

where

$$X_{0,h}^A(\Omega_i) = \{v \in X_h^A(\Omega_i) : \ \partial^\alpha u(p) = 0 \ \ \forall p \in \partial\Omega_{i,h} \ \ |\alpha| \le 1\}.$$

We next define $\tilde{V}_h^A = \{Hu : u \in V_h^A\}$, i.e. the subspace of discrete biharmonic functions of V_h^A. We can decompose u_h^A, the solution of (4.11), into $u_h^A = Pu_h^A + Hu_h^A$. We can compute Pu_h^A solving N independent local problems. The discrete biharmonic part of u_h^A further denoted by $\tilde{u}_h^A = Hu_h^A$ is the solution of the following variational discrete problem

$$a_H(\tilde{u}_h^A, v) = f(v) \ \ \forall v \in \tilde{V}_h^A. \tag{4.37}$$

We describe ASM for solving this problem in terms of decomposition of the space $\tilde{V}_h^A$ into several subspaces, cf. Section 1.4.

Let define a coarse space as follows

$$V_0^A = \{u \in \tilde{V}_h^A : \begin{array}{l} v_{k|\gamma_{m,k}} = v_{l|\delta_{m,l}} \in P_3(\Gamma_{kl}), \;\; I_{h_k}\partial_n v_{k|\gamma_{m,k}} = \\ I_{h_l}\partial_n v_{l|\delta_{m,l}} \in P_1(\Gamma_{kl}) \;\; and \;\; \partial^\alpha v \;\; is \\ continuous \;\; at \;\; all \;\; crosspoints, \;\; for \;\; |\alpha| \le 1 \end{array} \}.$$

Here I_{h_k}, I_{h_l} are piecewise linear interpolants defined over h_k and h_l meshes of master $\gamma_{m,k} = \Gamma_{kl}$ and slave $\delta_{m,l}$, respectively.

We next define $V_{x,\alpha}^A$, a one-dimensional space associated with a multi-index α of length one and a vertex $x \in \mathcal{V}(\Omega_k)$. We define $V_{x,\alpha}^A = \text{span}\{\phi_{k,x}^\alpha\}$, where $\phi_{k,x}^\alpha \in \tilde{V}_h^A$ is the locally discrete biharmonic vertex function associated with α degree of freedom and a vertex $x \in \partial\Omega_k$, defined analogously as the one in Section 4.5.1.

We next define subspaces associated with masters as in Section 4.5.1. Let $\gamma_{m,k}$ be a master and $\delta_{m,l}$ its associated slave. Then let V_m^A be a space formed by all functions $v \in V_m^A$ for which

$$\partial^\alpha v(x) = 0 \;\; for \;\; x \in \left\{ \bigcup_{\gamma_s \subset \Gamma} \gamma_{s,h} \cup \mathcal{V} \right\} \setminus \gamma_{m,k,h}.$$

We remind that $\mathcal{V} = \bigcup_{s=1}^N \mathcal{V}(\Omega_s)$ is the set of the vertices of all subdomains contained in Γ. Note that V_m^A contains functions which have nonzero degrees of freedom only at nodes of $\gamma_{m,k,h}$ and by the mortar conditions at ones of $\delta_{m,l,h}$, and in $\Omega_{k,h}$ and $\Omega_{l,h}$.

We obtain the following decomposition:

$$\tilde{V}_h^A = V_0^A + \sum_{\gamma_m \subset \Gamma} V_m^A + \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} V_{x,\alpha}^A.$$

We next define operators $T_0^A : \tilde{V}_h^A \to V_0^A$, $T_m^A : \tilde{V}_h^A \to V_m^A$ and $T_{x,\alpha}^A : \tilde{V}_h^A \to V_{x,\alpha}^A$ as orthogonal projections (in terms of $a_h(\cdot,\cdot)$) onto respective subspaces.

We define $T^A : \tilde{V}_h^A \to \tilde{V}_h^A$ by

$$T^A = T_0^A + \sum_{\gamma_m \subset \Gamma} T_m^A + \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} T_{x,\alpha}^A.$$

Then we replace problem (4.37) by

$$T^A \tilde{u}_h^A = g, \tag{4.38}$$

where $g = g_0 + \sum_{\gamma_m \subset \Gamma} g_m + \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} g_{x,\alpha}$ and $g_0 = T_0^A \tilde{u}_h^A$, $g_m = T_m^A \tilde{u}_h^A$ and $g_{x,\alpha} = T_{x,\alpha}^A \tilde{u}_h^A$.

We now state the main result of this subsection.

Theorem 4.5.5 *For any $u \in \tilde{V}_h^A$, holds*

$$(1 + \log(H/\underline{h}))^{-2} a_H(u,u) \preceq a_H(T^A u, u) \preceq a_H(u,u),$$

where $\underline{h} = \inf_k h_k$ and $H = \max_k H_k$.

The proof is given in Section 4.5.6.

4.5.5 Technical tools

In this subsection, we present some technical lemmas.

We first consider one subdomain Ω_k with boundary $\partial\Omega_k = \bigcup \overline{\gamma}_{m,k} \cup \bigcup \overline{\delta}_{l,k}$, where first sum is over all masters $\gamma_{m,k}$ and the second one of all slaves contained in $\partial\Omega_k$.

For proof of the first lemma, see e.g. Section 4, Lemma 4.1 in [82].

Lemma 4.5.1 *If $u \in H^{1/2}(\partial\Omega_k)$ satisfies condition $u = 0$ on $\partial\Omega_k \setminus \mathcal{E}$ and $\|\partial_s u\|_{L^\infty(\mathcal{E})} \preceq h_k^{-1} \|u\|_{L^\infty(\mathcal{E})}$, then holds*

$$|u|^2_{H^{1/2}(\partial\Omega_k)} \preceq |u|^2_{H^{1/2}(\mathcal{E})} + (1 + \log(|\mathcal{E}|/h_k)) \|u\|^2_{L^\infty(\mathcal{E})},$$

where $|\mathcal{E}|$ is the diameter of this edge.

The second lemma gives a Sobolev like inequality. The proof of this lemma can be found e.g. in [107], see Lemma 4.2.2, p.45 there. It can also be proved using the discrete Sobolev inequality for piecewise linear functions, see Lemma 7, p.170 in [20], cf. also Lemma 4.2 in [82].

Lemma 4.5.2 *If $u \in X_h^{RH}(\Omega_k)$, $(X_h^B(\Omega_k), X_h^H(\Omega_k))$, then holds*

$$|u|^2_{W^{1,\infty}(\Omega_k)} \preceq (1 + \log(H_k/h_k)) \left(H_k^{-2} |u|^2_{H^1(\Omega_k)} + |u|^2_{H^2(\Omega_k)} \right).$$

From the previous lemma, we get the following corollary.

Corollary 4.5.1 *If $\gamma_{m,i} \subset \partial\Omega_i$ is a master and $\delta_{m,j} \subset \partial\Omega_j$ is the corresponding slave, then*

$$\|\nabla(u_i - u_j)\|^2_{L^\infty(\Gamma_{ij})} \preceq (1 + log(H/\underline{h})) \sum_{s=i,j} |u|^2_{H^2(\Omega_s)} \ \ \forall u \in V_h^{RH}.$$

Proof. We have

$$\|\nabla(u_i - u_j)\|^2_{L^\infty(\Gamma_{ij})} \preceq \|\partial_s(u_i - u_j)\|^2_{L^\infty(\Gamma_{ij})} + \|\partial_n(u_i - u_j)\|^2_{L^\infty(\Gamma_{ij})}.$$

Let $j_s : L^2(\Gamma_{ij}) \to P_s(\Gamma_{ij})$ be the L^2 orthogonal projection onto the space of polynomials of degree s. By the mortar conditions (4.12) and (4.13), we know that

$$j_1 u_{i\,|\gamma_{m,i}} = j_1 u_{j\,|\delta_{m,j}} \ \ and \ \ j_0 \partial_n u_{i\,|\gamma_{m,i}} = j_0 \partial_n u_{j\,|\delta_{m,j}},$$

where $u_{i\,|\gamma_{m,i}}$, $u_{j\,|\delta_{m,j}}$, $\partial_n u_{i\,|\gamma_{m,i}}$ and $\partial_n u_{j\,|\delta_{m,j}}$ equal the respective traces of u_i, u_j, $\partial_n u_i$ and $\partial_n u_j$ onto the master $\gamma_{m,i}$ and the slave $\delta_{m,j}$, respectively.

Let estimate the term associated with tangential derivative. We have

$$\|\partial_s u_i - \partial_s u_j\|^2_{L^\infty(\Gamma_{ij})} \preceq \|\partial_s u_i - (j_1 u_i)'\|^2_{L^\infty(\Gamma_{ij})} + \|\partial_s u_j - (j_1 u_j)'\|^2_{L^\infty(\Gamma_{ij})} \preceq$$

$$\|\partial_s u_i + w_1'\|^2_{L^\infty(\Gamma_{ij})} + \|(j_1(u_i + w_1)'\|^2_{L^\infty(\Gamma_{ij})} + \|\partial_s u_j + v_1'\|^2_{L^\infty(\Gamma_{ij})} + \|(j_1(u_j + v_1)'\|^2_{L^\infty(\Gamma_{ij})},$$

where w_1, v_1 are arbitrary linear polynomials.

Using Lemma 4.5.2, we can bound the first term by

$$\|\partial_s u_i + w_1'\|^2_{L^\infty(\Gamma_{ij})} \preceq (1 + \log(H_i/h_i))(\sum_{s=1}^{2} H_i^{2s-4} |u_i + w_1|^2_{H^s(\Omega_i)}).$$

Since j_1 is the L^2 orthogonal projection onto the space of linear polynomials, we obtain

$$\|(j_1(u_i + w_1)'\|^2_{L^\infty(\Gamma_{ij})} \preceq H_i^{-3} \|j_1(u_i + w_1)\|^2_{L^2(\gamma_{m,i})} \preceq \sum_{s=0}^{2} H_i^{2s-4} |u + w_1|^2_{H^s(\Omega_i)}.$$

We have used an inverse inequality over the master $\gamma_{m,i}$, the trace theorem, e.g. see Theorem 1.5.2.1, p.42 in [71], and a scaling argument.

Since w_1 was an arbitrary linear polynomial, we get by a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47],

$$|\partial_s u_i(x) - (j_1 u_i)'(x)|^2 \preceq (1 + \log(H_i/h_i)) |u|^2_{H^2(\Omega_i)}.$$

The terms associated with the slave are estimated in the same way. Finally, the term associated with the normal derivative, we estimate in a very similar way and get the same bound.

□

The next lemma states property of vertex discrete biharmonic functions defined in Section 4.5.1.

Lemma 4.5.3 *For a vertex $x \in \mathcal{V}(\Omega_k) \cap \mathcal{V}(c_r)$ and a multi-index α of length one, holds*

$$|\phi^{\alpha}_{k,x}|^2_{H^2_H(\Omega)} \preceq (1 + \log(H_k/\underline{h}_x)),$$

where $\underline{h}_x = \inf_j h_j$ for j for which $c_r = x$ is an end of Γ_{kj} and $\phi^{\alpha}_{k,x} \in \tilde{V}^{RH}_h$ was defined in Section 4.5.1.

Proof. Note that on each edge denoted by $\mathcal{E}$ (that can be a master or a slave), $Tr\ u_{|\mathcal{E}} = (u_{|\mathcal{E}}, \nabla u_{|\mathcal{E}})$ can be represented as

$$Tr\ u_{|\mathcal{E}} = (u_{|\mathcal{E}}, \partial_s u, \partial_n u).$$

We use this observation afterwards in this proof.

We can have three situations, the first one, where x is the common end of two slaves, of two masters, and of a slave and a master of Ω_k. Let consider the first one, i.e. $x = \overline{\delta}_{m,k} \cap \overline{\delta}_{l,k}$. Then $\phi^{\alpha}_{k,x}$ is nonzero only in Ω_k. We also have $Tr\ \phi^{\alpha}_{k,x} = 0$ on $\partial\Omega_k \setminus (\delta_{m,k} \cup \delta_{l,k} \cup \{x\})$. Additionally, we can represent $Tr\ \phi^{\alpha}_{k,x}$ as

$$Tr\ \phi^{\alpha}_{k,x} = Tr\ \phi^{\alpha,N}_{k,x} + (\psi_{0,m}, \psi'_{0,m}, \psi_{1,m}) + (\psi_{0,l}, \psi'_{0,l}, \psi_{1,l}),$$

where $\phi^{\alpha,N}_{k,x}$ is the standard biharmonic nodal function associated with the α-degree of freedom at the vertex x. We have

$$Tr\ \phi^{\alpha,N}_{k,x\,|\delta_m} = (\phi^{\alpha,N}_{k,x\,|\delta_m}, \phi'^{\alpha,N}_{k,x\,|\delta_m}, \partial_n \phi^{\alpha,N}_{k,x\,|\delta_m}).$$

The functions $\psi_{0,m}$, $\psi_{1,m}$ are defined as follows

$$\psi_{0,m} = -\Pi^1_{m,k}\phi^{\alpha,N}_{k,x|\delta_m}, \quad \psi_{1,m} = -\Pi^0_{m,k,1}\partial_n\phi^{\alpha,N}_{k,x|\delta_m}$$

on the slave $\delta_{m,k}$ and as equal to zero on $\partial\Omega_k \setminus \delta_{m,k}$. Here $\Pi^1_{m,k}$ was defined by (4.25) and $\Pi^0_{m,k,1}$ by (4.28). The functions $\psi_{0,l}$, $\psi_{1,l}$ are defined in the same way for the second slave $\delta_{l,k}$. This representation of $Tr\ \phi^{\alpha}_{k,x}$ follows from the mortar conditions (4.12) and (4.13).

Next by Lemma 4.4.6 and (4.31), we obtain

$$|\phi^{\alpha}_{k,x}|_{H^2_H(\Omega)} = |\phi^{\alpha}_{k,x}|_{H^2(\Omega_k)} \preceq |\nabla\phi^{\alpha}_{k,x}|_{H^{1/2}(\partial\Omega_k)} \preceq |\nabla\phi^{\alpha,N}_{k,x}|_{H^{1/2}(\partial\Omega_k)}+$$

$$+\|\psi'_{0,m}\|_{H^{1/2}_{00}(\delta_{m,k})} + \|\psi_{1,m}\|_{H^{1/2}_{00}(\delta_{m,k})} + \|\psi'_{0,l}\|_{H^{1/2}_{00}(\delta_{l,k})} + \|\psi_{1,l}\|_{H^{1/2}_{00}(\delta_{l,k})}.$$

The first term by the trace theorem, see e.g. Theorem 1.5.1.2, p.37 in [71], and the properties of standard nodal functions can be bounded by a constant.

The remaining terms we estimate as follows

$$\|\psi'_{0,m}\|^2_{H^{1/2}_{00}(\delta_{m,k})} \preceq h_k^{-3}\|\psi_{0,m}\|^2_{L^2(\delta_{m,k})} = h_k^{-3}\|\Pi^1_{m,k}\phi^{\alpha,N}_{k,x|\delta_m}\|^2_{L^2(\delta_{m,k})} \preceq$$

$$\preceq h_k^{-3}\|\phi^{\alpha,N}_{k,x|\delta_m}\|^2_{L^2(\delta_{m,k})} \leq (Const).$$

We have used an inverse inequality, Lemma 4.4.5 and properties of standard nodal functions of reduced HCT element. Analogously, we get

$$\|\psi_{1,m}\|^2_{H^{1/2}_{00}(\delta_{m,k})} \preceq h_k^{-1}\|\Pi^0_{m,k,1}\partial_n\phi^{\alpha,N}_{k,x|\delta_m}\|^2_{L^2(\delta_{m,k})} \preceq$$

$$\preceq h_k^{-1}\|\partial_n\phi^{\alpha,N}_{k,x|\delta_m}\|^2_{L^2(\delta_{m,k})} \leq (Const).$$

We have used an inverse inequality, Lemma 4.4.9 and properties of standard nodal functions.

The two remaining terms associated with the slave $\delta_{s,k}$ we estimate in exactly the same way.

Let now consider the case, where x is the common end of two masters $\gamma_{m,k}$ and $\gamma_{s,k}$. Let $\delta_{m,i}$ and $\delta_{s,j}$ be the slaves associated with the masters $\gamma_{m,k}$ and $\gamma_{s,k}$, respectively. Then $\phi^{\alpha}_{k,x}$ is nonzero in three subdomains Ω_k, Ω_i and Ω_j because $Tr\ \phi^{\alpha}_{k,x} = Tr\ \phi^{\alpha,N}_{k,x}$ on $\partial\Omega_k$, but also $Tr\ \phi^{\alpha}_{k,x|\delta_{m,i}}$ and $Tr\ \phi^{\alpha}_{k,x|\delta_{s,j}}$ are nonzero.

On $\delta_{m,i}$, we have

$$Tr\ \phi^{\alpha}_{k,x|\partial\Omega_i} = (\psi_{0,m,i}, \psi'_{0,m,i}, \psi_{1,m,i}),$$

where

$$\psi_{0,m,i} = \Pi^1_{m,i}\phi^{\alpha,N}_{k,x|\gamma_m}, \quad \psi_{1,m,i} = \Pi^0_{m,i,1}(\partial_n\phi^{\alpha,N}_{k,x|\gamma_m}),$$

and on the master $\gamma_{m,k}$

$$Tr\ \phi^{\alpha}_{k,x|\gamma_{m,k}} = Tr\ \phi^{\alpha,N}_{k,x|\gamma_{m,k}} = (\phi^{\alpha,N}_{k,x|\gamma_{m,k}}, \phi'^{\alpha,N}_{k,x|\gamma_{m,k}}, \partial_n\phi^{\alpha,N}_{k,x|\gamma_{m,k}}).$$

We remind that $\phi^{\alpha,N}_{k,x}$ is the standard nodal function associated with the h_k triangulation of Ω_k and the α degree of freedom at the vertex $x \in \mathcal{V}(\Omega_k)$. In the same way, we can introduce the similar functions $\psi_{0,s,j}$ and $\psi_{1,s,j}$ for the second master $\gamma_{s,k}$ and its associated slave $\delta_{s,j}$.

Thus by Lemma 4.4.6 and (4.31), we obtain

$$|\phi^{\alpha}_{k,x}|^2_{H^2_H(\Omega)} = \sum_{l=k,i,j} |\phi^{\alpha}_{k,x}|^2_{H^2(\Omega_l)} \preceq \sum_{l=k,i,j} |\nabla \phi^{\alpha}_{k,x}|^2_{H^{1/2}(\partial\Omega_l)} \preceq$$

$$\preceq |\nabla \phi^{\alpha,N}_{k,x}|^2_{H^{1/2}(\partial\Omega_k)} + \|\psi'_{0,m,i}\|^2_{H^{1/2}_{00}(\delta_{m,i})} +$$

$$+ \|\psi_{1,m,i}\|^2_{H^{1/2}_{00}(\delta_{m,i})} + \|\psi'_{0,s,j}\|^2_{H^{1/2}_{00}(\delta_{s,j})} + \|\psi_{1,s,j}\|^2_{H^{1/2}_{00}(\delta_{s,j})}.$$

The first term by the trace theorem and the properties of standard nodal functions can be bounded by a constant.

We now estimate the two terms associated with slave $\delta_{m,i}$. If $h_k \le h_i$, the proof can be carried out as in the previous case. Thus we assume that $h_i \le h_k$. Additionally, let $\phi^{\alpha,N}_{i,y|\delta_{m,i}}$ be a trace of the nodal function $\phi^{\alpha,N}_{i,y}$ associated with the h_i triangulation of Ω_i, the α degree of freedom and y, the vertex of Ω_i such that $y \in \mathcal{V}(c_r)$. Note that y and x occupy the same geometrical position of c_r.

By an inverse inequality and Lemma 4.4.5, we have

$$\|\psi'_{0,m,i}\|^2_{H^{1/2}_{00}(\delta_{m,i})} \preceq \|\partial_s \Pi^1_{m,i}(\phi^{\alpha,N}_{k,x|\gamma_m} - \phi^{\alpha,N}_{i,y|\delta_{m,i}})\|^2_{H^{1/2}_{00}(\delta_{m,i})} +$$

$$+ \|\partial_s \Pi^1_{m,i} \phi^{\alpha,N}_{i,y|\delta_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,i})} \preceq \|\partial_s(\phi^{\alpha,N}_{k,x|\gamma_m} - \phi^{\alpha,N}_{i,y|\delta_{m,i}})\|^2_{H^{1/2}_{00}(\delta_{m,i})} +$$

$$+ h_i^{-3} \|\phi^{\alpha,N}_{i,y|\delta_{m,i}}\|^2_{L^2(\delta_{m,i})}.$$

The second term we estimate by constant in the same way as before. The first one we estimate by Lemma 4.5.1 and get

$$\|\partial_s(\phi^{\alpha,N}_{k,x|\gamma_m} - \phi^{\alpha,N}_{i,y|\delta_{m,i}})\|^2_{H^{1/2}_{00}(\delta_{m,i})} \preceq |\partial_s(\phi^{\alpha,N}_{k,x|\gamma_m} - \phi^{\alpha,N}_{i,y|\delta_{m,i}})|^2_{H^{1/2}(\delta_{m,i})} +$$

$$+ (1 + \log(|\Gamma_{ki}|/h_i)) \|\partial_s(\phi^{\alpha,N}_{k,x|\gamma_m} - \phi^{\alpha,N}_{i,y|\delta_{m,i}})\|^2_{L^\infty(\delta_{m,i})}.$$

Then by properties of nodal functions associated with degrees of freedom of the first order, we estimate all these terms by constant and finally get

$$\|\psi'_{0,m,i}\|^2_{H^{1/2}_{00}(\delta_{m,i})} \preceq (1 + \log(|\Gamma_{ki}|/h_i)).$$

Using Lemma 4.4.9 instead of Lemma 4.4.5 and similar arguments, we can get

$$\|\psi_{1,m,i}\|^2_{H^{1/2}_{00}(\delta_{m,i})} \preceq (1 + \log(|\Gamma_{ki}|/h_i)).$$

The two analogous terms associated with the second master we estimate in the same way. The proof of the third case can be done combining arguments used in the proofs of the first and second cases.

□

Remark 4.5.3 *The statement of Lemma 4.5.3 is also true for analogously defined vertex functions for mortar methods that use locally HCT or piecewise bicubic elements. The proofs proceed in the same way.*

Next lemma states stability of interpolation (grid transitional) M_h^{RH} in H_H^2 seminorm.

Lemma 4.5.4 *For the operator $M_h^{RH} : \hat{V}_H^{RH} \to V_h^{RH}$ defined in Section 4.5.2, holds*

$$|M_h^{RH} u|_{H^2(\Omega_k)} \preceq |u|_{H^2(\Omega_k)} \quad \forall u \in \hat{V}_H^{RH}.$$

Proof. Let $u \in \hat{V}_H^{RH}$ and let $w = M_h^{RH} u_{|\Omega_k}$. We have $u \in P_3(K_j), j = 1, 2, 3,$, where K_j is a triangle formed by connecting vertices of Ω_k to its centroid.

Let $\tau \in T_h(\Omega_k)$. Then we have from definition of M_h^{RH}

$$|w|^2_{H^2(\tau)} \preceq \sum_{p \neq q} \text{diam}(\tau)^{-2} |u(p) - L_q^H u(p)|^2 + \sum_{p \neq q} \sum_{|\alpha|=1} |\partial^\alpha u(p) - \partial^\alpha u(q)|^2,$$

where the first sum is over all vertices of this element that differs from q which is a vertex of τ. The vertex q can be chosen in an arbitrary way. Here L_q^H is the linear Hermitian interpolant defined by $L_q^H f(x) = f(q) + \sum_{k=1}^2 f_{x_k}(q)(x_k - q_k)$ for a point $q = (q_1, q_2) \in \Re^2$.

We first consider a case, where $\tau \subset K_j$ for a triangle K_j. Then $u_{|\tau} \in P_3(\tau)$ and using a scaling argument, equivalence of all norms over finite dimensional space $P_3(\hat{\tau})$ for a reference triangle $\hat{\tau}$, and a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], we get

$$\text{diam}(\tau)^{-2} |u(p) - L_q^H u(p)|^2 \preceq |u|^2_{H^2(\tau)}$$

and

$$|\partial^\alpha u(p) - \partial^\alpha u(q)|^2 \preceq |u|^2_{H^2(\tau)}.$$

In other cases, i.e. $\tau \cap K_j \neq \emptyset$ for more than one triangle K_j, the situation is more complicated. Let assume that τ cuts through two triangles K_1 and K_2. Let $q \in K_1$ and $p_1, p_2 \in K_2$. We consider all terms, where p_1 appears. This nodal point is the common end of some triangles τ'. We assume that p_1 and q are vertices of a triangle τ_1 and τ_q, respectively, such that $\tau_1 \subset K_2$ and $\tau_q \subset K_1$. If not, we would have to consider a triangle S of diameter $\operatorname{diam}(S) \asymp \operatorname{diam}(\tau)$ such that $p_1 \in S \subset K_2 \cap \tau'$ for an element τ' which have p_1 as a vertex and S satisfies the shape regularity condition of $T_h(\Omega_k)$. The existence of such a triangle follows from the shape regularity of the triangulation. Then instead of the triangle τ_1, we would consider S. The same is valid for q and τ_q. Next we can conclude that for any linear polynomial w_1, we obtain

$$|u(p_1) - L_q^H u(p_1)|^2 \preceq |(u + w_1)(p_1)|^2 + |L_q^H (u + w_1)|^2 \preceq$$

$$\preceq \sum_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} \sum_{s=0}^{2} \operatorname{diam}(\tau')^{2s-2} |u + w_1|^2_{H^s(\tau')} \preceq \sum_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} \operatorname{diam}(\tau')^2 |u|^2_{H^2(\tau')}.$$

We have used the fact that $u_{|\tau_1} \in P_3(\tau_1)$ and $u_{|\tau_q} \in P_3(\tau_q)$, a scaling argument, and a quotient space argument for the region $\overline{G} = \bigcup_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} \overline{\tau}'$, see e.g. Theorem 3.1.1, p.115 in [47]. In the same manner, we get

$$|\partial^\alpha u(p_1) - \partial^\alpha u(q)|^2 \preceq \sum_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} |u|^2_{H^2(\tau')}.$$

The terms associated with p_2 we estimate in the same way.

There is one triangle τ_c such that cuts through all K_j. But the estimate of norm of w over this element can be done in the same manner as above. We now sum over all elements τ and get the desired estimate. □

The next lemma is necessary in the proof of Theorem 4.5.3. In the proof, we use similar arguments to those of Lemma 4.3 in [82].

Lemma 4.5.5 *Under the assumptions Theorem 4.5.3, see Section 4.5.3, holds*

$$\sup_{u \in V_{0,i}^N} \frac{< S B_i u, B_i u >}{< S_i u, u >} \preceq (1 + \log(H_i/\underline{h}_i))^2,$$

where $\underline{h}_i = \min\{h_j : \partial\Omega_i \cap \partial\Omega_j \neq \emptyset\}$.

Proof. We first note that $B_i u$ is nonzero only in Ω_i and subdomains that share a common edge with Ω_i, it follows from the fact that $u(x) = 0$ for a vertex x of Ω_i.

By the definition of B_i, Lemma 4.4.6 and (4.31), we have

$$a_H(B_i u, B_i u) \preceq \sum_{|\alpha|=1} \sum_{x\in\mathcal{V}(\Omega_i)} |\partial^\alpha u(x)|^2 \ |\nabla \phi^\alpha_{i,x}|^2_{H^2_H(\Omega)} +$$

$$+ \sum_{|\alpha|=1} \sum_{\partial\Omega_j\cap\partial\Omega_i=\Gamma_{ij}} |\partial^\alpha u - \sum_{x\in\mathcal{V}(\Omega_i)} \partial^\alpha u(x) \partial^\alpha \phi^\alpha_{i,x}|^2_{H^{1/2}(\partial\Omega_j)}.$$

By the trace theorem, see e.g. Theorem 1.5.1.2, p.37 in [71], Remark 4.5.3, and Lemma 4.5.2, we can estimate the first double sum. The second sum we can estimate proceeding in the same way as in the proof of Theorem 4.5.1, see below, and we finally get

$$a_H(B_i u, B_i u) \preceq (1 + \log(H_i/\underline{h}_i))^2 \sum_{s=0}^{2} H_i^{2s-4} |u|^2_{H^s(\Omega_i)} \preceq$$

$$\preceq (1 + \log(H_i/\underline{h}_i))^2 |u|^2_{H^2(\Omega_i)}.$$

We have used a scaling argument and the fact that the seminorm $|\cdot|_{H^2(\Omega_i)}$ and the norm $\|\cdot\|_{H^2(\Omega_i)}$ are equivalent over a subspace of H^2 formed by functions which are equal to zero at three or more points. Here these points are the vertices of Ω_i. $\square$

We now state some technical lemmas for the mortar Adini method.

First we define $\tilde{\Pi}^1_{m,j} : L^2(\delta_{m,j}) \to H^2_0(\delta_{m,j}) \cap W^{h_j}(\delta_{m,j})$, an auxiliary operator associated with the slave $\delta_{m,j} \subset \partial\Omega_j$. Here $W^{h_j}(\delta_{m,j})$ is the space of traces of functions which are in the bicubic finite element space defined on $T_h(\Omega_j)$, the h_j triangulation of Ω_j, i.e. is the space of C^1 continuous piecewise cubic (in elements of the h_j triangulation of $\delta_{m,j}$) functions. These functions are uniquely defined by the their values and values of their derivatives at all nodes of $\delta_{m,j,h}$. Thus we set the values of $\tilde{\Pi}^1_{m,j}$ as follows

$$\tilde{\Pi}^1_{m,j} u(p) = \Pi^0_{m,j,1} u(p) \quad \textit{and} \quad \frac{d}{dt}\tilde{\Pi}^1_{m,j} u(p) = 0 \quad \forall p \in \delta_{m,j,h},$$

where $\Pi^0_{m,j,1}$ is the mortar projection defined in (4.28).

Note that for $u \in V_h^A$ hold

$$\partial_n \mathcal{M}^A_j u_{j|\delta_{m,j}} = \tilde{\Pi}^1_{m,j} I_{h_i} \partial_n u_{i|\gamma_{m,i}}, \qquad \mathcal{M}^A_j u_{j|\delta_{m,j}} = u_{j|\delta_{m,j}} = \Pi^1_{m,j} u_{i|\gamma_{m,i}}, \tag{4.39}$$

what follows from the mortar conditions (4.9) and (4.10), and the definition of $\mathcal{M}^A_j$ the local equivalence mapping, see Definition 4.3.1. Here I_{h_i} are linear interpolants onto the h_i mesh of a master $\gamma_{m,i}$, and $\delta_{m,j}$ is the associated slave, and $\Pi^1_{m,j}$ is the mortar projection defined in (4.25).

Lemma 4.5.6 *For* $u \in [L^2(\delta_{m,j}), H^1_0(\delta_{m,j})]_s,\ s \in [0,1]$, *holds*

$$\|\tilde{\Pi}^1_{m,j} u\|_{[L^2(\delta_{m,j}),H^1_0(\delta_{m,j})]_s} \preceq \|u\|_{[L^2(\delta_{m,j}),H^1_0(\delta_{m,j})]_s},$$

where $[L^2(\delta_{m,j}), H^1_0(\delta_{m,j})]_s$ *is a Hilbertian interpolation space between* $L^2(\delta_{m,j})$ *and* $H^1_0(\delta_{m,j})$.

Proof. The proof follows from the respective stability properties of the operator $\Pi^0_{m,j,1}$, see Lemma 4.4.9, an obvious observation that

$$|\tilde{\Pi}^1_{m,j} u|^2_{H^s(\delta_{m,j})} \asymp |\Pi^0_{m,j,1} u|^2_{H^s(\delta_{m,j})} \quad s = 0, 1,$$

and an interpolation argument, e.g. see Proposition 12.1.5, p.279 in [38]. □

Corollary 4.5.2 *For* $u = \{u_k\} \in \tilde{V}^A_h$ *for which* $Tr\mathcal{M}^A_i u_i = 0$ *on* $\partial\Omega_i \setminus \gamma_{m,i}$ *and* $Tr\mathcal{M}^A_j u_j = 0$ *on* $\partial\Omega_j \setminus \delta_{m,j} = 0$, *we have*

$$|u_j|_{H^2_h(\Omega_j)} \preceq |\nabla \mathcal{M}^A_i u_i|_{H^{1/2}(\partial\Omega_i)} \asymp |u_i|_{H^2_h(\Omega_i)},$$

Proof. By Lemma 4.3.3, Lemma 4.4.6, (4.31) and (4.39), we obtain

$$|u_j|^2_{H^2_h(\Omega_j)} \leq |(\mathcal{M}^A_j)^\dagger Ext(\mathcal{M}^A_j u)|^2_{H^2_h(\Omega_j)} \preceq |\nabla \mathcal{M}^A_j u|^2_{H^{1/2}(\partial\Omega_j)} \preceq$$

$$\preceq \|\partial_s \mathcal{M}^A_j u_j\|^2_{H^{1/2}_{00}(\delta_{m,j})} + \|\partial_n \mathcal{M}^A_j u_j\|^2_{H^{1/2}_{00}(\delta_{m,j})} =$$

$$= \|\partial_s \Pi^1_{m,j} u_i\|^2_{H^{1/2}_{00}(\delta_{m,j})} + \|\tilde{\Pi}^1_{m,j} I_{h_i} \partial_n u_{i|\gamma_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,j})}.$$

The first term we estimate with the help of Lemma 4.4.5, (4.39) and the trace theorem, e.g. see Theorem 1.5.1.2, p.37 in [71], and get

$$\|\partial_s \Pi^1_{m,j} u_i\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq \|\partial_s u_i\|^2_{H^{1/2}_{00}(\gamma_{m,i})} \preceq |\mathcal{M}^A_i u|^2_{H^2(\Omega_i)}.$$

The second term we estimate as follows

$$\|\tilde{\Pi}^1_{m,j} I_{h_i} \partial_n u_{i|\gamma_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq \|I_{h_i} \partial_n u_{i|\gamma_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq$$

$$\preceq h_i^{-1} \|(I - I_{h_i})\partial_n \mathcal{M}_i^A u_i\|^2_{L^2(\delta_{m,j})} + \|\partial_n \mathcal{M}_i^A u_{i|\gamma_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,j})}.$$

We have used Lemma 4.5.6, the fact that $I_{h_i}\partial_n u_{i|\gamma_{m,i}} = I_{h_i}\partial_n \mathcal{M}_i^A u_{i|\gamma_m}$, and an inverse inequality. The second term we estimate by the trace theorem and the first term by a scaling argument, the trace theorem applied to each element of the h_i triangulation of $\gamma_{m,i}$, see e.g. Theorem 1.5.2.1, p.42 in [71], and a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], and get

$$\|\tilde{\Pi}^1_{m,j} I_{h_i}\partial_n u_{i|\gamma_{m,i}}\|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq |\mathcal{M}_i^A u|^2_{H^2(\Omega_i)}.$$

Finally, Lemma 4.3.3 yields the desired estimate.

The equivalence of the norm $|\nabla \mathcal{M}_i^A u|_{H^{1/2}(\partial\Omega_i)}$ and the seminorm $|\mathcal{M}_i^A u|_{H^2(\Omega_i)}$ follows from Lemma 4.3.3, Lemma 4.4.6, and (4.31), cf. beginning of this proof. □

The next lemma states property of coarse grid space V_0^A defined in Section 4.5.4.

Lemma 4.5.7 *For any $u_0 \in V_0^A \subset \tilde{V}_h^A$, a coarse grid function, holds*

$$a_{h,k}(u_0, u_0) \preceq \sum_{p\in\mathcal{V}(\Omega_k)} \sum_{|\alpha|\le 1} H_k^{2|\alpha|-2} |\partial^\alpha u_0(p)|^2,$$

where $\mathcal{V}(\Omega_k) \subset \Gamma$ is the set of all vertices of $\Omega_k \cap \Gamma$.

Proof. We first note that u_0 is uniquely defined in Ω_k by the values of respective degrees of freedom at all vertices. We can conclude that

$$u_0 = \sum_{p\in\mathcal{V}(\Omega_k)} \sum_{|\alpha|\le 1} \partial^\alpha u_0(p)\, \phi^{Coar}_{k,p,\alpha},$$

where $\phi^{Coar}_{k,p,\alpha}$ is a function in V_0^A such that

$$\partial^\beta \phi^{Coar}_{k,p,\alpha}(q) = \begin{cases} 0 & p \ne q \quad or \quad \beta \ne \alpha, \\ 1 & p = q \quad and \quad \beta = \alpha. \end{cases}$$

Thus to finish the proof it suffices to prove the bound of H_h^2 seminorm of these functions.

Let first consider such a function for a multi-index α and a vertex $p \in \partial\Omega_k$. Then by Lemma 4.3.3, Lemma 4.4.6, and (4.31), we have

$$|\phi^{Coar}_{k,p,\alpha}|^2_{H_h^2(\Omega_k)} \le |(\mathcal{M}_j^A)^\dagger Ext(\mathcal{M}_j^A \phi^{Coar}_{k,p,\alpha})|^2_{H_h^2(\Omega_j)} \preceq |\nabla \mathcal{M}_k^A \phi^{Coar}_{k,p,\alpha}|^2_{H^{1/2}(\partial\Omega_k)}.$$

We next note that the trace of a function which corresponds to a vertex p is nonzero only on two interfaces Γ_{kl} and Γ_{kj} for which p is their common end.

Let consider $\phi^{Coar}_{k,p,\alpha}$ for $\alpha = (0,0)$. We have that $\partial_n \mathcal{M}^A_k \phi^{Coar}_{k,p,(0,0)} = 0$ on both interfaces. We can conclude that $Tr\ \mathcal{M}^A_k \phi^{Coar}_{k,p,(0,0)}$ on $\partial\Omega_k$ is equal to the trace of a coarse Adini nodal function which is defined on the rectangle Ω_k and has all degrees of freedom equal to zero at all vertices of Ω_k except of the one associated with multi-index $(0,0)$ and the vertex p. This function has H^2_h seminorm over Ω_k bounded by H_k^{-1}.

Hence by the standard trace theorem, e.g. see Theorem 1.5.1.2, p.37 in [71], we have

$$|\phi^{Coar}_{k,p,\alpha}|^2_{H^2_h(\Omega_k)} \preceq H_k^{-2}, \quad \alpha = (0,0).$$

Let consider the function associated with $\alpha = (1,0)$ (the norm of the one corresponding to $(0,1)$ we can estimate in the same way). We have that one of interfaces, say Γ_{kl} is parallel to axis OX_1, and the second one Γ_{kj} to OX_2.

We can note that

$$Tr\ \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} = (\mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}, \partial_s \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}, 0) \ \ on\ \ \Gamma_{kl}$$

and

$$Tr\ \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} = (0, 0, \partial_n \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}) \ \ on\ \ \Gamma_{kj}.$$

We introduce $\tilde{\phi}^{Coar}_{k,p,\alpha} \in Tr\ H^2(\Omega_k)$, an auxiliary function defined on $\partial\Omega_k$, as follows

$$Tr\ \tilde{\phi}^{Coar}_{k,p,\alpha} = \begin{cases} Tr\ \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} & on \quad \partial\Omega_k \setminus \Gamma_{kj}, \\ (0, 0, I_{h_k} \partial_n \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}) & on \qquad \Gamma_{kj}. \end{cases}$$

Note that $I_{h_k} \partial_n \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}$ is the linear function equal to one at p and to zero at the other end of Γ_{kj} what follows from the definition of V^A_0, i.e if we assume that $p = (0,0)$ and $\Gamma_{kj} = [0, H_k]$, then we have $\partial_n \tilde{\phi}^{Coar}_{k,p,\alpha}(t) = 1 - t/H_k$ for $t \in \Gamma_{kj}$.

Thus we can conclude that

$$|\phi^{Coar}_{k,p,\alpha}|^2_{H^2_h(\Omega_k)} \preceq |\nabla \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} - \nabla \tilde{\phi}^{Coar}_{k,p,\alpha}|^2_{H^{1/2}(\partial\Omega_k)} + |\nabla \tilde{\phi}^{Coar}_{k,p,\alpha}|^2_{H^{1/2}(\partial\Omega_k)}.$$

Using a reference rectangle we can estimate the second term by a constant. Note that at all nodal points p^k_i of the h_k mesh of Γ_{kj}, we have $\mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha}(p^k_i) = \tilde{\phi}^{Coar}_{k,p,\alpha}(p^k_i)$. Thus the first term we further estimate as follows

$$|\nabla \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} - \nabla \tilde{\phi}^{Coar}_{k,p,\alpha}|^2_{H^{1/2}(\partial\Omega_k)} \preceq \|\partial_n \mathcal{M}^A_k \phi^{Coar}_{k,p,\alpha} - \partial_n \tilde{\phi}^{Coar}_{k,p,\alpha}\|^2_{H^{1/2}_{00}(\Gamma_{kj})} \preceq$$

$$\preceq h_k^{-1} \|\partial_n \mathcal{M}_k^A \phi_{k,p,\alpha}^{Coar} - \partial_n \tilde{\phi}_{k,p,\alpha}^{Coar}\|_{L^2(\Gamma_{kj})}^2 \preceq \sum_{p_i^k \in \Gamma_{kj}} |\partial_n \tilde{\phi}_{k,p,\alpha}^{Coar}(p_{i+1}^k) -$$

$$-\partial_n \tilde{\phi}_{k,p,\alpha}^{Coar}(p_i^k)|^2 \preceq \max_i |p_{i+1}^k - p_i^k|^2 H_k^{-2}(H_k/h_k) \preceq (h_k/H_k) \leq 1.$$

We have used an inverse inequality, the fact that $\partial_n \tilde{\phi}_{k,p,\alpha}^{Coar}$ is the linear function on Γ_{kj} that equals one at p and zero at the other end of this interface and that the number of p_i^k is bounded by $(Const)H_k/h_k$ what follows from the quasi-uniformity of the h_k triangulation of Γ_{kj}. □

The next corollary is analogous to Lemma 4.5.3.

Corollary 4.5.3 *For $\phi_{k,x}^{\alpha} \in \tilde{V}_h^A$, the vertex function corresponding to a vertex $x \in \mathcal{V}(c_r) \cap \mathcal{V}(\Omega_k)$ and a multi-index α of length one, holds*

$$|\phi_{k,x}^{\alpha}|_{H_H^2(\Omega)} \preceq (1 + \log(H_k/\underline{h}_x)),$$

where $\underline{h}_x = \inf_j h_j$ for j for which c_r is an end of Γ_{kj}. The function $\phi_{k,x}^{\alpha} \in \tilde{V}_h^A$ was defined in Section 4.5.4.

The proof follows the lines of the ones of Lemma 4.5.3 and Corollary 4.5.2, therefore we omit it.

4.5.6 Proofs of the main theorems of ASM methods

In this subsection, we give the proofs of Theorems 4.5.1, 4.5.2 and 4.5.5.

Proof. (Theorem 4.5.1) Using the general ASM framework, we have to check three key assumptions, see Theorem 1.4.1 in Section 1.4.

Assumption (iii)
Here it is satisfied with $\omega = 1$ because we have set $a_H(\cdot,\cdot)$ as our local bilinear forms for the subspaces.

Assumption (ii)
It is satisfied with a constant independent of the number of subdomains since the supports of functions from both local subspaces are contained only in two (for spaces associated with masters) or in several (for the vertex spaces) subdomains.

Assumption (i)
We have to prove that there is a decomposition of $u \in \tilde{V}_h^{RH}$

$$u = u^0 + \sum_{\gamma_m \subset \Gamma} u^m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}, \tag{4.40}$$

where $u^0 \in V_0^{(1)}$, $u^m \in V_m^{(1)}$ and $u_{x,\alpha} \in V_{x,\alpha}^{(1)}$ are such that

$$a_H(u^0,u^0) + \sum_{\gamma_m \subset \Gamma} a_H(u^m,u^m) + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} a_H(u_{x,\alpha},u_{x,\alpha})$$
$$\preceq (1+\log(H/\underline{h}))^2 a_H(u,u). \tag{4.41}$$

We now define this decomposition. We first define $u^0 \in V_0^{(1)}$. Note that it is defined by the values of all degrees of freedom of u^0 associated with c_r for a crosspoint c_r.

We set

$$u^0(c_r) = u(c_r) \;\; and \;\; \partial^\alpha u^0(c_r) = \sum_{x \in \mathcal{V}(c_r)} (1/N_{c_r}) \partial^\alpha u_k(x),$$

where N_{c_r} is the number of subdomains with a crosspoint c_r as a common point. We use the fact that if vertices $x \in \partial\Omega_k$ and $y \in \partial\Omega_l$ are in $\mathcal{V}(c_r)$, then $u_k(x) = u_l(y)$.

Let $w = u - u^0$. We next define $u_{x,\alpha} \in V_{x,\alpha}^{(1)}$ by

$$u_{x,\alpha} = \partial^\alpha w(x)\, \phi_{k,x}^\alpha, \;\; |\alpha| = 1.$$

Let $\tilde{w} = w - \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}$. Then we have

$$\partial^\alpha \tilde{w}(x) = 0 \;\; for \;\; x \in \mathcal{V} \;\; and \;\; |\alpha| \le 1.$$

Let $\gamma_{m,k}$ be a master, then we define $u^m \in V_m^{(1)}$ by setting the values of all degrees of freedom at the nodal points of $\gamma_{m,k,h}$. We set

$$\partial^\alpha u^m(p) = \partial^\alpha \tilde{w}(p) \;\; \forall p \in \gamma_{m,k,h}, \; |\alpha| \le 1.$$

Note that (4.40) is satisfied.

We first estimate $a_H(u^0,u^0)$. We have

$$a_H(u^0,u^0) \asymp \sum_{k=1}^{N} |u^0|^2_{H^2(\Omega_k)}.$$

We now estimate each term separately. By Lemma 4.4.6 and (4.31)

$$|u^0|^2_{H^2(\Omega_k)} = |u^0 - L_k u^0|^2_{H^2(\Omega_k)} \preceq |\nabla(u^0 - L_k u^0)|^2_{H^{1/2}(\partial\Omega_k)},$$

where $L_k u^0$ is the linear interpolant of u^0 defined by the values of u^0 at three arbitrary vertices of Ω_k. We will also use the fact that $L_k u^0 = L_k u = L_k u_k$.

By the trace theorem, see e.g Theorem 1.5.1.2, p.37 in [71], and the definition of $V_0^{(1)}$, we conclude that

$$|u^0|^2_{H^2(\Omega_k)} \preceq \sum_{q \in \mathcal{V}(\Omega_k)} (H_k^{-2}|u^0(q) - L_k u^0(q)|^2 + \sum_{|\alpha|=1} |\partial^\alpha u^0(q) - \partial^\alpha L_k u^0(q)|^2). \quad (4.42)$$

Let now consider one vertex q. The term from the first sum corresponding to this vertex, we estimate by the Sobolev inequality of continuous embedding $C^0 \subset H^2$, e.g. see Theorem 1.4.4.1, p.27 in [71], a scaling argument, and a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], and get

$$H_k^{-2}|u^0(q) - L_k u^0(q)|^2 = H_k^{-2}|u_k(q) - L_k u_k(q)|^2 \preceq |u|^2_{H^2(\Omega_k)}.$$

We next estimate a term from the second sum in (4.42). We also introduce the Hermitian linear interpolant L_q^H associated with the vertex q defined by $L_q^H f(x) = f(q) + f_{x_1}(x_1 - q_1) + f_{x_2}(x_2 - q_2)$ for a function $f \in C^1(\overline{\Omega}_k)$.

Then we have

$$|\partial^\alpha u^0(q) - \partial^\alpha L_k u^0(q)|^2 \preceq |\partial^\alpha u^0(q) - \partial^\alpha L_q^H u_k(q)|^2 + |\partial^\alpha L_q^H u_k(q) - \partial^\alpha L_k u_k(q)|^2 \preceq$$

$$\preceq \sum_{y \in \mathcal{V}(q)} |\partial^\alpha u_i(y) - \partial^\alpha u_k(q)|^2 + |\partial^\alpha L_q^H u_k(q) - \partial^\alpha L_k u_k(q)|^2. \quad (4.43)$$

Here $\mathcal{V}(q) = \mathcal{V}(c_r)$, for a crosspoint c_r, is the set of all vertices of substructures which geometrically coincides with the vertex q and with the crosspoint c_r. The second term we estimate by Lemma 4.5.2, the Sobolev inequality of continuous embedding $C^0 \subset H^2$, e.g. see Theorem 1.4.4.1, p.27 in [71], a scaling argument and, a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], and get

$$|\partial^\alpha L_q^H u_k(q) - \partial^\alpha L_k u_k(q)|^2 \preceq (1 + \log(H_k/h_k))|u|^2_{H^2(\Omega_k)}, \quad |\alpha| = 1.$$

The first sum in (4.43), we can bound by

$$\sum_{y \in \mathcal{V}(q)} |\partial^\alpha u_i(y) - \partial^\alpha u_k(q)|^2 \preceq \sum_{x \neq y} |\partial^\alpha u_s(x) - \partial^\alpha u_r(y)|^2 \preceq$$

$$\preceq \sum_{x \neq y} \|\nabla(u_s - u_r)\|^2_{L^\infty(\Gamma_{sr})},$$

where the last sum is taken over all pairs of geometrically coinciding vertices $x, y \in \mathcal{V}(q)$ such that x is an end of the master $\gamma_{m,s}$ and y is an end of the associated slave $\delta_{m,r}$.

Using Corollary 4.5.1 and summing first over all masters with one end as q, then over all vertices of Ω_k and adding the resulting estimates to the previous ones, we get

$$a_{h,k}(u^0, u^0) \preceq (1 + \log(H/\underline{h})) \sum_{\partial\Omega_i \cap \partial\Omega_k \neq \emptyset} |u_i|^2_{H^2(\Omega_i)}. \tag{4.44}$$

Summing over all subdomains yields

$$a_H(u^0, u^0) \preceq (1 + \log(H/\underline{h}))|u|^2_{H^2_H(\Omega)}. \tag{4.45}$$

We next consider one crosspoint c_r, a vertex $x \in \mathcal{V}(c_r) \cap \mathcal{V}(\Omega_k)$ and a multi-index α of length one and we estimate $a_H(u_{x,\alpha}, u_{x,\alpha})$ for $u_{x,\alpha} \in V^{(1)}_{x,\alpha}$. We have

$$a_H(u_{x,\alpha}, u_{x,\alpha}) \preceq |\partial^\alpha w(x)|^2 |\phi^\alpha_{k,x}|^2_{H^2(\Omega)} \preceq (1 + \log(H/\underline{h}))|\partial^\alpha u_k(x) - \partial^\alpha u^0(x)|^2.$$

We have used Lemma 4.5.3. Note that $u^0(x) = L_k u^0(x) = L_k u_k(x)$, where L_k is the linear interpolant defined by the values of a function at x and two other vertices of $\partial\Omega_k$. Then we have

$$|\partial^\alpha u_k(x) - \partial^\alpha u^0(x)|^2 \preceq |\partial^\alpha L_k u^0(x) - \partial^\alpha u^0(x)|^2 + |\partial^\alpha u_k(x) - \partial^\alpha L_k u_k(x)|^2$$

and the first term has already appeared in (4.42). Hence by the arguments that we needed to prove (4.44), we can conclude that

$$a_H(u_{x,\alpha}, u_{x,\alpha}) \preceq (1 + \log(H/\underline{h}))^2 \sum_{c_r \in \partial\Omega_i} |u|^2_{H^2(\Omega_i)}.$$

The sum is taken over all subdomains that have c_r as a vertex. Next summing over all vertices and all multi-indices of length one, we get

$$\sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} a_H(u_{x,\alpha}, u_{x,\alpha}) \preceq (1 + \log(H/\underline{h}))^2 |u|^2_{H^2_H(\Omega)}. \tag{4.46}$$

It remains to prove a bound for $a_H(u^m, u^m)$. Let consider one term associated with a master $\gamma_{m,i}$ and the corresponding slave $\delta_{m,j}$. Then $u^m \in V^{(1)}_m$ can be nonzero only over Ω_i and Ω_j. Thus by Lemma 4.4.6 and (4.31), we get

$$a_H(u^m, u^m) \preceq |\nabla u^m_i|^2_{H^{1/2}_{00}(\gamma_{m,i})} + |\nabla u^m_j|^2_{H^{1/2}_{00}(\delta_{m,j})} \preceq |\nabla u^m_i|^2_{H^{1/2}_{00}(\gamma_{m,i})}.$$

The last bound follows from Lemmas 4.4.5 and 4.4.9.

By Lemma 4.5.1, we obtain

$$a_H(u^m, u^m) \preceq |\nabla u_i^m|^2_{H^{1/2}(\gamma_{m,i})} + (1 + \log(H/h_i))\|\nabla u_i^m\|^2_{L^\infty(\gamma_{m,i})}. \tag{4.47}$$

The first term in (4.47), we can estimate by the trace theorem, e.g. see Theorem 1.5.1.2, p.37 in [71], and obtain

$$|\nabla u_i^m|^2_{H^{1/2}(\gamma_{m,i})} \preceq |u_i|^2_{H^2(\Omega_i)} + |u_i^0|^2_{H^2(\Omega_i)} + |u_{x,\alpha}|^2_{H^2(\Omega_i)} + |u_{y,\alpha}|^2_{H^2(\Omega_i)} \preceq$$

$$\preceq (1 + \log(H/\underline{h}))^2 \sum_{\partial\Omega_i \cap \partial\Omega_k \neq \emptyset} |u_k|^2_{H^2_h(\Omega_k)},$$

where x, y are the ends of the master. We have used the fact that these terms have already been estimated.

We now estimate the second term in (4.47). Note that we have $Tr\ u_{x,\alpha\,|\gamma_m} = \partial^\alpha w(x) Tr \phi^\alpha_{k,x\,|\gamma_m}$ and that $Tr \phi^\alpha_{k,x\,|\gamma_m} = Tr \phi^{\alpha,N}_{k,x\,|\gamma_m}$, where $\phi^{\alpha,N}_{k,x\,|\gamma_m}$ is the standard nodal function of reduced HCT element method associated with the α degree of freedom and the vertex x. Thus we have

$$\|\nabla u_{x,\alpha}\|^2_{L^\infty(\gamma_{m,i})} \preceq |\partial^\alpha w(x)|^2 \preceq (1 + \log(H/\underline{h})) \sum_{c_r \in \partial\Omega_i} |u|^2_{H^2_h(\Omega_i)}.$$

The last bound is obtained in the same way as above. The same result holds for the term corresponding to y, the second end of this master.

We next estimate $\|u_i - u_i^0\|^2_{W^{1,\infty}(\gamma_{m,i})}$. We again use the linear interpolant L_i defined by the values of a function at three vertices of Ω_i and can conclude that

$$\|u_i - u_i^0\|^2_{W^{1,\infty}(\gamma_{m,i})} \preceq \|u_i - L_i u_i\|^2_{W^{1,\infty}(\gamma_{m,i})} + \|u_i^0 - L_i u_i^0\|^2_{W^{1,\infty}(\gamma_{m,i})}.$$

We used the fact that $L_i u_i = L_i u_i^0$, what follows from the definition of u^0. The first term we can estimate by Lemma 4.5.2 and a quotient space argument, cf. Theorem 3.1.1, p.115 in [47], and the second one in a similar way to the one that we have used to prove (4.44). Finally, summing all these bounds, we get

$$a_H(u^m, u^m) \preceq (1 + \log(H/\underline{h}))^2 \sum_{\partial\Omega_i \cap \partial\Omega_k \neq \emptyset} |u_k|^2_{H^2(\Omega_k)}.$$

Summing over all masters yields

$$\sum_{\gamma_m \subset \Gamma} a_H(u^m, u^m) \preceq (1 + \log(H/\underline{h}))^2 |u|^2_{H^2_H(\Omega)}. \tag{4.48}$$

To get (4.41), we add the estimates (4.45), (4.46), and (4.48) and this ends the proof. □

Proof. (Theorem 4.5.2) We now use again the general ASM framework. We have to check three key assumptions, see Theorem 1.4.1 in Section 1.4. Some parts of this proof are the same or very similar to the one of Theorem 4.5.1.

Assumption (iii)
Here it is satisfied with $\omega = 1$ for all subspaces except for $V_0^{(2)}$ as in the proof of Theorem 4.5.1. For the coarse space $V_0^{(2)}$, we obtain

$$a_H(u,u) \preceq b_0^{(2)}(u,u) \quad \forall u \in V_0^{(2)},$$

what follows from Lemma 4.5.4.

Assumption (ii)
As in the proof of Theorem 4.5.1, it is satisfied with a constant.

Assumption (i)
We first define a decomposition of $u \in V_h^{RH}$

$$u = u^0 + \sum_{\gamma_m \subset \Gamma} u^m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha},$$

where $u^0 \in V_0^{(2)}$, $u^m \in V_m^{(2)}$ and $u_{x,\alpha} \in V_{x,\alpha}^{(2)}$ are such that

$$b_0^{(2)}(u^0,u^0) + \sum_{\gamma_m \subset \Gamma} a_H(u^m,u^m) + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} a_H(u_{x,\alpha},u_{x,\alpha}) \preceq$$
$$\preceq (1 + \log(H/\underline{h}))^2 a_H(u,u). \tag{4.49}$$

We set $u^0 = M_h^{RH} \hat{u}^0$, where $\hat{u}^0 \in \hat{V}_H^{RH}$ is defined by

$$\hat{u}^0(c_r) = u(c_r), \qquad \partial^\alpha \hat{u}^0(c_r) = \sum_{x \in \mathcal{V}(c_r)} (1/N_{c_r}) \partial^\alpha u_k(x),$$

where $N_{c_r} = \mathrm{card}(\mathcal{V}(c_r))$.

We next define $u_{x,\alpha} \in V_{x,\alpha}^{(2)}$ by

$$u_{x,\alpha} = (\partial^\alpha u - \partial^\alpha u^0)(x)\, \psi_{k,x}^{\alpha}.$$

Let $w = u - u^0 - \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}$.

We next decompose w into two parts $w = Pw + Hw$, see Section 4.5.1 and (4.30). Note that for all respective multi-indices α and nodes $p \in \partial\Omega_{k,h}$, we have $\partial^\alpha w(p) = \partial^\alpha Hw(p)$.

Let $\gamma_{m,k}$ be a master and $\delta_{m,l}$ the corresponding slave, then we define an auxiliary discrete biharmonic function $\tilde{u}^m \in V_m^{(2)}$ by setting its values of all degrees of freedom at all nodal points of $\gamma_{m,k,h}$. We set

$$\partial^\alpha \tilde{u}^m(p) = \partial^\alpha w(p) \ \ for \ \ p \in \gamma_{m,k,h}.$$

Finally, we define $u^m \in V_m^{(2)}$ as

$$u^m = \tilde{u}^m + (1/N_k)P_k w_k + (1/N_l)P_l w_l, \tag{4.50}$$

where N_i, $i = k, l$, is the number of edges Γ_{ij} contained in $\partial\Omega_i \cap \Gamma$ and $P_i w$ was defined in (4.30) and extended as zero onto other subdomains, i.e. we identify $P_i w$ with $(0, \ldots, 0, P_i w_i, 0, \ldots, 0)$.

Note that

$$u = u^0 + \sum_{\gamma_m \subset \Gamma} u^m + \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}.$$

We first estimate $b_0^{(2)}(u^0, u^0)$. By the definition of $V_0^{(2)} = M_h^{RH} \hat{V}_H^{RH}$ and using properties of coarse nodal functions of reduced HCT method, we have

$$b_0^{(2)}(u^0, u^0) = a(\hat{u}^0, \hat{u}^0) \preceq \sum_{k=1}^{N} \{ \sum_{q \in \mathcal{V}(\Omega_k)} (H_k^{-2} |\hat{u}^0(q) - L_k \hat{u}^0(p)|^2 +$$

$$+ \sum_{|\alpha|=1} |\partial^\alpha \hat{u}^0(q) - \partial^\alpha L_k \hat{u}^0(q)|^2) \},$$

where $L_k \hat{u}^0$ is the linear interpolant of $\hat{u}^0$ defined by the values of $\hat{u}^0$ at the vertices of Ω_k.

Proceeding in the same way as in the proof of Theorem 4.5.1, we obtain

$$b_0^{(2)}(u^0, u^0) \preceq (1 + \log(H/\underline{h})) |u|^2_{H_H^2(\Omega)}. \tag{4.51}$$

The bound for vertex functions we can get noting that the estimates of Lemma 4.5.3 are also valid for vertex functions $\psi_{k,x}^\alpha$ and further following the proof of Theorem 4.5.1. Hence

$$\sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} a_H(u_{x,\alpha}, u_{x,\alpha}) \preceq (1 + \log(H/\underline{h}))^2 |u|^2_{H_H^2(\Omega)}. \tag{4.52}$$

It remains to prove a bound for the sum of $a_H(u^m, u^m)$. Let consider one term associated with a master $\gamma_{m,i}$ and the corresponding slave $\delta_{m,j}$. Then $u^m \in V_m^{(2)}$ is nonzero only over Ω_i and Ω_j.

We have, see (4.50),

$$a_H(u^m, u^m) \preceq a_H(\tilde{u}^m, \tilde{u}^m) + \sum_{s=i,j} a_{s,h}(P_s w, P_s w) \leq a_H(\tilde{u}^m, \tilde{u}^m) + \sum_{s=i,j} a_{s,h}(w, w).$$

We have used the fact that P_s, $s = i, j$, are the orthogonal projections in terms of $a_{s,h}(\cdot, \cdot)$, $s = i, j$, respectively. Using the definition of w, we can estimate the second term by

$$a_{s,h}(w, w) \preceq a_{s,h}(u, u) + a_{s,h}(u^0, u^0) + \sum_{x \in \partial\Omega_s} \sum_{|\alpha|=1} a_{s,h}(u_{x,\alpha}, u_{x,\alpha}), \quad s = i, j.$$

Summing over all masters and utilizing Lemma 4.5.4, we obtain

$$\sum_{\gamma_m \subset \Gamma} a_H(u^m, u^m) \preceq a_H(u, u) + b_0^{(2)}(u^0, u^0) +$$

$$+ \sum_{k=1}^{N} \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} a_H(u_{x,\alpha}, u_{x,\alpha}) + \sum_{\gamma_m \subset \Gamma} a_H(\tilde{u}^m, \tilde{u}^m).$$

The second term is estimated by (4.51), the third term - the triple sum by (4.52), and the last term - the sum over all masters proceeding in a very similar way to that of the proof of Theorem 4.5.1. Finally, we obtain

$$\sum_{\gamma_m \subset \Gamma} a_H(u^m, u^m) \preceq (1 + \log(H/\underline{h}))^2 |u|^2_{H^2_H(\Omega)}. \tag{4.53}$$

Adding (4.51), (4.52), and (4.53), we get (4.49). The proof of Theorem 4.5.2 is completed. □

Proof. (Theorem 4.5.5) The lines of this proof are similar to that of Theorem 4.5.1. **Assumptions (ii) and (iii)** are satisfied with constant independent of H and all h_k, as in Theorem 4.5.1.

Assumption (i)
We have to prove the existence of decomposition of $u \in \tilde{V}_h^A$ that satisfies an inequality analogous to (4.41) in the proof of Theorem 4.5.1.

We set $u_0 \in V_0^A$ by

$$u^0(c_r) = u(c_r) \ \ and \ \ \partial^\alpha u^0(c_r) = \sum_{x \in \mathcal{V}(c_r)} (1/N_{c_r}) \partial^\alpha u_k(x),$$

where $N_{c_r} = \mathrm{card}(\{k : k \in \mathcal{N}(c_r)\}) = \mathrm{card}(\{x : x \in \mathcal{V}(c_r)\})$.

Let $w = u - u^0$. We next define $u_{x,\alpha} \in V^A_{x,\alpha}$ by

$$u_{x,\alpha} = \partial^\alpha w(x)\, \phi^\alpha_{k,x}$$

and introduce $\tilde{w} = w - \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}$.

Let $\gamma_{m,k}$ be a master, then $u^m \in V^A_m$ is defined by setting its values of all degrees of freedom at the nodal points of $\gamma_{m,k,h}$, as follows

$$\partial^\alpha u^m(p) = \partial^\alpha \tilde{w}(p) \quad \forall p \in \gamma_{m,k,h},\ |\alpha| \le 1.$$

Note that

$$u = u^0 + \sum_{\gamma_m \subset \Gamma} u^m + \sum_{k=1}^N \sum_{x \in \mathcal{V}(\Omega_k)} \sum_{|\alpha|=1} u_{x,\alpha}.$$

We now prove an estimate that is analogous to (4.41). Lemma 4.5.7 yields that

$$a_H(u^0,u^0) \preceq \sum_{k=1}^N \sum_{q \in \mathcal{V}(\Omega_k)} (H_k^{-2}|u^0(q) - L_k u^0(q)|^2 + \sum_{|\alpha|=1} |\partial^\alpha u^0(q) - \partial^\alpha L_k u^0(q)|^2).$$

Here $L_k u^0$ is the linear interpolant of u^0 defined by the values of u^0 at three arbitrary vertices of Ω_k. Note that $L_k \mathcal{M}^A_k u_k = L_k u_k = L_k u^0$, cf. Definition 4.3.1.

Let consider a vertex q. The term $H_k^{-2}|u^0(q) - L_k u^0(q)|^2$ corresponding to this vertex is estimated utilizing Lemma 4.3.3, the Sobolev inequality of continuous embedding $C^0 \subset H^2$, e.g. see Theorem 1.4.4.1, p.27 in [71], a scaling argument, and a quotient space argument, e.g. see Theorem 3.1.1, p.115 in [47], and we get

$$H_k^{-2}|u^0(q) - L_k u^0(q)|^2 = H_k^{-2}|\mathcal{M}^A_k u^0(q) - L_k \mathcal{M}^A_k u^0(q)|^2 \preceq |u|^2_{H^2_h(\Omega_k)}.$$

We next estimate one term from the second sum in which derivatives of the first order appear. We also introduce the Hermitian linear interpolant L^H_q associated with this vertex q and defined by $L^H_q f(x) = f(q) + f_{x_1}(x_1 - q_1) + f_{x_2}(x_2 - q_2)$ for a function $f \in C^1(\overline{\Omega}_k)$.

Note that $L^H_q \mathcal{M}^A_k u_k = L^H_q u_k$. We will use this fact below. We also have $\partial^\alpha L^H_q u_k(q) = \partial^\alpha u_k(q)$. Hence

$$|\partial^\alpha u^0(q) - \partial^\alpha L_k u^0(q)|^2 \preceq \sum_{y \in \mathcal{V}(q)} |\partial^\alpha u_i(y) - \partial^\alpha u_k(q)|^2 + |\partial^\alpha L^H_q u_k(q) - \partial^\alpha L_k u_k(q)|^2. \tag{4.54}$$

The second term is estimated by Lemma 4.3.3, Lemma 4.5.2, a scaling argument, and a quotient space argument, see e.g. Theorem 3.1.1, p.115 in [47], and it gives

$$|\partial^\alpha L_q^H u_k(q) - \partial^\alpha L_k u_k(q)|^2 \preceq (1 + \log(H_k/h_k))|u|^2_{H_h^2(\Omega_k)}.$$

The first sum in (4.54) is bounded by

$$\sum_{y \in \mathcal{V}(q)} |\partial^\alpha u_i(y) - \partial^\alpha u_k(q)|^2 \preceq \sum_{x \neq y} (|\partial_n u_i(x) - \partial_n u_j(y)|^2 + |\partial_s u_i(x) - \partial_s u_j(y)|^2),$$

where the sum is taken over all pairs of vertices $x, y \in \mathcal{V}(q)$ such that $x \in \mathcal{V}(\Omega_i)$ is an end of master $\gamma_{m,i}$ and $y \in \mathcal{V}(\Omega_j)$ is an end of the associated slave $\delta_{m,j}$, both vertices geometrically occupy the same place.

Let now consider one term at the right hand-side corresponding to a master $\gamma_{m,i}$. The terms in which the tangential derivative appears can be estimated as in the proof of Corollary 4.5.1. Let consider a term corresponding to normal derivatives. We introduce $j_0 : L^2(\Gamma_{ij}) \to P_0(\Gamma_{ij})$ the orthogonal projections onto the space of polynomials of degree zero. By the mortar condition (4.10), we have

$$j_0 I_{h_i} \partial_n u_{i,\gamma_{m,i}} = j_0 I_{h_j} \partial_n u_{j,\delta_{m,j}},$$

where I_{h_i} and I_{h_j} are piecewise linear interpolants defined on the h_i and h_j triangulations of Γ_{ij}, respectively.

Then we have

$$|\partial_n u_i(x) - \partial_n u_j(y)|^2 \preceq |\partial_n u_i(x) - j_0 I_{h_i} \partial_n u_i(x)|^2 + |\partial_n u_j(y) - j_0 I_{h_j} \partial_n u_j(y)|^2 \preceq$$

$$|\partial_n (u_i + w_1)(x)|^2 + |j_0 I_{h_i} \partial_n (u_i + w_1)(x)|^2 + |\partial_n (u_j + v_1)(y)|^2 + |j_0 I_{h_j} \partial_n (u_j + v_1)(y)|^2,$$

where w_1, v_1 are arbitrary linear polynomials.

By Lemma 4.5.2, we bound the first term as

$$|\partial_n (u_i + w_1)(x)|^2 = |\partial_n \mathcal{M}_i^A (u_i + w_1)(x)|^2 \preceq$$

$$\preceq (1 + \log(H_i/h_i)) \sum_{s=1}^{2} H_i^{2s-4} |\mathcal{M}_i^A (u_i + w_1)|^2_{H^s(\Omega_i)}.$$

Since j_0 is an L^2 orthogonal projection, we have

$$|j_0 I_{h_i} \partial_n (u_i + w_1)(x)|^2 \asymp H_i^{-1} \| j_0 I_{h_i} \partial_n (u_i + w_1) \|^2_{L^2(\gamma_{m,i})} \leq$$

$$\leq H_i^{-1}\|I_{h_i}\partial_n(u_i+w_1)\|^2_{L^2(\gamma_{m,i})} \asymp H_i^{-1}\|\partial_n(\mathcal{M}_i^A u_i+w_1)\|^2_{L^2(\gamma_{m,i})} \preceq$$

$$\preceq \sum_{s=1}^{2} H_i^{2s-4}|\mathcal{M}_i^A u_i+w_1|^2_{H^s(\Omega_i)}.$$

We have used the trace theorem. see e.g. Theorem 1.5.2.1, p.42 in [71], and the properties of $\mathcal{M}_i^A$, see Definition 4.3.1. Using the fact that w_1 is an arbitrary linear polynomial, a quotient space argument, see e.g. Theorem 3.1.1, p.115 in [47], and Lemma 4.3.3, we obtain

$$|\partial_n u_i(x) - j_0 I_{h_i}\partial_n u_i(x)|^2 \preceq (1+\log(H_i/h_i))|u|^2_{H_h^2(\Omega_i)}.$$

The terms associated with the slave are estimated in the same way. Next summing first over all masters which have q as one end and next over all vertices of Ω_k yields the following bound:

$$a_{h,k}(u^0,u^0) \preceq (1+\log(H/\underline{h})) \sum_{\partial\Omega_i\cap\partial\Omega_k\neq\emptyset} |u_i|^2_{H_h^2(\Omega_i)}.$$

Finally, summing over all subdomains yields

$$a_H(u^0,u^0) \preceq (1+\log(H/\underline{h}))|u|^2_{H_H^2(\Omega)}. \tag{4.55}$$

Following the proof of Theorem 4.5.1 and utilizing Corollary 4.5.3, we can bound the terms associated with vertices by

$$\sum_{k=1}^{N}\sum_{x\in\mathcal{V}(\Omega_k)}\sum_{|\alpha|=1} a_H(u_{x,\alpha},u_{x,\alpha}) \preceq (1+\log(H/\underline{h}))^2\sum_{k=1}^{N}|u|^2_{H_h^2(\Omega_i)}. \tag{4.56}$$

It remains to prove a bound for the sum of $a_H(u^m,u^m)$. Let consider one term associated with a master $\gamma_{m,i}$ and the corresponding slave $\delta_{m,j}$. Then $u^m \in V_m^A$ can be nonzero only in Ω_i and Ω_j. Hence by Corollary 4.5.2 and Lemma 4.5.1, we get

$$a_H(u^m,u^m) \preceq \|\nabla\mathcal{M}_i^A u^m\|^2_{H_{00}^{1/2}(\gamma_{m,i})} \preceq$$

$$\preceq |\nabla\mathcal{M}_i^A u^m|^2_{H^{1/2}(\gamma_{m,i})} + (1+\log(H/h_i))\|\nabla\mathcal{M}_i^A u_i^m\|^2_{L^\infty(\gamma_{m,i})}.$$

The first term can be estimated by the trace theorem, see e.g. Theorem 1.5.2.1, p.42 in [71], and the previously obtained estimates, cf. the proof of Theorem 4.5.1, and the second one can be estimated in a very similar way to that of proof of Theorem 4.5.1, additionally applying Lemma 4.3.3.

Finally, summing all these bounds, we get

$$a_H(u^m, u^m) \preceq (1 + \log(H/\underline{h}))^2 \sum_{\partial\Omega_i \cap \partial\Omega_k \neq \emptyset} |u_k|^2_{H^2_h(\Omega_k)}.$$

Summing over all masters yields

$$\sum_{\gamma_m \subset \Gamma} a_H(u^m, u^m) \preceq (1 + \log(H/\underline{h}))^2 |u|^2_{H^2_H(\Omega)}. \tag{4.57}$$

Adding (4.55), (4.56), and (4.57), we get the estimate needed in Assumption (i). The proof is completed. □

Bibliography

[1] Y. Achdou and Y. A. Kuznetsov. Substructuring preconditioners for finite element methods on nonmatching grids. *East-West J. Numer. Math.*, 3(1):1–28, 1995.

[2] Y. Achdou, Y. A. Kuznetsov, and O. Pironneau. Substructuring preconditioners for the Q_1 mortar element method. *Numer. Math.*, 71(4):419–449, 1995.

[3] Y. Achdou, Y. Maday, and O. Widlund. Méthode itérative de sous-structuration pour les éléments avec joints. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(2):185–190, 1996.

[4] Y. Achdou, Y. Maday, and O. B. Widlund. Iterative substructuring preconditioners for mortar element methods in two dimensions. *SIAM J. Numer. Anal.*, 36(2):551–580, 1999.

[5] Y. Achdou and F. Nataf. Preconditioners for the mortar method based on local approximations of the Steklov-Poincaré operator. *Math. Models Methods Appl. Sci.*, 5(7):967–997, 1995.

[6] R. A. Adams. *Sobolev spaces*, volume 65 of *Pure and Applied Mathematics*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[7] A. Adini and R. Clough. Analysis of plate bending problem by the finite element method. Report G. 7337, NSF, 1961.

[8] Z. Belhachmi. Nonconforming mortar element methods for the spectral discretization of two-dimensional fourth-order problems. *SIAM J. Numer. Anal.*, 34(4):1545–1573, 1997.

[9] F. Ben Belgacem. The mortar finite element method with lagrange multipliers. *Numer. Math.*, to appear.

[10] F. Ben Belgacem and Y. Maday. Non-conforming spectral method for second order elliptic problems in 3D. *East-West J. Numer. Math.*, 1(4):235–251, 1993.

[11] F. Ben Belgacem and Y. Maday. The mortar element method for three-dimensional finite elements. *RAIRO Modél. Math. Anal. Numér.*, 31(2):289–302, 1997.

[12] A. Berger, R. Scott, and G. Strang. Approximate boundary conditions in the finite element method. In *Symposia Mathematica, Vol. X (Convegno di Analisi Numerica, INDAM, Rome, 1972)*, pages 295–313, London, 1972. Academic Press.

[13] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*, volume 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1976.

[14] C. Bernardi, N. Debit, and Y. Maday. Couplage de méthodes spectrale et d'éléments finis: premiers résultats d'approximation. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(8):353–356, 1987.

[15] C. Bernardi, N. Debit, and Y. Maday. Coupling finite element and spectral methods: first results. *Math. Comp.*, 54(189):21–39, 1990.

[16] C. Bernardi and Y. Maday. Mesh adaptivity in finite elements by the mortar mesh. Technical report, Laboratoire d' Analyse Numerique, Universite Pierre et Marie Curie - Center National de la Recherche Scientifique, January 1995.

[17] C. Bernardi and Y. Maday. Raffinement de maillage en éléments finis par la méthode des joints. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(3):373–377, 1995.

[18] C. Bernardi, Y. Maday, and A. T. Patera. Domain decomposition by the mortar element method. In *Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992)*, volume 384 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 269–286. Kluwer Acad. Publ., Dordrecht, 1993.

[19] C. Bernardi, Y. Maday, and A. T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. In *Nonlinear partial*

differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), volume 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.

[20] P. E. Bjørstad, W. D. Gropp, and B. F. Smith. *Domain decomposition: Parallel multilevel methods for elliptic partial differential equations.* Cambridge University Press, Cambridge, 1996.

[21] P. E. Bjørstad and O. B. Widlund. Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.*, 23(6):1097–1120, 1986.

[22] F. Bogner, R. Fox, and L. Schmidt. The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas. In *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Ohio, 1965. Wright Patterson A.F.B.

[23] J.-F. Bourgat, R. Glowinski, P. Le Tallec, and M. Vidrascu. Variational formulation and algorithm for trace operator in domain decomposition calculations. In *Domain decomposition methods (Los Angeles, CA, 1988)*, pages 3–16. SIAM, Philadelphia, PA, 1989.

[24] D. Braess. *Finite elements.* Cambridge University Press, Cambridge, 1997. Theory, fast solvers, and applications in solid mechanics, Translated from the 1992 German original by Larry L. Schumaker.

[25] D. Braess, W. Dahmen, and W. Christian. A multigrid algorithm for mortar finite element method. Preprint 231, Fakultät für Mathematik der Ruhr-Universität Bochum, Bochum, March 1998.

[26] J. H. Bramble. *Multigrid methods.* Longman Scientific & Technical, Harlow, 1993.

[27] J. H. Bramble, J. E. Pasciak, and A. H. Schatz. The construction of preconditioners for elliptic problems by substructuring. I. *Math. Comp.*, 47(175):103–134, 1986.

[28] J. H. Bramble, J. E. Pasciak, and A. H. Schatz. The construction of preconditioners for elliptic problems by substructuring. II. *Math. Comp.*, 49(179):1–16, 1987.

[29] J. H. Bramble, J. E. Pasciak, and A. H. Schatz. The construction of preconditioners for elliptic problems by substructuring. III. *Math. Comp.*, 51(184):415–430, 1988.

[30] J. H. Bramble, J. E. Pasciak, and A. H. Schatz. The construction of preconditioners for elliptic problems by substructuring. IV. *Math. Comp.*, 53(187):1–24, 1989.

[31] J. H. Bramble, J. E. Pasciak, J. P. Wang, and J. Xu. Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.*, 57(195):1–21, 1991.

[32] J. H. Bramble and J. Xu. Some estimates for a weighted L^2 projection. *Math. Comp.*, 56(194):463–476, 1991.

[33] S. C. Brenner. Multigrid methods for nonconforming finite elements. In *Proceedings of the Fourth Copper Mountain Conference on Multigrid Methods (Copper Mountain, CO, 1989)*, pages 54–65, Philadelphia, PA, 1989. SIAM.

[34] S. C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite elements. In *Domain decomposition methods in scientific and engineering computing (University Park, PA, 1993)*, volume 180 of *Contemp. Math.*, pages 9–14. Amer. Math. Soc., Providence, RI, 1994.

[35] S. C. Brenner. A two-level additive Schwarz preconditioner for nonconforming plate elements. *Numer. Math.*, 72(4):419–447, 1996.

[36] S. C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65(215):897–921, 1996.

[37] S. C. Brenner. The condition number of the schur complement in domain decomposition. IMI Research Report 97:05, Department of Mathematics, University of South Carolina, 1997.

[38] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1994.

[39] S. C. Brenner and L.-Y. Sung. Balancing domain decomposition for nonconforming plate problem. IMA Preprint Series 1512, University of Minnesota, October 1997.

[40] X.-C. Cai. An optimal two-level overlapping domain decomposition method for elliptic problems in two and three dimensions. *SIAM J. Sci. Comput.*, 14(1):239–247, 1993.

[41] X.-C. Cai and M. Dryja. Domain decomposition methods for monotone nonlinear elliptic problems. In *Domain decomposition methods in scientific and engineering computing (University Park, PA, 1993)*, volume 180 of *Contemp. Math.*, pages 21–27. Amer. Math. Soc., Providence, RI, 1994.

[42] X.-C. Cai and O. B. Widlund. Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems. *SIAM J. Numer. Anal.*, 30(4):936–952, 1993.

[43] M. A. Casarin. *Schwarz preconditioners for spectral and mortar finite element methods with applications to incompressible fluids*. PhD thesis, Courant Institute, May 1996.

[44] M. A. Casarin and O. B. Widlund. A hierarchical preconditioner for the mortar finite element method. *Electron. Trans. Numer. Anal.*, 4(June):75–88 (electronic), 1996.

[45] T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, editors. *Domain decomposition methods*, Philadelphia, PA, 1989. Society for Industrial and Applied Mathematics (SIAM).

[46] T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, editors. *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1990. Society for Industrial and Applied Mathematics (SIAM).

[47] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 4 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.

[48] P. G. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of numerical analysis, Vol. II*, pages 17–351. North-Holland, Amsterdam, 1991.

[49] R. Clough and J. Tocher. Finite element stiffness matrices for analysis of plates in bending. In *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Ohio, 1965. Wright Patterson A.F.B.

[50] L. C. Cowsar. *Some domain decomposition and multigrid preconditioners for hybrid mixed finite elements*. PhD thesis, Rice University, April 1994.

[51] L. C. Cowsar, J. Mandel, and M. F. Wheeler. Balancing domain decomposition for mixed finite elements. *Math. Comp.*, 64(211):989–1015, 1995.

[52] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 7(R-3):33–75, 1973.

[53] Y.-H. De Roeck and P. Le Tallec. Analysis and test of a local domain-decomposition preconditioner. In *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow, 1990)*, pages 112–128. SIAM, Philadelphia, PA, 1991.

[54] Q. V. Dihn, R. Glowinski, and J. Périaux. Solving elliptic problems by domain decomposition methods with applications. In *Elliptic problem solvers, II (Monterey, Calif., 1983)*, pages 395–426. Academic Press, Orlando, FL, 1984.

[55] M. Dryja. A finite element-capacitance method for elliptic problems on regions partitioned into subregions. *Numer. Math.*, 44(2):153–168, 1984.

[56] M. Dryja. A method of domain decomposition for three-dimensional finite element elliptic problems. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 43–61. SIAM, Philadelphia, PA, 1988.

[57] M. Dryja. Additive Schwarz methods for elliptic mortar finite element problems. In *Modelling and optimization of distributed parameter systems (Warsaw, 1995)*, pages 31–50. Chapman & Hall, New York, 1996.

[58] M. Dryja. An iterative substructuring method for elliptic mortar finite element problems with a new coarse space. *East-West J. Numer. Math.*, 5(2):79–98, 1997.

[59] M. Dryja and W. Hackbusch. On the nonlinear domain decomposition method. *BIT*, 37(2):296–311, 1997.

[60] M. Dryja, B. F. Smith, and O. B. Widlund. Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. *SIAM J. Numer. Anal.*, 31(6):1662–1694, 1994.

[61] M. Dryja and O. B. Widlund. An additive variant of the schwarz alternating method for the case of many subregions. Tech. Report 339, Courant Institute of Math. Sciences, New York, 1987.

[62] M. Dryja and O. B. Widlund. Some domain decomposition algorithms for elliptic problems. In *Iterative methods for large linear systems (Austin, TX, 1988)*, pages 273–291. Academic Press, Boston, MA, 1990.

[63] M. Dryja and O. B. Widlund. Multilevel additive methods for elliptic finite element problems. In *Parallel algorithms for partial differential equations (Kiel, 1990)*, volume 31 of *Notes Numer. Fluid Mech.*, pages 58–69. Vieweg, Braunschweig, 1991.

[64] M. Dryja and O. B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, 1995.

[65] E. G. D'yakonov. *Optimization in solving elliptic problems.* CRC Press, Boca Raton, FL, 1996. Translated from the 1989 Russian original, Translation edited and with a preface by Steve McCormick.

[66] R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors. *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1988. Society for Industrial and Applied Mathematics (SIAM).

[67] R. Glowinski, Y. A. Kuznetsov, G. Meurant, J. Périaux, and O. B. Widlund, editors. *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1991. Society for Industrial and Applied Mathematics (SIAM).

[68] R. Glowinski, J. Périaux, Z.-C. Shi, and O. B. Widlund, editors. *Domain Decomposition Methods in Sciences and Engineering*, Strasbourg, France, 1997. John Wiley & Sons, Ltd.

[69] G. H. Golub and C. F. Van Loan. *Matrix computations.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, third edition, 1996.

[70] M. Griebel and P. Oswald. On the abstract theory of additive and multiplicative Schwarz algorithms. *Numer. Math.*, 70(2):163–180, 1995.

[71] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1985.

[72] W. Hackbusch. *Iterative solution of large sparse systems of equations*, volume 95 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994. Translated and revised from the 1991 German original.

[73] P. R. Halmos. *Measure Theory*. D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[74] D. E. Keyes, T. F. Chan, G. Meurant, J. S. Scroggs, and R. G. Voigt, editors. *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1992. Society for Industrial and Applied Mathematics (SIAM).

[75] D. E. Keyes and J. Xu, editors. *Domain decomposition methods in scientific and engineering computing*, volume 180 of *Contemporary Mathematics*, Providence, RI, 1994. American Mathematical Society.

[76] C. Lacour. Non-conforming domain decomposition method for plate and shell problems. In *Domain decomposition methods, 10 (Boulder, CO, 1997)*, pages 304–310. Amer. Math. Soc., Providence, RI, 1998.

[77] C. Lacour and Y. Maday. La méthode des éléments avec joint appliquée aux méthodes d'approximations discrete Kirchhoff triangles. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(10):1237–1242, 1998.

[78] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Academic Press, New York, 1968. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis.

[79] P. Lascaux and P. Lesaint. Some nonconforming finite elements for the plate bending problem. *Rev. Française Automat. Informat. Recherche Operationnelle Sér. Rouge Anal. Numér.*, 9(R-1):9–53, 1975.

[80] P. Le Tallec. Neumann-Neumann domain decomposition algorithms for solving 2D elliptic problems with nonmatching grids. *East-West J. Numer. Math.*, 1(2):129–146, 1993.

[81] P. Le Tallec, Y. H. De Roeck, and M. Vidrascu. Domain decomposition methods for large linearly elliptic three-dimensional problems. *J. Comput. Appl. Math.*, 34(1):93–117, 1991.

[82] P. Le Tallec, J. Mandel, and M. Vidrascu. A Neumann-Neumann domain decomposition algorithm for solving plate and shell problems. *SIAM J. Numer. Anal.*, 35(2):836–867 (electronic), 1998.

[83] P. Le Tallec, T. Sassi, and M. Vidrascu. Three-dimensional domain decomposition methods with nonmatching grids and unstructured coarse solvers. In *Domain decomposition methods in scientific and engineering computing (University Park, PA, 1993)*, volume 180 of *Contemp. Math.*, pages 61–74. Amer. Math. Soc., Providence, RI, 1994.

[84] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I, II.* Die Grundlehren der mathematischen Wissenschaften, Band 181,182. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth.

[85] P.-L. Lions. On the Schwarz alternating method. I. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 1–42. SIAM, Philadelphia, PA, 1988.

[86] L. Marcinkowski. Metoda dekompozycji obszaru dla quasiliniowych eliptycznych równań różniczkowych cząstkowych. Master's thesis, Uniwersytet Warszawski, Warszawa, 1994. In polish.

[87] L. Marcinkowski. Additive schwarz method for quasilinear elliptic partial differential equations. Tech. Report RW96-01 13, Institute of Applied Mathematics and Mechanics, Warsaw University, January 1996.

[88] L. Marcinkowski. Mortar element method for quasilinear elliptic boundary value problems. *East-West J. Numer. Math.*, 4(4):293–309, 1996.

[89] L. Marcinkowski. The mortar element method with locally nonconforming elements. *BIT*, 39(4):716–739, 1999.

[90] L. S. D. Morley. The triangular equilibrium problem in the solution of plate bending problems. *Aero. Quart.*, 23(19):149–169, 1968.

[91] S. V. Nepomnyaschikh. *Domain decomposition and Schwarz methods in a subspace for the approximate solution of elliptic boundary value problem.* PhD thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.

[92] A. Quarteroni, J. Périaux, Y. A. Kuznetsov, and O. B. Widlund, editors. *Domain decomposition methods in science and engineering*, volume 157 of *Contemporary Mathematics*, Providence, RI, 1994. American Mathematical Society.

[93] V. J. Rivkind and N. N. Ural'ceva. A priori estimates for quasilinear parabolic equations with discontinuous coefficients and their application in approximation methods. *Dokl. Akad. Nauk SSSR*, 185:271–274, 1969.

[94] V. J. Rivkind and N. N. Ural'ceva. Projection difference schemes for quasilinear elliptic and parabolic equations. *Vestnik Leningrad. Univ.*, 19 Mat. Meh. Astronom. Vyp. 4:66–69, 147, 1972.

[95] W. Rudin. *Functional analysis.* McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York, 1973.

[96] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[97] Y. Saad and M. H. Schultz. GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.*, 7(3):856–869, 1986.

[98] M. Sarkis. Multilevel methods for P_1 nonconforming finite elements and discontinuous coefficients in three dimensions. In *Domain decomposition methods in scientific and engineering computing (University Park, PA, 1993)*, volume 180 of *Contemp. Math.*, pages 119–124. Amer. Math. Soc., Providence, RI, 1994.

[99] M. Sarkis. *Schwarz preconditioners for elliptic problems with discontinuous coefficients using conforming and non-conforming elements.* PhD thesis, Courant Institute, September 1994.

[100] M. Sarkis. Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements. *Numer. Math.*, 77(3):383–406, 1997.

[101] H. A. Schwarz. *Gesammelte mathematisches abhandlungen. Vol. II.* Springer-Verlag, Berlin, 1890.

[102] O. B. Widlund. Iterative substructuring methods: algorithms and theory for elliptic problems in the plane. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 113–128. SIAM, Philadelphia, PA, 1988.

[103] O. B. Widlund. Iterative substructuring methods: the general elliptic case. In *Computational processes and systems, No. 6 (Russian)*, pages 76–89. "Nauka", Moscow, 1988.

[104] B. I. Wohlmuth. A residual based error estimator for mortar finite element discretizations. Report 370, Mathematical Institute, University of Augsburg, 1997.

[105] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34(4):581–613, 1992.

[106] A. Ženíšek. *Nonlinear elliptic and evolution problems and their finite element approximations*. Computational Mathematics and Applications. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1990. With a foreword by P.-A. Raviart.

[107] X. Zhang. *Studies in domain decomposition: Multilevel methods and the biharmonic Dirichlet problem*. PhD thesis, Courant Institute, September 1991.

[108] X. Zhang. Multilevel Schwarz methods. *Numer. Math.*, 63(4):521–539, 1992.