Weighted quadrature formulas and approximation by zonal function networks on the sphere

H. N. Mhaskar[a]

[a]Department of Mathematics, California State University, Los Angeles, California, 90032, U.S.A.

Let $q \geq 1$ be an integer, $\mathbb{S}^q$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$. A Zonal Function (ZF) network with an activation function $\phi : [-1, 1] \to \mathbb{R}$ and n neurons is a function on $\mathbb{S}^q$ of the form $\mathbf{x} \mapsto \sum_{k=1}^{n} a_k \phi(\mathbf{x} \cdot \xi_k)$, where a_k's are real numbers, ξ_k's are points on $\mathbb{S}^q$. We consider the activation functions ϕ for which the coefficients $\{\hat{\phi}(\ell)\}$ in the appropriate ultraspherical polynomial expansion decay as a power of $(\ell + 1)^{-1}$. We construct ZF networks to approximate functions in the Sobolev classes on the unit sphere embedded in a Euclidean space, yielding an optimal degree of approximation in terms of n, compared with the nonlinear n-widths of these classes. Our networks are explicitly defined as linear operators, and do not require training in the traditional sense. In the case of uniform approximation, our construction utilizes values of the target function at scattered sites. The approximation bounds are used to obtain error bounds on a very general class of quadrature formulas that are exact for the integration of high degree polynomials with respect to a weighted integral. The bounds are better than those expected from a straightforward application of the Sobolev embeddings.