Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings I: General Concepts

Stephan Dahlke[a], Erich Novak[b], Winfried Sickel[c]

[a]Philipps-Universität Marburg, FB12 Mathematik und Informatik, Hans-Meerwein Straße, Lahnberge, 35032 Marburg, Germany, e-mail: dahlke@mathematik.uni-marburg.de

[b]Friedrich-Schiller-Universität Jena, Mathematisches Institut, Ernst-Abbe-Platz 2, 07743 Jena, Germany, e-mail: novak@math.uni-jena.de

[c]Friedrich-Schiller-Universität Jena, Mathematisches Institut, Ernst-Abbe-Platz 2, 07743 Jena, Germany, e-mail: sickel@math.uni-jena.de

We study the optimal approximation of the solution of an operator equation $\mathcal{A}(u) = f$ by four types of mappings: a) linear mappings of rank n; b) n-term approximation with respect to a Riesz basis; c) approximation based on linear information about the right hand side f; d) continuous mappings. We consider worst case errors, where f is an element of the unit ball of a Sobolev or Besov space $B_q^r(L_p(\Omega))$ and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain; the error is always measured in the H^s-norm. The respective widths are the linear widths (or approximation numbers), the nonlinear widths, the Gelfand widths, and the manifold widths. As a technical tool we also study the Bernstein numbers. Our main results are the following. If $p \geq 2$ then the order of convergence is the same for all four classes of approximations. In particular, the best linear approximations are of the same order as the best nonlinear ones. The best linear approximation can be quite difficult to realize as a numerical algorithm since the optimal Galerkin space usually depends on the operator and of the shape of the domain Ω. For $p < 2$ there is an essential difference, nonlinear approximations are better than linear ones. Also in this case it turns out, however, that linear information about the right hand side f is optimal. As a main theoretical tool we study best n-term approximation with respect to an optimal Riesz basis and related nonlinear widths. The main results are about approximation, not about computation. However, we also discuss consequences of the results for the numerical complexity of operator equations.