SIAM J. IMAGING SCIENCES
Vol. 10, No. 4, pp. 1691–1723

Total Variation Denoising in l^1 Anisotropy*

Michał Łasica†, Salvador Moll‡, and Piotr B. Mucha†

Abstract. We aim at constructing solutions to the minimizing problem for the variant of the Rudin–Osher–Fatemi denoising model with rectilinear anisotropy and to the gradient flow of its underlying anisotropic total variation functional. We consider a naturally defined class of functions piecewise constant on rectangles (PCR). This class forms a strictly dense subset of the space of functions of bounded variation with an anisotropic norm. The main result shows that if the given noisy image is a PCR function, then solutions to both considered problems also have this property. For PCR data the problem of finding the solution is reduced to a finite algorithm. We discuss some implications of this result; for instance, we use it to prove that continuity is preserved by both considered problems.

Key words. denoising, Rudin–Osher–Fatemi model, total variation flow, anisotropy, rectangles, rectilinear polygons, piecewise constant solutions, regularity, tetris

AMS subject classifications. 68U10, 35K67, 35C05, 49N60, 35B65

DOI. 10.1137/16M1103610

1. Introduction.

In [20], the authors introduced the anisotropic version of the celebrated model by Rudin, Osher, and Fatemi [31] of total-variation-based noise removal from a corrupted image. The idea was to substitute the total variation term in the energy functional,

$$\int_\Omega |Du| + \frac{1}{2\lambda}\int_\Omega (u_0 - u)^2 \, \mathrm{d}\mathcal{L}^2, \tag{1}$$

by an anisotropic total variation term suitably chosen for a particular given image,

$$\int_\Omega |Du|_\varphi + \frac{1}{2\lambda}\int_\Omega (u_0 - u)^2 \, \mathrm{d}\mathcal{L}^2. \tag{2}$$

Although from the point of view of image processing it is most natural to consider the domain Ω being a rectangle, in principle, it can be any open bounded set with reasonably regular (e.g., Lipschitz) boundary, or the whole plane $\mathbb{R}^2$. The function $|\cdot|_\varphi \colon \mathbb{R}^2 \to [0,+\infty[$ encoding the anisotropy is assumed to be convex, positively 1-homogeneous, and such that $|\boldsymbol{x}|_\varphi > 0$ if $\boldsymbol{x} \neq 0$. Observe that (2) is a generalization of (1) for which $|\cdot|_\varphi$ is the Euclidean norm, $|\cdot|$.

*Received by the editors November 15, 2016; accepted for publication (in revised form) July 18, 2017; published electronically October 10, 2017.
http://www.siam.org/journals/siims/10-4/M110361.html
Funding: The work of the first author was supported by the grant of the National Science Centre, Poland 2014/13/N/ST1/02622. The work of the second author was supported by the Spanish MINECO and FEDER project MTM2015-70227-P as well as the Simons Foundation grant 346300 and the Polish Government MNiSW 2015-2019 matching fund.
†Institute of Applied Mathematics and Mechanics, University of Warsaw, Warszawa 02-097, Poland (lasica@mimuw.edu.pl, mucha@hydra.mimuw.edu.pl).
‡Departament d'Anàlisi Matemàtica, Universitat de València, Burjassot 46100, Spain (j.salvador.moll@uv.es).

In this case, the associated *Wulff shape*,

$$W_\varphi := \left\{ \boldsymbol{y} \in \mathbb{R}^2 \colon \boldsymbol{y} \cdot \boldsymbol{x} \le |\boldsymbol{x}|_\varphi \text{ for all } \boldsymbol{x} \in \mathbb{R}^2 \right\},$$

is exactly the unit ball with respect to the Euclidean distance. Because of that, minimizers of (1) give rise to convex shapes which are smooth (as is the Euclidean ball). If, instead, $|\cdot|_\varphi$ is a crystalline anisotropy (in the sense that the Wulff shape is a polygon), then minimizers of (2) give rise to convex shapes which are compatible with the Wulff shape and, therefore, are not smooth anymore.

This new approach has been successfully applied to most of the classical problems in image processing including denoising (see [32], [23], and [24]), cartoon extraction [8], inpainting [17], deblurring [18], and denoising and deblurring of 2-dimensional bar codes [19]. In most of these works, the chosen anisotropy is the l^1 norm in the plane, i.e., $|\boldsymbol{x}|_\varphi = |\boldsymbol{x}|_1 := |x_1| + |x_2|$. In this case, the corresponding Wulff shape is the unit ball with respect to the l^∞ distance, i.e., a square.

In the present paper, we focus on the case $|\boldsymbol{x}|_\varphi = |\boldsymbol{x}|_1$. We give an explicit expression for the minimizer when the corrupted image u_0 belongs to the class of functions piecewise constant on rectangles (PCR) (as in the case of applications), denoted by $PCR(\Omega)$ (see section 2.3 for precise definitions). The minimizer turns out to belong to $PCR(\Omega)$.

Let us briefly explain the algorithm for construction of the minimizer. Given a function $u_0 \in PCR(\Omega)$, we consider G_{u_0}, the minimal grid associated with the level sets of u_0 (which are rectilinear polygons; precise definitions are given in section 2.3). Then, we construct level sets F_k of u, starting with highest values of u, as follows:

- *Step* 1. Take as F_1 the largest minimizer of the following *Cheeger quotient* $\mathcal{J}_1$ among all possible rectilinear polygons E contained in $\overline{\Omega}$ subordinate to G_{u_0}:

$$\mathcal{J}_1(E) = \frac{\mathcal{H}^1(\partial E \cap \Omega) - \frac{1}{\lambda}\int_E u_0 \,\mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}.$$

- *Step* k. Denote $\check{F}_k = \bigcup_{i=1}^{k-1} F_i$. If $\overline{\Omega} = \check{F}_k$, stop. Otherwise, denote by F_k the largest minimizer of the following Cheeger quotient $\mathcal{J}_k$ among all possible rectilinear polygons contained in $\overline{\Omega \setminus \check{F}_k}$ subordinate to G_{u_0}:

$$\mathcal{J}_k(E) = \frac{\mathcal{H}^1(\partial E \cap \Omega \setminus \check{F}_k) - \mathcal{H}^1(\partial E \cap \partial \check{F}_k) - \frac{1}{\lambda}\int_E u_0 \,\mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}.$$

In each rectilinear polygon F_k of resulting decomposition of Ω, we define

$$u|_{F_k} = -\lambda \mathcal{J}_k(F_k). \tag{3}$$

In order to prove that u given by this algorithm is in fact the minimizer, we perform a rather involved mathematical analysis starting from the following observation (the Euler–Lagrange equation for (2)): u is a minimizer of (2) if and only if $\frac{u-u_0}{\lambda}$ belongs to negative subdifferential of the energy functional $TV_{\varphi,\Omega}$ on $L^2(\Omega)$ defined by

$$TV_{\varphi,\Omega}(u) = \begin{cases} \int_\Omega |Du|_\varphi & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

This anisotropic energy functional was studied in [28] in the cases that $\Omega = \mathbb{R}^N$ or Ω is a bounded, open, smooth subset of $\mathbb{R}^N$, coupled with Dirichlet boundary conditions. The author characterized the subdifferential as the set of elements of form $\operatorname{div} \boldsymbol{\xi}$ with $\boldsymbol{\xi}$ satisfying certain conditions (see Theorem 1). In the case that Ω is a rectilinear polygon and $u_0 \in PCR(\Omega)$, the condition that the solution u to

$$\min_{u\in BV(\Omega)} TV_{1,\Omega}(u) + \frac{1}{2\lambda}\int_\Omega (u-u_0)^2 \,\mathrm{d}\mathcal{L}^2 \tag{4}$$

also belongs to $PCR(\Omega)$ follows from finding a vector field $\boldsymbol{\xi} \in L^\infty(\Omega)$ such that $|\boldsymbol{\xi}|_\infty \le 1$ $\mathcal{L}^2$-a.e., a suitable compatibility condition on the jump set of u is satisfied (Lemma 1), and $\operatorname{div} \boldsymbol{\xi}$ is PCR. Note that once we know that there are such u and $\boldsymbol{\xi}$, and rectilinear polygons F_k are ordered level sets of u, (3) follows by averaging $u = u_0 + \lambda \operatorname{div} \boldsymbol{\xi}$ over each F_k.

We construct the vector field $\boldsymbol{\xi}$ together with u in Theorem 5 by minimizing the L^2 norm of the divergence over vector fields satisfying compatibility conditions. Then, in order to show that the divergence is PCR, we rely on an auxiliary result (Theorem 3) in which we prove that a certain anisotropic Cheeger-type functional on sets (related to the algorithm sketched above) is indeed minimized by a rectilinear polygon. In the proof of that result, an important point is that, due to the structure of the Cheeger quotient, we construct approximate minimizers that belong to a finite class of rectilinear polygons subordinate to G_{u_0}. As the set of characteristic functions of rectilinear polygons is not compact with respect to $TV_{1,\Omega}$, this finiteness is essential.

On the other hand, our analysis shows that any function PCR belongs to the domain of the subdifferential of $TV_{1,\Omega}$ (Lemma 1), and that this class of functions is preserved by the gradient descent flow of this functional,

$$\begin{cases} u_t \in -\partial TV_{1,\Omega}, \\ u|_{t=0} = u_0. \end{cases} \tag{5}$$

In principle, one could try to deduce this result from Theorem 5 by analyzing the discretization of (5) with respect to the time variable (which coincides with a sequence of problems of form (4) with $\lambda = \Delta t$). Instead, in Theorems 6 and 7 we do this directly, constructing the vector field that encodes the solution by means of a number of variational problems. This way we obtain a finite explicit algorithm for obtaining u, with a different structure than that in Theorem 5. In particular, we use the semigroup property of solutions to (5). The results in this case say that in any of a finite number of intervals between two subsequent time instances of *merging*, u_t is a fixed function in $PCR(\Omega)$. The exact form of u_t is (again) determined by solving a number of (different) Cheeger-type problems. In this case, the algorithm is slightly more complicated. Given a function $w \in PCR(\Omega)$, we consider again G_w as the minimal grid associated with the level sets of w. For each level set Q, we label each part of the boundary as positive, and we say that it belongs to ∂Q^+ (resp., negative, ∂Q^-) if the value of $u(t,\cdot)$ inside Q is higher than (resp., lower than) the value of the level set adjacent to this part of the boundary (thus, we define a consistent signature; see subsection 2.3). Then, we produce a decomposition of each level set into a family of rectilinear polygons and related consistent signature by means of an algorithm similar to the one for the minimizer. Finally, to each rectilinear polygon in the decomposition, a constant related to the signature is assigned. It is proved that u_t coincides with this exact constant (up to the next merging time, when the algorithm has to be reinitialized).

We stress that the problem of determining evolution is nontrivial, as at time instances of merging, *breaking* may occur along certain line segments, leading to expansion of the jump set of the solution.

As $\partial TV_{1,\Omega}$ is a monotone operator, for any datum $u_0 \in L^2(\Omega)$ and a sequence $u_{0,n} \in L^2(\Omega)$, $n = 1, 2, \dots$ such that $u_{0,n} \to u_0$ in $L^2(\Omega)$, solutions[1] u_n with datum $u_{0,n}$ converge to the solution with datum u_0. It is easy to check that $PCR(\Omega)$ is dense in $L^2(\Omega)$. In fact, $PCR(\Omega)$ is even strictly dense in $BV(\Omega)$ (in the sense of seminorm $\int_\Omega |\nabla u|_1$); see [12, Theorem 3.4]. Therefore, we do not only give the explicit solution when an initial datum belongs to $PCR(\Omega)$, but we provide an algorithm to compute the solution for any initial corrupted image with the most natural approximation to it (with functions belonging to the domain of the subdifferential).

The idea of a finite dimensional approximation of problem (4) based on PCR functions is already present in the literature. For instance, in [21], the authors prove that the solutions to (4) where the functional is replaced with its restrictions (discretizations) to functions piecewise constant on finer and finer grids (and datum u_0 replaced by its suitable projections) converge to the solution to (4). Our result implies that minimizers to those discrete problems are themselves actual solutions to (4) (with projected datum).

For a typical example, the space of PCR functions associated with the $(M+1) \times (N+1)$ Cartesian grid in a rectangle $\Omega = [0, M] \times [0, N]$ is isomorphic to $\mathbb{R}^{M \times N} = (u_{i,j};\ i = 1, \dots, M;\ j = 1, \dots, N)$. Our result shows that if u_0 belongs to this space, the functional $TV_{1,\Omega}$ in (4) can be changed to the discrete Ising-type functional

$$\sum \{|u_{i,j} - u_{k,l}| \colon i, k = 1, \dots, M;\ j, l = 1, \dots, N;\ |i-k| + |j-m| = 1\}. \tag{6}$$

without altering the result of minimization. This information means that one can use one of many efficient algorithms, such as graph-cut-based algorithms (see, e.g., [16, 25] and references therein) devised for minimizing discrete functionals involving terms of type (6) to obtain the exact (up to machine error) solution to (4). We note here that the algorithm proposed by us is of theoretical significance as a tool allowing us to prove Theorem 5.

Our approach allows us to prove some continuity results about solutions to (4) and (5). In particular, if Ω is a rectangle or the plane, we prove that if the datum u_0 admits a modulus of continuity of a certain form, then solutions do as well (Theorems 8 and 9). Analogous results were obtained in the isotropic case in [14, 15]. The method there involves considering the distance between level sets of solution. First, the authors show that the jump set of the solution is contained (up to an $\mathcal{H}^1$-negligible set) in the jump set of the initial datum. We point out that such a result is not true in our case since breaking may appear (see Example 1). Very recently, the continuity result for minimizers of (2) in convex domains has been proved to hold in the case of general anisotropies with different methods [27]. We still choose to include continuity results in this paper, mainly because the technique used here is very much

[1] Note that both problems (4) and (5) give rise to one parameter families of functions in $BV(\Omega)$ (in one case indexed by λ, in the other, by t). In many cases they coincide, at least for a range of the parameter (see Theorem 10). If we refer to *solutions* without precise context, we mean both solutions to (4) and (5).

different. The results are basically corollaries of Lemmas 5 and 6, which assert nonincrease of the maximal jump on any length scale for PCR data. The lemmas are of independent interest from the point of view of computations, as discrete versions of continuity estimates. Example 3 shows that continuity is not preserved in general, either by (4) or (5), if Ω is not convex.

At this point we note that our results can be seen as a generalization of observations concerning 1-dimensional (1D) problems with total variation. Indeed, in 1 dimension it is easy to see that piecewise constant data are preserved (and, consequently, that continuity is preserved); in fact, much more is true, as in that case $|\nabla u| \leq |\nabla u_0|$ as measures [10, Corollary 3.2]. The particular simplicity of the 1D case allows for very detailed description of solutions (see, e.g., [9, 26, 30]).

Finally, we remark that our description of solutions shares a connection on the formal level to ideas in [11] (see, also, other papers referenced therein). In [11], the authors show that solutions to gradient flow equations of typical discretizations of linear growth functionals are piecewise linear in time, and give an explicit expression involving a suitably defined nonlinear spectral decomposition. Having obtained finite reduction of (5) in Theorem 7, we can use it to recover spectral decomposition of $u_0 \in PCR(\Omega)$ with respect to (5) via [11, Conclusion 2 and Theorem 4.13].

The plan of the paper is as follows. In section 2, we give some notation and preliminaries on rectilinear geometry, bounded variation (BV) functions, L^2-divergence vector fields, as well as the anisotropic total variation and its gradient descent flow in L^2. In section 3, we study an auxiliary Cheeger-type problem in rectilinear geometry. Next, in sections 4 and 5 we give explicit solutions to (4) and (5) in the case that $u_0 \in PCR(\Omega)$, where Ω is a rectilinear polygon. In section 6, we transfer the results to the case $\Omega = \mathbb{R}^2$. The idealized setting of the whole plane $\mathbb{R}^2$ is convenient for discussing examples (from the point of view of images, it corresponds to a discrete feature set against a uniform background). Since the construction of solutions is similar to the previous cases, we only point out the main differences and state the results. Section 7 is devoted to the study of preservation of moduli of continuity. Finally, in section 8, we show the power of our approach by explicitly computing the solutions for some data, including the effects of bending and creation of singularities. After that we end up with some conclusions.

2. Notation and preliminaries.

2.1. Balls. By $B_\varphi(\boldsymbol{x}, r)$ we denote the ball in $\mathbb{R}^N$ with respect to the norm $|\cdot|_\varphi$, centered at $\boldsymbol{x}$ of radius r. For the ball with respect to the Euclidean norm, we write simply $B(\boldsymbol{x}, r)$. Symbols $B_\varphi(r), B(r)$ stand for balls centered at the origin.

2.2. Measures. Lebesgue and Bochner spaces. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. If $A \subset \mathbb{R}^N$ is a set of positive (possibly inifinite) $\mathcal{L}^N$ measure, we denote by $L^p(A)$, $1 \leq p \leq \infty$, the Lebesgue space of functions integrable with power p with respect to $\mathcal{L}^N$. On the other hand, if $A \subset \mathbb{R}^N$ has finite $\mathcal{H}^{N-1}$ measure (e.g., A is the boundary of a Lipschitz domain), $L^p(A)$ denotes the Lebesgue space of functions integrable with power p with respect to $\mathcal{H}^{N-1}$. We adopt similar notation for spaces $L^p(A, \mathbb{R}^k)$, $k = 2, 3, \ldots$. Whenever it is clear, we adopt the convention that an equality or inequality between two measurable functions holds

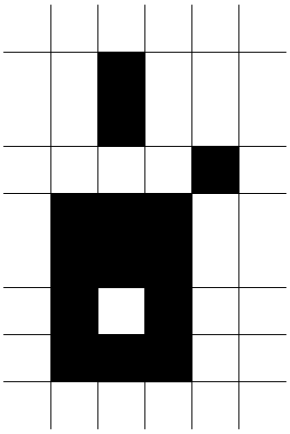


Figure 1. *An example of a rectilinear polygon F and a grid G. There holds $G = G(F)$ and $F \in \mathcal{F}(G)$.*

in the sense of Lebesgue spaces, i.e., almost everywhere with respect to the corresponding (implicitly specified) measure, unless otherwise stated.

If $]T_1, T_2[\subset \mathbb{R}$ and X is a Banach space, we denote by $L^p(]T_1, T_2[, X)$ the usual space of Bochner measurable functions $f\colon]T_1, T_2[\to X$ s.t. $\int_{T_1}^{T_2} \|f\|_X^p < \infty$. By $L^p_w(]T_1, T_2[, X)$ we denote the analogous space of weakly measurable functions (see [4, Chapter I]).

2.3. Rectilinear polygons. We denote by $\mathcal{R}$ the set of closed rectangles in the plane whose sides are parallel to the coordinate axes, and by $\mathcal{I}$, the set of all horizontal and vertical closed line segments of finite length in the plane.

We call $F \subset \mathbb{R}^2$ a *rectilinear polygon* if $F = \bigcup \mathcal{R}_F$ with a finite $\mathcal{R}_F \subset \mathcal{R}$. We denote by $\mathcal{F}$ the family of all rectilinear polygons. Similarly, we call $C \subset \mathbb{R}^2$ a *rectilinear curve* if $C = \bigcup \mathcal{I}_C$ with a finite $\mathcal{I}_C \subset \mathcal{I}$. We denote by $\mathcal{C}$ the set of all recilinear curves.

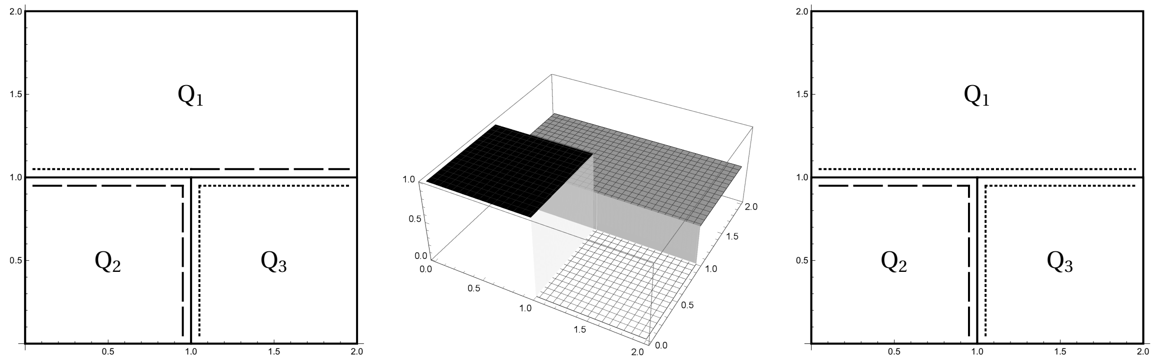
We call any finite set G of horizontal and vertical lines in the plane a *grid*. If F is a rectilinear polygon, we denote by $G(F)$ the minimal grid such that each side of F is contained in a line belonging to $G(F)$. If C is a rectilinear curve, we denote by $G(C)$ the minimal grid with the property that there exists $\mathcal{I}_C \subset \mathcal{I}$, $C = \bigcup \mathcal{I}_C$, such that all endpoints of intervals in $\mathcal{I}_C$ are vertices of $G(C)$.

Given a grid G, we denote

- by $\mathcal{I}(G)$ the set of line segments connecting adjacent vertices of G;
- by $\mathcal{R}(G)$ the set of rectangles whose sides belong to $\mathcal{I}(G)$;
- by $\mathcal{F}(G)$ the set of rectilinear polygons of form $\bigcup \mathcal{R}_F$ with a finite nonempty $\mathcal{R}_F \subset \mathcal{R}(G)$.

Note that all of the above are finite sets. For a generic example of the use of this notation, see Figure 1.

It is also convenient to introduce the following notions of partitions of rectilinear polygons and signatures for their boundaries. Let Ω be a rectilinear polygon. We say that a finite family $\mathcal{Q}$ of rectilinear polygons with disjoint interiors is a partition of Ω if $\Omega = \bigcup \mathcal{Q}$. If G is a grid, we say that a partition $\mathcal{Q}$ of Ω is subordinate to G if $\mathcal{Q} \subset \mathcal{F}(G)$. Let F be a rectilinear polygon and let G be a grid. We say that $(\partial F^+, \partial F^-) \in \mathcal{C} \times \mathcal{C}$ is a *signature* for ∂F (or for F) if $\partial F^\pm \subset \partial F$ and $\mathcal{H}^1(\partial F^+ \cap \partial F^-) = 0$. We say that a signature $(\partial F^+, \partial F^-)$ for ∂F is



(a) A consistent singature $\mathcal{S}\colon Q_i \mapsto (\partial Q_i^+, \partial Q_i^-)$ for $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$. Dashed lines denote ∂Q_i^+, dotted lines denote ∂Q_i^-, $i = 1, 2, 3$.

(b) The graph of a function $w \in PCR([0,2]^2)$ such that $\mathcal{S} = \mathcal{S}_w$.

(c) Not a consistent signature.

Figure 2. *Examples of signatures.*

subordinate to G if both $\partial F^\pm$ are subordinate to G. We say that

$$\mathcal{S}\colon \mathcal{Q} \ni Q \mapsto \mathcal{S}(Q) = (\partial Q^+, \partial Q^-) \in \mathcal{C} \times \mathcal{C}$$

is a *consistent signature* for $\mathcal{Q}$ if

- for each $Q \in \mathcal{Q}$, $\mathcal{S}(Q) = (\partial Q^+, \partial Q^-)$ is a signature for ∂Q and
- for each pair $Q, Q' \in \mathcal{Q}$, if $\boldsymbol{x} \in \partial Q^\pm \cap Q'$, then $\boldsymbol{x} \in \partial Q'^\mp$.

We say that a consistent signature $\mathcal{S}$ for $\mathcal{Q}$ is subordinate to G if for each $Q \in \mathcal{Q}$, $\mathcal{S}(Q)$ is subordinate to G.

Now, we give a precise definition of the class of functions PCR that we will work with. Let Ω be a rectangle and let $w \in L^1(\Omega)$. We write $w \in PCR(\Omega)$ if w has a finite number of level sets of positive $\mathcal{L}^2$ measure, and each one is a rectilinear polygon up to an $\mathcal{L}^2$-null set. We denote the family of level sets of a function $w \in PCR(\Omega)$ by $\mathcal{Q}_w$. $\mathcal{Q}_w$ is a partition of Ω in the sense of the definition in the previous paragraph.

Furthermore, we put $G_w = \bigcup_{Q \in \mathcal{Q}_w} G(Q)$, $\mathcal{I}_w = \mathcal{I}(G_w)$, $\mathcal{R}_w = \mathcal{R}(G_w)$, $\mathcal{F}_w = \mathcal{F}(G_w)$. Again, these are all finite sets.

Given $w \in PCR(\Omega)$ we define the *signature induced by* w, $\mathcal{S}_w\colon \mathcal{Q}_w \ni Q \mapsto (\partial Q^+, \partial Q^-)$, setting

$$\begin{aligned} \partial Q^+ &= \{\boldsymbol{x} \in \partial Q\colon \boldsymbol{x} \in Q' \in \mathcal{Q}_w, w|_{Q'} < w|_Q\}, \\ \partial Q^- &= \{\boldsymbol{x} \in \partial Q\colon \boldsymbol{x} \in Q' \in \mathcal{Q}_w, w|_{Q'} > w|_Q\} \end{aligned}$$

for each $Q \in \mathcal{Q}_w$. Here and in many other places we abuse notation slightly, identifying the constant function $w|_Q$ with its value. The signature induced by w is a consistent signature for $\mathcal{Q}_w$ subordinate to G_w (see Figure 2).

2.4. Functions of BV and sets of finite perimeter. We use standard notation and concepts related to BV functions as in [2]; in particular, given $u \in BV(\Omega)$, we write $\nabla u \mathcal{L}^N$

and $D^s u$ for the absolutely continuous and singular part of Du with respect to the Lebesgue measure $\mathcal{L}^N$, $u^{\pm}(x)$ for the lower and upper approximate limits of u at $x \in \Omega$ and J_u for its jump set, i.e., the set of points where $u^+ \neq u^-$. Finally, $\frac{Du}{|Du|}$ denotes the Radon–Nikodym derivative of Du with respect to its total variation $|Du|$.

The family $PCR(\Omega)$ introduced in the previous subsection is a linear subspace of $BV(\Omega)$. If $w \in PCR(\Omega)$, we have

$$J_w = \bigcup_{Q \in \mathcal{Q}_w} \partial Q^+ = \bigcup_{Q \in \mathcal{Q}_w} \partial Q^-, \qquad w^{\pm}|_{\partial Q^{\pm}} = w|_Q\,,$$

where $Q \mapsto (\partial Q^+, \partial Q^-)$ is the signature induced by w. Furthermore, we have

$$|Dw| = \left(w^+ - w^-\right) \mathcal{H}^1 \llcorner J_w = \bigcup_{Q \in \mathcal{Q}_w} w|_Q \left(\mathcal{H}^1 \llcorner \partial Q^+ - \mathcal{H}^1 \llcorner \partial Q^-\right). \tag{7}$$

Given an open set $\Omega \subseteq \mathbb{R}^N$ and a Lebesgue measurable subset E of $\mathbb{R}^N$, we say that E has finite perimeter in Ω if $\chi_E \in BV(\Omega)$ and we write $\mathrm{Per}(E,\Omega) := |D\chi_E|(\Omega)$. If E has finite perimeter in $\mathbb{R}^N$, we write $\mathrm{Per}(E) := \mathrm{Per}(E, \mathbb{R}^N)$.

If E is a set of finite perimeter in $\mathbb{R}^N$, the jump set of χ_E is $\mathcal{H}^{N-1}$-equivalent to the *reduced boundary* $\partial^* E$ defined by the following.[2] We say a point $\boldsymbol{x} \in \mathbb{R}^N$ belongs to $\partial^* E$ if $|D\chi_E|(B(\boldsymbol{x},\varrho)) > 0$ for all $\varrho > 0$ and quantity $\frac{D\chi_E(B(\boldsymbol{x},\varrho))}{|D\chi_E|(B(\boldsymbol{x},\varrho))}$ has a limit that belongs to $\mathbb{S}^{N-1}$ as $\varrho \to 0^+$. If these conditions hold, we denote this limit by $\boldsymbol{\nu}^E(\boldsymbol{x})$. There holds

$$\partial^* E \subset \partial^{\frac{1}{2}} E = \left\{ \boldsymbol{x} \in \mathbb{R}^N : \lim_{\varrho \to 0^+} \frac{\mathcal{L}^N(B(\boldsymbol{x},\varrho) \cap E)}{\mathcal{L}^N(B(\boldsymbol{x},\varrho))} = \frac{1}{2} \right\},$$

also $\mathcal{H}^{N-1}(\partial^{\frac{1}{2}} E \setminus \partial^* E) = 0$ and $\mathcal{H}^{N-1}$-almost every point in $\mathbb{R}^N$ is either a Lebesgue point for χ_E or belongs to $\partial^* E$.

2.5. Traces of L^2-divergence vector fields. We consider the space

$$X_\Omega = \left\{ \boldsymbol{z} \in L^\infty\left(\Omega, \mathbb{R}^N\right) : \ \operatorname{div} \boldsymbol{z} \in L^2(\Omega) \right\}. \tag{8}$$

In [3, Theorem 1.2], the weak trace on the boundary of a bounded Lipschitz domain Ω of the normal component of $\boldsymbol{z} \in X_\Omega$ is defined. Namely, it is proved that the formula

$$\left\langle \left[\boldsymbol{z}, \boldsymbol{\nu}^\Omega\right], \rho \right\rangle := \int_\Omega \rho \operatorname{div} \boldsymbol{z} \, \mathrm{d}\mathcal{L}^N + \int_\Omega \boldsymbol{z} \cdot \nabla \rho \, \mathrm{d}\mathcal{L}^N \qquad \left(\rho \in C^1\left(\overline{\Omega}\right)\right) \tag{9}$$

defines a linear operator $[\cdot, \boldsymbol{\nu}^\Omega] : X_\Omega \to L^\infty(\partial\Omega)$ such that

$$\left\| \left[\boldsymbol{z}, \boldsymbol{\nu}^\Omega\right] \right\|_{L^\infty(\partial\Omega)} \leq \|\boldsymbol{z}\|_{L^\infty(\Omega)} \tag{10}$$

for all $\boldsymbol{z} \in X_\Omega$ and $[\boldsymbol{z}, \boldsymbol{\nu}^\Omega]$ coincides with the pointwise trace of the normal component if $\boldsymbol{z}$ is smooth.

[2] This is the one point where we choose the notation as in, e.g., [22, 7] over [2].

2.6. The anisotropic total variation. The anisotropic perimeter. We recall here the notion of anisotropic total variation introduced in [1]. Given an open set $\Omega \subseteq \mathbb{R}^N$, a norm $|\cdot|_\varphi$ on $\mathbb{R}^N$, and a function $u \in L^2(\Omega)$, we define

$$TV_{\varphi,\Omega}(u) := \sup\left\{ \int_\Omega u \operatorname{div} \boldsymbol{\eta} \, \mathrm{d}\mathcal{L}^N : \boldsymbol{\eta} \in C_c^1\left(\Omega, \mathbb{R}^N\right),\ |\boldsymbol{\eta}|_\varphi^* \leq 1 \right\},$$

where $|\cdot|_\varphi^*$ denotes the dual norm associated with $|\cdot|_\varphi$. This is a proper, lower semicontinuous functional on $L^2(\Omega)$ with values in $[0,\infty]$. We have $TV_{\varphi,\Omega}(u) < +\infty$ if and only if $u \in BV(\Omega)$, in which case we use notation $|Du|_\varphi(\Omega) = TV_{\varphi,\Omega}(u)$. This is an equivalent seminorm on $BV(\Omega)$.

In the analysis of differential equations associated with the functional $TV_{\varphi,\Omega}$, a crucial role is played by the following result characterizing the subdifferential of $TV_{\varphi,\Omega}$, whose proof can easily be obtained by adapting that of [28, Theorem 12].

Theorem 1. *Let Ω be a bounded Lipschitz domain and let $w \in \mathcal{D}(TV_{\varphi,\Omega}) = BV(\Omega)$. There holds $v \in -\partial TV_{\varphi,\Omega}(w)$ if and only if $v \in L^2(\Omega)$ and there exists $\boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^N)$ such that $v = \operatorname{div} \boldsymbol{\xi}$ and*

$$-\int_\Omega w \operatorname{div} \boldsymbol{\xi} = \int_\Omega |Dw|_\varphi, \quad |\boldsymbol{\xi}|_\varphi^* \leq 1, \quad \left[\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega\right] = 0. \tag{11}$$

We denote by $X_{\varphi,\Omega}(w)$ the set of $\boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^N)$ with $\operatorname{div} \boldsymbol{\xi} \in L^2(\Omega)$ satisfying (11).

In the present paper, we are concerned with the case $\varphi = |\cdot|_1$. Hence, we have

$$|Du|_1(\Omega) = \int_\Omega |\nabla u|_1 \, dx + \int_\Omega \left| \frac{Du}{|Du|} \right|_1 d|D^s u|$$

for each $u \in BV(\Omega)$.

Given a set of finite perimeter E in Ω (resp., in $\mathbb{R}^N$) we denote $\operatorname{Per}_1(E,\Omega) = |D\chi_E|_1(\Omega)$ and $\operatorname{Per}_1(E) = \operatorname{Per}_1(E, \mathbb{R}^N)$. If E has finite perimeter in Ω, then

$$\operatorname{Per}_1(E,\Omega) = \int_{\partial^* E \cap \Omega} \left|\boldsymbol{\nu}^E\right|_1 \mathrm{d}\mathcal{H}^1. \tag{12}$$

If ∂E is Lipschitz, we can drop the star in $\partial^* E$, and $\boldsymbol{\nu}^E$ is the pointwise $\mathcal{H}^1$-a.e. defined outer Euclidean normal to E. Observe that, in the particular case that F is a rectilinear polygon,

$$\operatorname{Per}_1(F,\Omega) = \operatorname{Per}(F,\Omega).$$

Given $\lambda > 0$, a rectangle Ω, and $u_0 \in BV(\Omega)$ we consider the minimization problem (4). The problem has a unique solution, which is also the unique solution to the Euler–Lagrange equation

$$u = u_0 + \lambda \operatorname{div} \boldsymbol{z}, \qquad \boldsymbol{z} \in X_{1,\Omega}(u). \tag{13}$$

The following result is an easy corollary of Theorem 1 for the case of PCR functions.

Lemma 1. *Let Ω be a rectangle and $w \in PCR(\Omega)$. Then, $X_{1,\Omega}(w)$ consists of vector fields $\boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^2)$ such that, for any $Q \in \mathcal{Q}(w)$, $\boldsymbol{\xi}|_Q \in X_Q$ satisfies*

$$[\boldsymbol{\xi}, \boldsymbol{\nu}^Q]\big|_{\partial Q^\pm} = \mp 1, \qquad |\boldsymbol{\xi}|_\infty \leq 1 \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^Q]\big|_{\partial Q \cap \partial\Omega} = 0. \tag{14}$$

Furthermore, this set is nonempty.

Proof. It is easy to check that with any $\boldsymbol{\xi}$ satisfying (14), (11) holds. On the other hand, suppose that $\boldsymbol{\xi} \in X_{1,\Omega}$. Then, integrating by parts in each $Q \in \mathcal{Q}_w$ on the left-hand side of the first item in (11) and noting that $\boldsymbol{\nu}^Q|_{\partial Q^\pm} = \mp \frac{Dw}{|Dw|}$ we get

$$\int_{J_w} \left[\boldsymbol{\xi}, \tfrac{Dw}{|Dw|}\right] \left(w^+ - w^-\right) \mathrm{d}\mathcal{H}^1 = \int_{J_w} \left(w^+ - w^-\right) \mathrm{d}\mathcal{H}^1.$$

Together with the condition $|\boldsymbol{\xi}|_\infty \leq 1$ and (10) this implies the first item in (14).

One way to point out a field in $X_{1,\Omega}(w)$ is to extend it from $J_w = \bigcup_{Q \in \mathcal{Q}_w} \partial Q$, where one of its components is fixed by (14), by componentwise linear interpolation. ■

2.7. Anisotropic total variation flows. Another class of natural differential equations associated with functional $TV_{\varphi,\Omega}$ are anisotropic total variation flows that formally correspond to Neumann problems

$$\begin{cases} u_t = \operatorname{div} \partial|\cdot|_\varphi(\nabla u) & \text{in } \Omega, \\ \left[\partial|\cdot|_\varphi(\nabla u), \boldsymbol{\nu}^\Omega\right] = 0 & \text{on } \partial\Omega \end{cases} \tag{15}$$

with $\boldsymbol{\nu}^\Omega$ denoting the outer unit normal to $\partial\Omega$. In our case, $\varphi = 1$. Let us recall the notion of a *(strong) solution* to a general φ-anisotropic total variation flow, which is an adaptation of [28, Definition 4] for a bounded Lipschitz domain Ω.

Definition 1. *Let $0 \leq T_0 < T_* \leq \infty$. A function $u \in C([T_0, T_*[, L^2(\Omega))$ is called a strong solution to (15) in $[T_0, T_*[$ if $u_t \in L^2_{loc}(]T_0, T_*[, L^2(\Omega))$, $u \in L^1_w(]T_0, T_*[, BV(\Omega))$, and there exists $\boldsymbol{z} \in L^\infty(]T_0, T_*[\times \Omega, \mathbb{R}^2)$ such that*

$$u_t = \operatorname{div} \boldsymbol{z} \quad \textit{in } \mathcal{D}'(]T_0, T_*[\times \Omega), \tag{16}$$

$$|\boldsymbol{z}|^*_\varphi \leq 1 \quad \textit{a.e. in }]T_0, T_*[\times \Omega, \tag{17}$$

$$\left[\boldsymbol{z}(t), \boldsymbol{\nu}^\Omega\right] = 0, \ \textit{and} \tag{18}$$

$$-\int_\Omega u \operatorname{div} \boldsymbol{z} \, \mathrm{d}\mathcal{L}^N = \int_\Omega |Du(t,\cdot)|_\varphi \quad \textit{for a.e. } t \in]T_0, T_*[. \tag{19}$$

It can be proved as in [28, Theorem 11] that, given any $u_0 \in L^2(\Omega)$ and $0 \leq T_0 < T_* \leq \infty$, there exists a unique strong solution u to (15) in $[T_0, T_*[$ with $u(T_0, \cdot) = u_0$. Clearly, if $0 \leq T_0 < T_1 < T_2 \leq \infty$ and

$$u \in C\left([T_0, T_2[, L^2(\Omega)\right) \cap L^1_w(]T_0, T_*[, BV(\Omega)), \quad u_t \in L^2_{loc}\left(]T_0, T_*[, L^2(\Omega)\right),$$

is such that $u|_{[T_0,T_1[\times\Omega}$ a strong solution to (15) in $[T_0, T_1[$ and $u|_{[T_1,T_2[\times\Omega}$ a strong solution to (15) in $[T_1, T_2[$, then u is a strong solution to (15) in $[T_0, T_2[$.

In fact, this existential result is a characterization of the Crandall–Ligett semigroup generated by the negative subdifferential of $TV_{\varphi,\Omega}$. In the present paper we are concerned with evolution of regular (with respect to the operator $-\partial TV_{\varphi,\Omega}$) initial data. In such a case, semigroup theory yields the following result [4, Chapter III].

Theorem 2. *Let $u_0 \in \mathcal{D}(\partial TV_{\varphi,\Omega})$ and let u be the strong solution to (15) in $[0,\infty[$ starting with u_0. Then, every $\boldsymbol{z} \in L^\infty(]0,\infty[\times\Omega,\mathbb{R}^N)$ satisfying (16)–(19) has a representative (denoted henceforth $\boldsymbol{z}$) such that*

(1) *in every $t \in [0,\infty[$, $\boldsymbol{z}(t,\cdot)$ minimizes*

$$\mathcal{F}_\Omega(\boldsymbol{\xi}) = \int_\Omega (\operatorname{div}\boldsymbol{\xi})^2 \,\mathrm{d}\mathcal{L}^N$$

in $X_{\varphi,\Omega}(u(t,\cdot))$ and this condition uniquely defines $\operatorname{div}\boldsymbol{z}(t,\cdot)$;

(2) *the function*

$$[0,\infty[\ni t \mapsto \operatorname{div}\boldsymbol{z}(t,\cdot) \in L^2(\Omega) \quad \textit{is right continuous;}$$

(3) *the function*

$$[0,\infty[\ni t \mapsto \|\operatorname{div}\boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)} \quad \textit{is nonincreasing;}$$

(4) *the function $[0,\infty[\ni t \mapsto u(t,\cdot) \in L^2(\Omega)$ is right differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t}^{+} u(t,\cdot) = \operatorname{div}\boldsymbol{z}(t,\cdot) \quad \textit{in every } t \in [0,\infty[.$$

3. Cheeger problems in rectilinear geometry. Let F_0 be a rectilinear polygon, let $f \in PCR(F_0)$, and let $(\partial F_0^+, \partial F_0^-)$ be a signature for ∂F_0. We denote $G = G_f \cup G(\partial F_0^+) \cup G(\partial F_0^-)$.

We introduce a functional $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$ with values in $]-\infty,+\infty]$ defined on subsets of F_0 of positive area given by

$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) = \frac{\mathrm{Per}_1(E,\operatorname{int}F_0) + \mathcal{H}^1(\partial^* E \cap \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-) - \int_E f\,\mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}$$

if E has finite perimeter and $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) = +\infty$ otherwise. Note that for each measurable $E \subset F_0$ of positive area and finite perimeter, we have

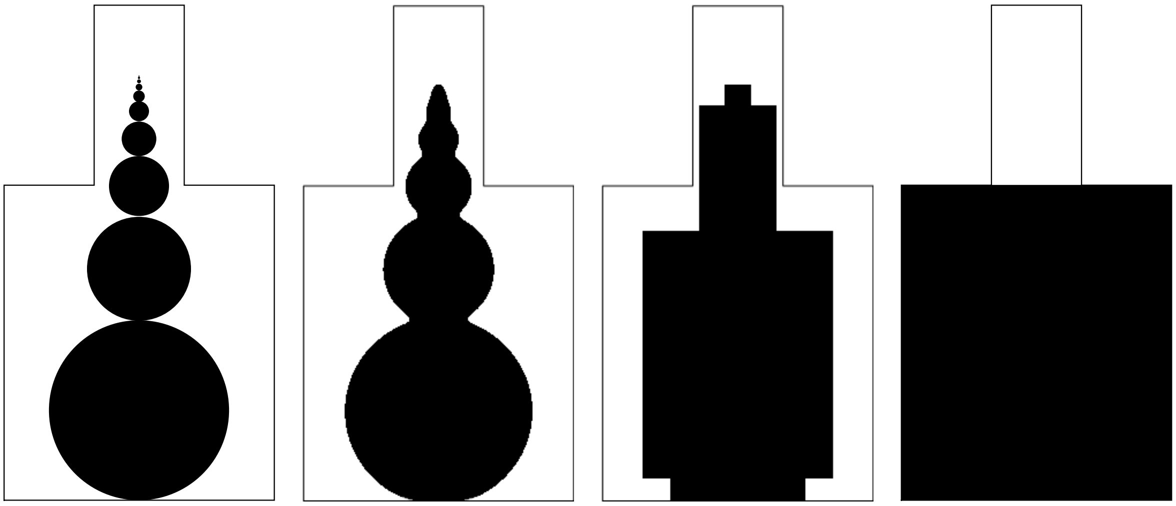
$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) = \frac{\mathrm{Per}_1(E) - \mathcal{H}^1(\partial^* E \cap \partial F_0 \setminus \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-) - \int_E f\,\mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}.$$

Lemma 2. *Let $E \subset F_0$ be a set of finite perimeter with $\mathcal{L}^2(E) > 0$. Then for every $\varepsilon > 0$ there exists a rectilinear polygon $F \in \mathcal{F}(G)$ such that*

$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(F) < \mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) + \varepsilon.$$

Proof. We construct F in three steps (the outline of the construction is represented in Figure 3). Throughout the proof, we write $\mathcal{J} = \mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$ for short.

Step 1. *Smoothing:* First, given $\varepsilon > 0$, we obtain a smooth closed set $\widetilde{E} \subset F$ such that $\mathcal{J}(\widetilde{E}) \le \mathcal{J}(E) + \varepsilon$ and $\widetilde{E}$ does not contain any vertices of F_0. For this purpose, we adapt the



(a) A BV set E. (b) Smoothed set $\widetilde{E}$. (c) Squared set $\widehat{E}_0$. (d) Aligned set F.

Figure 3. *The construction in Lemma 2 applied to an example set of finite perimeter contained in a rectilinear polygon (in the case $f = 0$).*

standard method of smooth approximation of sets of finite perimeter. Namely, we consider superlevels of smooth functions $\psi_\delta * \chi_E$, $\delta > 0$. Here, ψ_δ is a standard smooth approximation of unity. Using Sard's lemma on regular values of smooth functions and the coarea formula for anisotropic total variation [1, Remark 4.4], we obtain, reasoning as in the proof of [22, Theorem 1.24], a number $0 < t < \frac{1}{2}$ and a sequence $\delta_j \to 0^+$ such that

$$\widetilde{E}_j = \{\psi_{\delta_j} * \chi_E \geq t\}$$

is a smooth set for each $j = 1, 2, \dots$ and

$$\text{(20)} \quad \mathcal{L}^2(\widetilde{E}_j \triangle E) \to 0, \qquad \liminf_{j\to\infty} \operatorname{Per}_1(\widetilde{E}_j) = \operatorname{Per}_1(E),$$
$$\mathcal{H}^1\left((\partial^* E) \setminus \widetilde{E}_j\right) \to 0, \qquad \left| \int_{\widetilde{E}_j} f \, \mathrm{d}\mathcal{L}^2 - \int_E f \, \mathrm{d}\mathcal{L}^2 \right| \to 0.$$

Here and in the following we denote by $\triangle$ the symmetric difference. The first two items in (20) are covered explicitly in [22]. The last one is clear since $f \in L^\infty(F_0)$. It remains to justify the third item. Since $\partial^* E \subset \partial^{\frac{1}{2}} E$, for each $\boldsymbol{x} \in \partial^* E$ there is a natural number j_0 such that for every $j > j_0$ there holds $\boldsymbol{x} \in \widetilde{E}_j$. Thus, as $\mathcal{H}^1(\partial^* E) = \operatorname{Per}(E)$ is finite, the assertion follows by continuity of measures.

Perturbing each $\widetilde{E}_j$ a little, we can require that $\partial \widetilde{E}_j$ is transverse to every line in G. Then, $\partial(F_0 \cap \widetilde{E}_j)$ are piecewise smooth curves and it is visible that all items in (20) remain true if we substitute $F_0 \cap \widetilde{E}_j$ for $\widetilde{E}_j$. Therefore, for any given $\varepsilon' > 0$ we choose a number j such that

$$
\begin{aligned}
(21)\quad & \mathcal{L}^2(F_0 \cap \widetilde{E}_j) > \mathcal{L}^2(E) - \varepsilon', \qquad \mathrm{Per}_1(F_0 \cap \widetilde{E}_j) < \mathrm{Per}_1(E) + \varepsilon', \\
& \mathcal{H}^1\left(\partial(\widetilde{E}_j \cap F) \cap \partial F_0 \setminus \partial F_0^+\right) > \mathcal{H}^1\left(\partial^* E \cap \partial F_0 \setminus \partial F_0^+\right) - \varepsilon', \\
& \text{and} \quad \mathcal{H}^1\left(\partial(\widetilde{E}_j \cap F) \cap \partial F_0^-\right) > \mathcal{H}^1\left(\partial^* E \cap \partial F_0^-\right) - \varepsilon'.
\end{aligned}
$$

Taking ε' small enough we obtain

$$
\mathcal{J}(F_0 \cap \widetilde{E}_j) < \mathcal{J}(E) + \varepsilon. \tag{22}
$$

Due to transversality, there are at most a finite number of points where the piecewise smooth curve $\partial(F_0 \cap \widetilde{E}_j)$ is not infinitely differentiable. Thus, we can smooth out the set $F_0 \cap \widetilde{E}_j$ in such a way that (21) and, consequently, (22), still hold. We denote the resulting set by $\widetilde{E}$. Possibly adjusting $\widetilde{E}$ slightly, we can require that it does not contain any vertices of F_0.

Step 2. *Squaring:* Let now $\varepsilon'' > 0$. For each $\boldsymbol{x} \in \partial\widetilde{E}$ there is an open square $U(\boldsymbol{x}) = I(\boldsymbol{x}) \times J(\boldsymbol{x})$ of side smaller than ε'' such that $\widetilde{E} \cap U(\boldsymbol{x})$ coincides with a subgraph of a smooth function $g\colon I(\boldsymbol{x}) \to J(\boldsymbol{x})$ or $g\colon J(\boldsymbol{x}) \to I(\boldsymbol{x})$ and that $U(\boldsymbol{x})$ intersects at most one edge of F_0 (contained in the supergraph of g). The family $\{U(\boldsymbol{x})\colon \boldsymbol{x} \in \partial\widetilde{E}\}$ is an open cover of $\partial\widetilde{E}$. We extract a finite cover $\{U_1, \dots, U_l\}$, $l = l(\varepsilon'')$ out of it. We assume that $\{U_1, \dots, U_l\}$ is minimal in the sense that none of its proper subsets cover $\partial\widetilde{E}$. Let us take $\widehat{E}_0 = \widetilde{E} \cup \bigcup_{i=1}^{l} W_i$, where $W_i \subset F_0$ is the smallest closed rectangle containing $U_i \cap \widetilde{E}$. The operation of taking a union of $\widetilde{E}$ with W_1 increases volume while not increasing the l^1-perimeter. Indeed, denoting $U_1 =]a_1, b_1[\times]a_2, b_2[$ and assuming without loss of generality that $\widetilde{E} \cap U_1$ coincides with the subgraph of a smooth function $g_1 \colon]a_1, b_1[\to]a_2, b_2[$, we have

$$
\int_{W_1 \cap \partial\widetilde{E}} \left|\boldsymbol{\nu}^{\widetilde{E}}\right|_1 \mathrm{d}\mathcal{H}^1 = \int_{]a_1,b_1[} 1 + |g_1'| \,\mathrm{d}\mathcal{L}^1 \geq |\sup g_1 - g_1(a_1)| + |b_1 - a_1| + |\sup g_1 - g_1(b_1)|
$$

and, consequently,

$$
\mathrm{Per}_1(\widetilde{E}) = \mathcal{H}^1(\partial\widetilde{E} \setminus W_1) + \int_{W_1 \cap \partial\widetilde{E}} \left|\boldsymbol{\nu}^{\widetilde{E}}\right|_1 \mathrm{d}\mathcal{H}^1 \geq \mathrm{Per}_1(\widetilde{E} \cup W_1).
$$

Similarly, we show that taking the union of $\widetilde{E} \cup W_1$ with W_2 does not increase the perimeter, and so on. Furthermore, clearly $\partial\widetilde{E}_0 \cap \partial F_0 \setminus \partial F_0^+ \subset \partial\widehat{E}_0 \cap \partial F_0 \setminus \partial F_0^+$ and $\partial\widetilde{E}_0 \cap \partial F_0^- \subset \partial\widehat{E}_0 \cap \partial F_0^-$. Summing up, we have

$$
\begin{aligned}
(23)\quad & \mathcal{L}^2(\widehat{E}_0) \geq \mathcal{L}^2(\widetilde{E}), \qquad \mathrm{Per}_1(\widehat{E}_0) \leq \mathrm{Per}_1(\widetilde{E}), \\
& \mathcal{H}^1\left(\partial\widehat{E}_0 \cap \partial F_0 \setminus \partial F_0^+\right) \geq \mathcal{H}^1\left(\partial\widetilde{E} \cap \partial F_0 \setminus \partial F_0^+\right), \qquad \mathcal{H}^1(\partial\widehat{E}_0 \cap \partial F_0^-) \geq \mathcal{H}^1(\partial\widetilde{E} \cap \partial F_0^-), \\
& \left|\int_{\widehat{E}_0} f \,\mathrm{d}\mathcal{L}^2 - \int_{\widetilde{E}} f \,\mathrm{d}\mathcal{L}^2\right| \leq 2 \,\mathrm{ess\,max}\, f \cdot l(\varepsilon'') \cdot (\varepsilon'')^2.
\end{aligned}
$$

No point $\boldsymbol{x} \in F_0$ is contained in more than two of $W_1, \dots, W_l$, and so

$$
l(\varepsilon'') \cdot (\varepsilon'')^2 \leq 2\mathcal{L}^2\left(\{\boldsymbol{x}\colon \mathrm{dist}(\boldsymbol{x}, \partial\widetilde{E}) \leq \varepsilon''\}\right) \to 0
$$

as $\varepsilon'' \to 0$. Thus, fixing a small enough ε'', $\mathcal{J}(\widehat{E}_0) < \mathcal{J}(E) + \varepsilon$ holds.

Step 3. *Aligning:* Take any line $L_0 \subset G(\widehat{E}_0)$ that is not contained in G. We assume for clarity that L is horizontal, i.e., $L = \mathbb{R} \times \{y_0\}$, $y_0 \in \mathbb{R}$. We denote $L_0 \cap \partial \widehat{E}_0 = C_0 \times \{y_0\}$ and observe that $C_0 \subset \mathbb{R}$ necessarily contains an interval. Let $L_+ = \mathbb{R}\times\{y_+\}$ and $L_- = \mathbb{R}\times\{y_-\}$ be the lines in $G \cup G(\widehat{E}_0)$ situated above and below L_0 closest to L_0. We have $C_0 \times [y_-, y_+] \subset F_0$. Let us first assume that $\widehat{E}_0 \neq C_0 \times [y_0, y_+]$, $\widehat{E}_0 \neq C_0 \times [y_-, y_0]$. For $y \in [y_-, y_+]$, we define

$$\overline{\mathcal{J}}(y) = \begin{cases} \mathcal{J}(\widehat{E}_0 \triangle (C_0 \times [y_0, y])) & \text{if } y > y_0, \\ \mathcal{J}(\widehat{E}_0 \triangle (C_0 \times [y, y_0])) & \text{otherwise.} \end{cases} \tag{24}$$

Denoting $\mathcal{Q}_f = \{Q_1, \ldots, Q_n\}$, we observe that $Q_i \cap (C_0 \times [y_-, y_+]) = C_i \times [y_-, y_+]$ with $C_i \subset \mathbb{R}$ for $i = 1, \ldots, n$. This follows from the choice of L_0 and $L_\pm$. Similarly, each one of the line segments constituting $\partial C_0 \times [y_-, y_+]$ is contained (up to a finite number of points) in $\partial F_+ \cup \operatorname{int} F$, ∂F_-, or $\partial F \setminus (\partial F^+ \cup \partial F^-)$. Therefore, $\overline{\mathcal{J}}$ is a homography and hence monotone on $]y_-, y_+[$.

However, $\overline{\mathcal{J}}$ might be discontinuous at the endpoints of its domain. This is only possible if, as y attains y_+ (or y_-), a pair of edges of $\widehat{E}_0 \triangle (C_0 \times [y_0, y])$ (resp., $\widehat{E}_0 \triangle (C_0 \times [y, y_0])$) vanishes, or an edge touches the boundary of F_0. In either case, there still holds $\lim_{y\to y_\pm} \overline{\mathcal{J}}(y) \geq \overline{\mathcal{J}}(y_\pm)$.

Thus, whether $\overline{\mathcal{J}}$ is continuous or not, either $\overline{\mathcal{J}}(y_+)$ or $\overline{\mathcal{J}}(y_-)$ (or both) is not larger than $\overline{\mathcal{J}}(y_0) = \mathcal{J}(\widehat{E}_0)$. In accordance with that, we denote either $\widehat{E}_1 = \widehat{E}_0 \triangle (C_0 \times [y_0, y_+])$ or $\widehat{E}_1 = \widehat{E}_0 \triangle (C_0 \times [y_-, y_0])$ and perform the same argument with $\widehat{E}_1$ instead of $\widehat{E}_0$.

Now, let us go back to the excluded cases and suppose, without loss of generality, that $\widehat{E}_0 = C_0 \times [y_0, y_+]$. Then, $\overline{\mathcal{J}}$ is still a well-defined homography in $[y_-, y_+[$ and $\lim_{y\to y_+} \overline{\mathcal{J}}(y) = +\infty$. Hence, $\overline{\mathcal{J}}(y_-) \leq \overline{\mathcal{J}}(y_0) = \mathcal{J}(\widehat{E}_0)$ and we put $\widehat{E}_1 = \widehat{E}_0 \triangle (C_0 \times [y_-, y_0]) = C_0 \times [y_-, y_+]$ and continue the procedure.

For each i, $G(\widehat{E}_{i+1})$ contains at least one line not contained in G less than $G(\widehat{E}_i)$, so this procedure terminates in a finite number s of steps and we obtain $F = \widehat{E}_s$ all of whose edges are contained in G and $\mathcal{J}(F) \leq \mathcal{J}(\widehat{E}_0) < \mathcal{J}(E) + \varepsilon$. ∎

Theorem 3. *The functional $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$ is bounded from below and is minimized by a rectilinear polygon $F \subset F_0$ such that $F \in \mathcal{F}(G)$.*

Proof. Suppose that $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(E_n) \to -\infty$. Then, due to Lemma 2 there exist rectilinear polygons $F_n \subset F_0$, $n = 1, 2, \ldots$ such that $F_n \in \mathcal{F}(G)$ and $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}(F_n) \to -\infty$, an impossibility.

Now, consider any minimizing sequence (E_n) of $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$. By means of Lemma 2 we find a minimizing sequence of rectilinear polygons $F_n \in F_0$ such that $F \in \mathcal{F}(G)$. As the set of such rectilinear polygons is finite, (F_n) has a constant subsequence $(F_{n_k}) \equiv (F)$. Clearly, F minimizes $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$. ∎

Instead of $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-,f}$ we can consider

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f}(E)\colon \{E \subset F_0 \text{ — measurable s.t. } \mathcal{L}^2(E) > 0\} \to [-\infty, +\infty[$$

defined by

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) = \frac{-\operatorname{Per}_1(E, \operatorname{int} F_0) + \mathcal{H}^1(\partial^* E \cap \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-) - \int_E f \, \mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}$$

if E has finite perimeter and $-\infty$ otherwise. Then, noticing that

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f} = -\mathcal{J}_{F_0,\partial F_0^-,\partial F_0^+,-f}$$

we obtain analogous versions of Lemma 2 and Theorem 3.

Lemma 3. *Let $E \subset F_0$ be a set of finite perimeter with $\mathcal{L}^2(E) > 0$. Then for every $\varepsilon > 0$ there exists a rectilinear polygon $F \in \mathcal{F}(G)$ such that*

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f}(F) > \check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f}(E) - \varepsilon.$$

Theorem 4. *The functional $\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-,f}$ is bounded from above and is maximized by a rectilinear polygon $F \subset F_0$ such that $F \in \mathcal{F}(G)$.*

4. The minimization problem for TV_1 with PCR datum. Let Ω be a rectilinear polygon and let u_0 belong to $PCR(\Omega)$. Given $\lambda > 0$, we use Lemma 3 to prove that the solution u to the problem (4) is encoded in the partition $\{F_1, \dots, F_l\}$ of Ω (partition into level sets of u) and consistent signature $F_k \mapsto (\partial F_k^+, \partial F_k^-)$, $k = 1, \dots, l$ (signature induced by u), both subordinate to G_{u_0}, produced by the following algorithm:

- First, denote by F_1 the largest minimizer of $\mathcal{J}_{\Omega,\emptyset,\emptyset,\frac{u_0}{\lambda}}$, and put $\partial F_1^+ = \partial F_1 \setminus \partial\Omega$, $\partial F_1^- = \emptyset$.
- At the kth step, denote $\check{F}_k = \bigcup_{i=1}^{k-1} F_i$. If $\Omega = \check{F}_k$, stop. Otherwise, denote by F_k the largest minimizer of $\mathcal{J}_{\Omega\setminus\check{F}_k,\emptyset,\partial\check{F}_k,\frac{u_0}{\lambda}}$ and put $\partial F_k^+ = \partial F_k \setminus (\partial\Omega \cup \partial\check{F}_k)$, $\partial F_k^- = \partial F_k \cap \partial\check{F}_k$.

We take the largest minimizer, because we want to construct the whole level set in one turn. As Cheeger quotients are subadditive, the largest minimizer is the sum of all minimizers. Note that the algorithm ends after a finite number of steps, as $\check{F}_k$ is larger than $\check{F}_{k-1}$ by F_k, a nonempty rectilinear polygon subordinate to $G(u_0)$ and contained in Ω. These collectively make a finite set.

Remark. $\partial F_k^\pm$ are defined in such a way that each $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-,\frac{u_0}{\lambda}}$ is the restriction of $\mathcal{J}_{\Omega\setminus\check{F}_k,\emptyset,\partial\check{F}_k,\frac{u_0}{\lambda}}$ to subsets of F_k. In particular, F_k is a minimizer of $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-,\frac{u_0}{\lambda}}$.

Theorem 5. *Let F_k, $(\partial F_k^+, \partial F_k^-)$, $k = 1, \dots, l$, be as above. Let $u \in PCR(\Omega)$ be given by*

$$u|_{F_k} = \frac{1}{\mathcal{L}^2(F_k)} \left(\int_{F_k} u_0 \, d\mathcal{L}^2 - \lambda(\mathcal{H}^1(\partial F_k^+) - \mathcal{H}^1(\partial F_k^-)) \right) \tag{25}$$

for $k = 1, \dots, l$. Then u is the solution to (4). Furthermore, $\mathcal{Q}_u = \{F_1, \dots, F_l\}$, $\mathcal{S}_u(F_k) = (\partial F_k^+, \partial F_k^-)$, $k = 1, \dots, l$.

Proof. We now adapt the reasoning in the proof of [5, Theorem 5]. For given $k = 1, \dots, l$, we consider the functional $\mathcal{F}_k$ defined by

$$\mathcal{F}_k(\boldsymbol{\xi}) = \int_{F_k} \left(\operatorname{div} \boldsymbol{\xi} + \frac{u_0}{\lambda} \right)^2 d\mathcal{L}^2$$

on the set of vector fields $\boldsymbol{\eta} \in X_{F_k}$ satisfying

$$|\boldsymbol{\eta}|_\infty \le 1, \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^{F_k}]|_{\partial F_k^\pm} = \mp 1, \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^{F_k}]|_{\partial F_k \cap \partial\Omega} = 0.$$

We first prove that any vector field $\boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^2)$, such that $\boldsymbol{\xi}|_{F_k}$ satisfies the above conditions for each $k = 1, \dots, l$, belongs to $X_{1,\Omega}(u)$, with u defined by (25). This follows immediately by

Lemma 1 once we know that $F_k \mapsto (\partial F_k^+, \partial F_k^-)$, $k = 1, \ldots, l$, is the signature induced by u, i.e., that the inequality

(26)
$$\frac{1}{\mathcal{L}^2(F_{k+1})}\left(\int_{F_{k+1}} u_0 \,\mathrm{d}\mathcal{L}^2 - \lambda\left(\mathcal{H}^1\left(\partial F_{k+1} \setminus (\partial\Omega \cup \partial\check{F}_{k+1})\right) - \mathcal{H}^1(\partial F_{k+1} \cap \partial \check{F}_{k+1})\right)\right) = u|_{F_{k+1}}$$
$$< u|_{F_k} = \frac{1}{\mathcal{L}^2(F_k)}\left(\int_{F_k} u_0 \,\mathrm{d}\mathcal{L}^2 - \lambda\left(\mathcal{H}^1\left(\partial F_k \setminus (\partial\Omega \cup \partial\check{F}_k)\right) - \mathcal{H}^1(\partial F_k \cap \partial \check{F}_k)\right)\right)$$

is satisfied for $k = 1, \ldots, l-1$ with u defined in (25). Note that we have
$$\mathcal{H}^1(\partial F_{k+1} \cap \partial \check{F}_k) = \mathcal{H}^1(\partial F_{k+1} \cap \partial\check{F}_{k+1}) - \mathcal{H}^1(\partial F_{k+1} \cap \partial F_k)$$
$$\text{and} \quad \mathcal{H}^1\left(\partial F_{k+1} \setminus (\partial\Omega \cup \check{F}_k)\right) = \mathcal{H}^1\left(\partial F_{k+1} \setminus (\partial\Omega \cup \check{F}_{k+1})\right) + \mathcal{H}^1(\partial F_{k+1} \cap \partial F_k).$$
Thus, were (26) not the case, we would have (using the inequality $\frac{x_1+x_2}{y_1+y_2} \le \frac{x_1}{y_1}$ that holds whenever $\frac{x_2}{y_2} \le \frac{x_1}{y_1}$ for positive numbers x_1, x_2, y_1, y_2),

(27)
$$\begin{aligned}
&\mathcal{J}_{\Omega\setminus\check{F}_k, \emptyset, \partial\check{F}_k, \frac{u_0}{\lambda}}(F_k \cup F_{k+1}) \\
&\quad\le \frac{1}{\mathcal{L}^2(F_k) + \mathcal{L}^2(F_{k+1})}\left(\mathcal{H}^1(\partial F_k \cap \partial\check{F}_k) - \mathcal{H}^1(\partial F_k \setminus (\partial\Omega \cup \partial\check{F}_k)) - \frac{1}{\lambda}\int_{F_k} u_0 \,\mathrm{d}\mathcal{L}^2\right. \\
&\qquad \left. + \mathcal{H}^1(\partial F_{k+1} \cap \partial\check{F}_{k+1}) - \mathcal{H}^1(\partial F_{k+1} \setminus (\partial\Omega \cup \partial\check{F}_{k+1})) - \frac{1}{\lambda}\int_{F_{k+1}} u_0 \,\mathrm{d}\mathcal{L}^2\right) \\
&\quad\le \mathcal{J}_{\Omega\setminus\check{F}_k, \emptyset, \partial\check{F}_k, \frac{u_0}{\lambda}}(F_k),
\end{aligned}$$
in contradiction to the choice of F_k.

Proceeding as in [6, Proposition 6.1], we see that $\mathcal{F}_k$ attains a minimum and for any two minimizers $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ we have $\operatorname{div}\boldsymbol{\eta}_1 = \operatorname{div}\boldsymbol{\eta}_2$ in F_k. Let us take any minimizer and denote it $\boldsymbol{\xi}_{F_k}$. Arguing as in [6, Theorem 6.7] and [7, Theorem 5.3], $\operatorname{div}\boldsymbol{\xi}_{F_k} \in L^\infty(F_k) \cap BV(F_k)$. Let $\nu = \mathcal{J}_{F_k, \partial F_k^+, \partial F_k^-, \frac{u_0}{\lambda}}(F_k)$. We have
$$\frac{1}{\mathcal{L}^2(F_k)}\int_{F_k}\left(\operatorname{div}\boldsymbol{\xi}_{F_k} + \frac{u_0}{\lambda}\right)\mathrm{d}\mathcal{L}^2 = -\frac{1}{\mathcal{L}^2(F_k)}\left(\mathcal{H}^1(\partial F_k^+) - \mathcal{H}^1(\partial F_k^-) - \frac{1}{\lambda}\int_{F_k} u_0 \,\mathrm{d}\mathcal{L}^2\right) = -\nu.$$
Were $\operatorname{div}\boldsymbol{\xi}_{F_k} + \frac{u_0}{\lambda}$ not constant in F_k, there would exist $\mu < \nu$ such that
$$A_\mu = \left\{\boldsymbol{x} \in F_k \colon -\left(\operatorname{div}\boldsymbol{\xi}_{F_k}(\boldsymbol{x}) + \frac{u_0}{\lambda}\right) < \mu\right\}$$
has positive measure and finite perimeter. Employing [7, Proposition 3.5],
$$\begin{aligned}
-\nu < -\mu &< \frac{1}{\mathcal{L}^2(A_\mu)}\int_{A_\mu}\left(\operatorname{div}\boldsymbol{\xi}_{F_k} + \frac{u_0}{\lambda}\right)\mathrm{d}\mathcal{L}^2 \\
&= -\frac{1}{\mathcal{L}^2(A_\mu)}\left(\operatorname{Per}_1(A_\mu, F_k) + \mathcal{H}^1(\partial F_k^+ \cap \partial^* A_\mu) - \mathcal{H}^1(\partial F_k^- \cap \partial^* A_\mu) - \frac{1}{\lambda}\int_{F_k} u_0 \,\mathrm{d}\mathcal{L}^2\right) \\
&= -\mathcal{J}_{F_k, \partial F_k^+, \partial F_k^-, \frac{u_0}{\lambda}}(A_\mu),
\end{aligned}$$

which would contradict that F_k minimizes $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-,\frac{u_0}{\lambda}}$ (see the remark before the statement of the theorem), hence, $u_0 + \lambda \operatorname{div} \boldsymbol{\xi}_{F_k}$ is constant in F_k and therefore equal to its mean value:

$$u_0 + \lambda \operatorname{div} \boldsymbol{\xi}_{F_k} = \frac{1}{\mathcal{L}^2(F_k)} \left(\int_{F_k} u_0 \, \mathrm{d}\mathcal{L}^2 - \lambda(\mathcal{H}^1(\partial F_k^+) - \mathcal{H}^1(\partial F_k^-)) \right) = u,$$

i.e., u satisfies the Euler–Lagrange equation (13).

The last sentence of the assertion follows from (26). ■

Remark. Instead of considering the minimization problem for $\mathcal{J}$, one can consider at each step the maximization problem for $\check{\mathcal{J}}$ (see Theorem 4).

5. The TV_1 flow with PCR initial datum.

In what now follows, we are concerned with the identification of the evolution of the initial datum $w \in PCR(\Omega)$ with Ω a rectilinear polygon under the l^1-anisotropic total variation flow (5). The result below determines the initial evolution, prescribing possible breaking of initial facets.

Theorem 6. *Let $w \in PCR(\Omega)$ and let G be any grid such that $\mathcal{Q}_w$ is subordinate to G. Then, there exists a field $\boldsymbol{\eta} \in X_{1,\Omega}(w)$ and, for each $Q \in \mathcal{Q}_w$, a partition $\mathcal{T}_Q$ of Q and a consistent signature $\mathcal{S}_Q$ for $\mathcal{T}_Q$ subordinate to G, $\mathcal{S}_Q \colon F \mapsto (\partial F^+, \partial F^-)$ for $F \in \mathcal{T}_Q$, such that*

(1) *$\mathcal{T} = \bigcup_{Q \in \mathcal{Q}_w} \mathcal{T}_Q$ is a partition of Ω subordinate to G, $\mathcal{S} \colon \mathcal{T} \to \mathcal{C} \times \mathcal{C}$ given by $\mathcal{S}(F) = \mathcal{S}_Q(F)$ for $F \in \mathcal{T}_Q$ is a consistent signature for $\mathcal{T}$ subordinate to G;*

(2) *for each $F \in \mathcal{T}$,*

$$\operatorname{div} \boldsymbol{\eta}|_F = -\frac{1}{\mathcal{L}^2(F)} \left(\mathcal{H}^1(\partial F^+) - \mathcal{H}^1(\partial F^-) \right), \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^F]|_{\partial F^\pm} = \mp 1.$$

Proof. We fix $Q \in \mathcal{Q}_w$ and produce the partition $\mathcal{T}_Q$ of Q and consistent signature $\mathcal{S}_Q$ for $\mathcal{T}_Q$ by means of an inductive procedure analogous to the one in section 4. First, by virtue of Theorem 3, the functional $\mathcal{J}_{Q,\partial Q^+,\partial Q^-,0}$ attains its minimum value on a rectilinear polygon $F_1 \in \mathcal{F}(G)$. We define

$$\partial F_1^- = \partial Q^- \cap \partial F_1 \quad \text{and} \quad \partial F_1^+ = (\partial F_1 \cap \partial Q^+) \cup (\partial F_1 \setminus \partial Q).$$

Next, in the kth step, we put $\check{F}_k = \bigcup_{j=1}^{k-1} F_j$. If $\check{F}_k = Q$ we stop and put $\mathcal{T}_Q = \{F_1, \ldots, F_{k-1}\}$, $\mathcal{S}_Q(F_j) = (\partial F_j^+, \partial F_j^-)$ for $j = 1, \ldots, k-1$. Otherwise we define F_k as any minimizer of

$$\mathcal{J}_{Q \setminus \check{F}_k, \partial Q^+, \partial Q^- \cup \partial \check{F}_k, 0}$$

and

$$\begin{aligned} \partial F_k^- &= \partial F_k \cap \left(\partial Q^- \cup \partial \check{F}_k \right), \\ \partial F_k^+ &= (\partial F_k \cap \partial Q^+) \cup \left(\partial F_k \setminus \left(\partial Q \cup \partial \check{F}_k \right) \right). \end{aligned}$$

Now, for each $F_k \in \mathcal{T}_Q$, we define $\boldsymbol{\eta}_{F_k}$ as any minimizer of the functional $\mathcal{F}_k$ defined on the set of vector fields $\boldsymbol{\xi} \in L^\infty(F_k, \mathbb{R}^2)$ satisfying

$$\operatorname{div} \boldsymbol{\xi} \in L^2(F_k), \qquad |\boldsymbol{\xi}|_\infty \leq 1, \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^{F_k}]|_{\partial F_k^\pm} = \mp 1, \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^{F_k}]|_{\partial F_k \cap \partial \Omega} = 0$$

by $\mathcal{F}_k(\boldsymbol{\xi}) = \int_{F_k} (\operatorname{div} \boldsymbol{\xi})^2 \, \mathrm{d}\mathcal{L}^2$. As in the proof of Theorem 5, we prove that $\operatorname{div} \boldsymbol{\eta}_{F_k}$ is constant in each $F_k \in \mathcal{T}_Q$.

Next, we repeat the procedure for the rest of $Q \in \mathcal{Q}_w$ and define $\boldsymbol{\eta}$ by $\boldsymbol{\eta}|_{F_k} = \boldsymbol{\eta}_{F_k}$ for every $F_k \in \mathcal{T}_Q$, $Q \in \mathcal{Q}_w$. Clearly, $\boldsymbol{\eta} \in X_{1,\Omega}(w)$. ■

Theorem 7. *Let $u_0 \in PCR(\Omega)$ and denote $G = G_{u_0}$. Let u be the global strong solution to (5). Then there exist a finite sequence of time instances $0 = t_0 < t_1 < \cdots < t_n$, partitions $\mathcal{Q}_0, \dots, \mathcal{Q}_{n-1}$ of Ω, and consistent signatures $\mathcal{S}_k$ for $\mathcal{Q}_k$ subordinate to G, $k = 0, \dots, n-1$, such that*

$$u_t = -\frac{1}{\mathcal{L}^2(F)} \left(\mathcal{H}^1(\partial F^+) - \mathcal{H}^1(\partial F^-) \right) \quad \text{in }]t_k, t_{k+1}[\times F \quad \text{for each } F \in \mathcal{Q}_k, \tag{28}$$

$k = 0, 1, \dots, n-1$, and $u(t,\cdot) = \frac{1}{\mathcal{L}^2(\Omega)} \int_\Omega u_0 \, \mathrm{d}\mathcal{L}^2$ for $t \geq t_n$. In particular, $u(t,\cdot) \in PCR(\Omega)$ and $\mathcal{Q}_{u(t,\cdot)}$ is subordinate to G for all $t > 0$. Furthermore,

$$t_n \leq \frac{1}{2} \left(\min_{F \subsetneq \Omega, F \in \mathcal{F}(G)} \frac{\operatorname{Per}_1(F,\Omega)}{\mathcal{L}^2(F)} \right)^{-1} \cdot \max \left| u_0 - \frac{1}{\mathcal{L}^2(\Omega)} \int_\Omega u_0 \, \mathrm{d}\mathcal{L}^2 \right|. \tag{29}$$

Remark. Theorem 7 implies that $t \mapsto u(t,\cdot)$ has a representative that is Lipschitz with values in $BV(\Omega)$.

Proof. We proceed inductively, starting with $j = 0$. Suppose we have proved that there exist time instances $0 = t_0 < \cdots < t_j$, partitions $\mathcal{Q}_k$ of Ω, and consistent signatures $\mathcal{S}_k$ for $\mathcal{Q}_k$ subordinate to G, $k = 0, \dots, j-1$, such that (28) holds for $k = 1, \dots, j$ (for $j = 0$ this assumption is vacuously satisfied). This implies that $u(t,\cdot) \in PCR(\Omega)$ and $\mathcal{Q}_{u(t,\cdot)}$ is subordinate to G for $t \in [0, t_j]$. Let $\mathcal{Q}_j = \mathcal{T}$, $\mathcal{S}_j = \mathcal{S}$, $\boldsymbol{z}_j = \boldsymbol{\eta}$, where $\mathcal{T}$, $\mathcal{S}$, and $\boldsymbol{\eta}$ are the partition of Ω, the consistent signature for $\mathcal{T}$ subordinate to G, and the vector field produced by Theorem 6 given $w = u(t_j,\cdot)$. For $T > 0$ let us define a function $\widetilde{u}_j \in C([t_j, T[, BV(\Omega))$ by

$$\begin{aligned} &\widetilde{u}_{j,t}(t,\cdot) = \operatorname{div} \boldsymbol{z}_j \text{ for } t \in]t_j, T[, \\ &\widetilde{u}_j(t_j,\cdot) = u(t_j,\cdot). \end{aligned}$$

Clearly, the pair $(\widetilde{u}_j, \boldsymbol{z}_j)$ satisfies regularity conditions as well as (16), (17) from Definition 1. Let us choose $T = t_{j+1}$ as the first time instance $t > t_j$ such that

$$\widetilde{u}_j(t,\cdot)|_F = \widetilde{u}_j(t,\cdot)|_{F'},$$

where $F, F' \in \mathcal{Q}_j$, $F \neq F'$, $\mathcal{H}^1(\partial F \cap \partial F') > 0$, i.e., the first moment of merging of facets after time t_j.

Due to condition (4) of Theorem 6, one can show, similarly to the proof of Theorem 5, that condition (19) of Definition 1 is satisfied for $(\widetilde{u}_j, \boldsymbol{z}_j)$ and $\widetilde{u}_j$ is the solution to (15) with initial datum $u(t_j,\cdot)$ in $[t_j, t_{j+1}[$. Then, due to continuity, u is necessarily equal to $\widetilde{u}_j$ in $[t_j, t_{j+1}]$, in particular, for $t \in [t_j, t_{j+1}]$, $u(t,\cdot) \in PCR(\Omega)$, and $\mathcal{Q}_{u(t,\cdot)}$ is subordinate to G. This completes the proof of the induction step.

Now, let us prove that this procedure terminates after a finite number of steps. For this purpose, we rely on Theorem 2. In fact, we prove that there exists a constant $\gamma = \gamma(G) > 0$

such that at each t_j, $j > 0$, the nonincreasing function $t \mapsto \|\operatorname{div} \boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)}$ has a jump of size at least γ. Here $\boldsymbol{z} \in L^\infty(]0,\infty[\times\Omega)$ satisfies the conditions in Definition 1 with u being the strong solution to (5) starting with u_0.

First, we argue that $\|\operatorname{div} \boldsymbol{z}(t_j,\cdot)\|_{L^2(\Omega)} < \|\operatorname{div} \boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)}$ for each $t \in [t_{j-1}, t_j[$ (in this interval $t \mapsto \operatorname{div} \boldsymbol{z}(t,\cdot)$ is a constant function). We will reason by contradiction. If $\|\operatorname{div} \boldsymbol{z}(t_j,\cdot)\|_{L^2(\Omega)} = \|\operatorname{div} \boldsymbol{z}(t_{j-1},\cdot)\|_{L^2(\Omega)}$, then $\boldsymbol{z}(t_{j-1},\cdot)$ is a minimizer of $\mathcal{F}_\Omega$ in $X_{1,\Omega}(u(t_j,\cdot))$ and consequently $\operatorname{div} \boldsymbol{z}(t_{j-1},\cdot) = \operatorname{div} \boldsymbol{z}(t_j,\cdot) = \operatorname{div} \boldsymbol{z}(t,\cdot)$ for $t \in [t_{j-1}, t_{j+1}[$ (see Theorem 2). According to Lemma 1, the minimization problem for $\mathcal{F}_\Omega$ in $X_{1,\Omega}(u(t_j,\cdot))$ is equivalent to minimization of functionals $\mathcal{F}_Q$ defined by $\mathcal{F}_Q(\boldsymbol{\eta}) = \int_Q (\operatorname{div} \boldsymbol{\eta})^2 \, \mathrm{d}\mathcal{L}^2$ on the set of vector fields $\boldsymbol{\eta} \in L^\infty(Q,\mathbb{R}^2)$ satisfying

$$\operatorname{div} \boldsymbol{\eta} \in L^2(Q), \qquad |\boldsymbol{\eta}|_\infty \le 1, \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^Q]\big|_{\partial Q^\pm} = \mp 1, \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^Q]\big|_{\partial Q \cap \partial\Omega} = 0$$

separately for each $Q \in \mathcal{Q}_{u(t_j,\cdot)}$, where $(\partial Q^+, \partial Q^+) = \mathcal{S}_{u(t_j,\cdot)}(Q)$. Let us take $Q \in \mathcal{Q}_{u(t_j,\cdot)}$ such that there exist F_1, F_2 in $\mathcal{Q}_{j-1}$, $F_1 \ne F_2$, $\mathcal{H}^1(\partial F_1 \cap \partial F_2) > 0$, with $F_1, F_2 \subset Q$. Denote by $\mathcal{Q}_{j-1,Q}$ the maximal subset of $\mathcal{Q}_{j-1}$ with the properties

- F_1, F_2 belong to $\mathcal{Q}_{j-1,Q}$;
- if F belongs to $\mathcal{Q}_{j-1}$, then $F \subset Q$;
- if F belongs to $\mathcal{Q}_{j-1,Q}$, then there exists $F' \in \mathcal{Q}_{j-1,Q}$, $F' \ne F$, with $\mathcal{H}^1(\partial F \cap \partial F') \ne 0$.

Let now F_0 be a minimizer of $F \mapsto u_t|_{]t_{j-1},t_{j+1}[\times F} = \operatorname{div} \boldsymbol{z}|_{]t_{j-1},t_{j+1}[\times F}$ among $F \in \mathcal{Q}_{j-1,Q}$. Then, due to (19) and the way Du changes after the moment of merging, we necessarily have

$$[\boldsymbol{z}, \boldsymbol{\nu}^{F_0}]\big|_{]t_j,t_{j+1}[\times \partial F_0^- \cup (\partial F_0^+ \setminus \partial Q)} = +1.$$

Due to the choice of F_0, $\mathcal{H}^1(\partial F_0^+ \setminus \partial Q) > 0$, hence,

$$\frac{1}{\mathcal{L}^2(F_0)} \int_{F_0} \operatorname{div} \boldsymbol{z} \, \mathrm{d}\mathcal{L}^2 \Big|_{[t_j,t_{j+1}[} > \frac{1}{\mathcal{L}^2(F_0)} \int_{F_0} \operatorname{div} \boldsymbol{z} \, \mathrm{d}\mathcal{L}^2 \Big|_{[t_{j-1},t_j[},$$

a desired contradiction.

Next, we observe that there is only a finite set of values, depending only on G, that $\|\operatorname{div} \boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)}$ can achieve. Indeed, for all $t \ge 0$, $\operatorname{div} \boldsymbol{z}(t,\cdot)$ is the unique result of minimization problems for $\mathcal{F}_Q$ with $Q \in \mathcal{Q}_{u(t_j,\cdot)}$, $(\partial Q^+, \partial Q^-) = \mathcal{S}_{u(t_j,\cdot)}(Q)$, $j = 0, 1, \ldots$. Each $\mathcal{Q}_{u(t_j,\cdot)}$ is a partition of Ω subordinate to G, each $\mathcal{S}_{u(t_j,\cdot)}$ is a consistent signature for $\mathcal{Q}_{u(t_j,\cdot)}$ subordinate to G. There are only a finite number of these.

It remains to prove the estimate on t_n. In any time instance $t \ge 0$ the maximum (minimum) value of $u(t,\cdot)$ is attained in a rectilinear polygon $F_+(t)$ ($F_-(t)$). In all but a finite number of t we have

$$\mp u_t(t,\cdot)|_{F_\pm(t)} = \mp \operatorname{div} \boldsymbol{z}(t,\cdot)|_{F_\pm(t)} = \frac{\operatorname{Per}_1(F_\pm(t),\Omega)}{\mathcal{L}^2(F_\pm(t))} \ge \min_{F \subsetneq \Omega, F \in \mathcal{F}(G)} \frac{\operatorname{Per}_1(F,\Omega)}{\mathcal{L}^2(F)},$$

unless $F^+ = F^- = \Omega$. Furthermore, testing (15) with χ_Ω yields $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(t,\cdot) \, \mathrm{d}\mathcal{L}^2 = 0$ in a.e. $t \ge 0$ and, due to continuity of the semigroup in L^2,

$$\frac{1}{|\Omega|} \int_\Omega u(t,\cdot) \, \mathrm{d}\mathcal{L}^2 = \frac{1}{|\Omega|} \int_\Omega u_0 \, \mathrm{d}\mathcal{L}^2$$

in all $t > 0$. This concludes the proof. ∎

6. The case $\Omega = \mathbb{R}^2$. In this section we transfer previous results to the case $\Omega = \mathbb{R}^2$. First, we note that all the definitions and theorems in subsections 2.6 and 2.7 carry over without change (the Neumann boundary condition becomes void) to this case (see [28]). As for the definitions in subsection 2.3, it turns out that the statements of our results transfer nicely to the case of the whole plane if we allow for certain *unbounded rectilinear polygons*. Accordingly, in this section a subset $F \subset \mathbb{R}^2$ will be called a rectilinear polygon if either

- $F = \bigcup \mathcal{R}_F$ with a finite $\mathcal{R}_F \subset \mathcal{R}$ (in which case we say that F is a bounded rectilinear polygon)
- or $F = \overline{\mathbb{R}^2 \setminus \bigcup \mathcal{R}_F}$ with a finite $\mathcal{R}_F \subset \mathcal{R}$ (in which case we say that F is an unbounded rectilinear polygon).

Next, we restrict ourselves to nonnegative compactly supported initial data. We say that a nonnegative compactly supported function $w \in BV(\mathbb{R}^2)$ belongs to $PCR_+(\mathbb{R}^2)$ if there exists a partition $\mathcal{Q}$ of $\mathbb{R}^2$ such that w is constant in the interior of each $Q \in \mathcal{Q}$. Note that any such $\mathcal{Q}$ contains exactly one unbounded set Q_0 and $w|_{Q_0} = 0$.

The essential difficulty in obtaining results analogous to Theorems 5 and 7 lies in dealing with unbounded sets that one expects to be produced by a suitable version of the algorithm in section 4. For this purpose, we need the following.

Lemma 4. *Let $f \in PCR_+(\mathbb{R}^2)$ and let F be an unbounded rectilinear polygon. Then, there exists a vector field $\boldsymbol{\xi}_F \in X_F$ such that*

$$|\boldsymbol{\xi}_F|_\infty \le 1, \qquad \operatorname{div} \boldsymbol{\xi}_F + f|_F = const., \qquad [\boldsymbol{\xi}_F, \boldsymbol{\nu}^F] = 1 \tag{30}$$

if and only if

$$\mathcal{H}^1(\partial^* E \cap \partial F) + \int_E f \, \mathrm{d}\mathcal{L}^2 \le \mathrm{Per}_1(E, \operatorname{int} F) \tag{31}$$

for all $E \subset F$ bounded of finite perimeter. Moreover, in this case $\operatorname{div} \boldsymbol{\xi}_F + f = 0$ in F for any vector field $\boldsymbol{\xi}_F$ satisfying (30).

This is a version of [5, Theorem 5 and Lemma 6] where an analogous statement is proved for isotropic perimeter in case $f = 0$. The idea of the proof is to consider the auxiliary problem in a large enough ball. The proof of Lemma 4 follows along similar lines; however, we decided to put it here, also because it seems that there is a small gap in the proof of [5, Theorem 5] that we patch. Namely, the first inequality in [5, line 12, p. 511] (corresponding to (36) here) does not seem to be satisfied in general.

Proof. It is easy to see that if $\boldsymbol{\xi}_F$ satisfies (30), then $\operatorname{div} \boldsymbol{\xi}_F + f = 0$ in F (see [5, Lemma 6]). Thus, if a vector field $\boldsymbol{\xi}_F \in X_F$ satisfies (30), then we have for any bounded set $E \subset \mathbb{R}^2$ of finite perimeter

$$0 = \int_E \operatorname{div} \boldsymbol{\xi}_F + f \, \mathrm{d}\mathcal{L}^2 \ge \mathcal{H}^1(\partial^* E \cap \partial F) + \int_E f \, \mathrm{d}\mathcal{L}^2 - \mathrm{Per}_1(E, \operatorname{int} F).$$

Now assume that (31) holds. Let us take $R > 0$ large enough that

$$2 \operatorname{dist}(\partial B_\infty(R), \partial F \cup \operatorname{supp} f) \ge \mathcal{H}^1(\partial F) + \int_F f \, \mathrm{d}\mathcal{L}^2. \tag{32}$$

Put $c(R) = -\frac{\mathcal{H}^1(\partial F)+\int_F f\,\mathrm{d}\mathcal{L}^2}{\mathcal{H}^1(\partial B_\infty(R))}$. Denote by $\boldsymbol{\xi}_R$ the minimizer of functional $\mathcal{F}$ defined by $\mathcal{F}(\boldsymbol{\eta}) = \int_{F\cap B_\infty(R)} (\operatorname{div}\boldsymbol{\eta} + f)^2\,\mathrm{d}\mathcal{L}^2$ on the set of vector fields $\boldsymbol{\eta} \in X_{F\cap B_\infty(R)}$ satisfying

$$|\boldsymbol{\eta}|_\infty \le 1, \qquad \left[\boldsymbol{\eta}, \boldsymbol{\nu}^F\right] = 1, \qquad \left[\boldsymbol{\eta}, \boldsymbol{\nu}^{B_\infty(R)}\right] = c(R).$$

If $\operatorname{div}\boldsymbol{\xi}_R + f$ is constant in $F\cap B_\infty(R)$, then, due to the choice of $c(R)$, $\operatorname{div}\boldsymbol{\xi}_R + f \equiv 0$ in $F\cap B_\infty(R)$. Supposing that the opposite is true, we obtain, as in the proof of Theorem 5, that there exists $\lambda > 0$ such that

$$Q_\lambda = \{\boldsymbol{x} \in F\cap B_\infty(R)\colon \operatorname{div}\boldsymbol{\xi}_R + f > \lambda\}$$

is a set of positive measure and finite perimeter, and we have

$$\begin{aligned}(33)\quad -\mathrm{Per}_1(Q_\lambda, \operatorname{int} F\cap B_\infty(R)) + \mathcal{H}^1(\partial^* Q_\lambda \cap \partial F) + \int_{Q_\lambda} f\,\mathrm{d}\mathcal{L}^2 + c(R)\mathcal{H}^1(\partial^* Q_\lambda \cap \partial B_\infty(R))\\ \ge \lambda\mathcal{L}^2(Q_\lambda) > 0\end{aligned}$$

which can be rewritten as

$$(34)\qquad -\mathrm{Per}_1(Q_\lambda) + 2\mathcal{H}^1(\partial^* Q_\lambda\cap\partial F) + \int_{Q_\lambda} f\,\mathrm{d}\mathcal{L}^2 + (1+c(R))\mathcal{H}^1(\partial^* Q_\lambda \cap \partial B_\infty(R)) > 0.$$

Assumption (32) implies that $c(R) > -1$, so we approximate Q_λ with a closed smooth set as in the proof of Lemma 2 in such a way that (34) still holds. Due to additivity of the left-hand side of (34), there is a connected component $\widetilde{Q}_\lambda$ of this smooth set that also satisfies (34) or, equivalently,

$$(35)\quad -\mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) + \mathcal{H}^1(\partial\widetilde{Q}_\lambda\cap\partial F) + \int_{\widetilde{Q}_\lambda} f\,\mathrm{d}\mathcal{L}^2 + c(R)\mathcal{H}^1(\partial\widetilde{Q}_\lambda\cap\partial B_\infty(R)) > 0.$$

If $\partial\widetilde{Q}_\lambda\cap\partial B_\infty(R) = \emptyset$, (35) contradicts (31). On the other hand, if $\partial\widetilde{Q}_\lambda\cap(\partial F\cup\operatorname{supp} f) = \emptyset$, (35) itself is a contradiction (recall that $c(R) \le 0$). Taking these observations into account, there necessarily holds

$$(36)\qquad \mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) \ge 2\operatorname{dist}(\partial B_\infty(R), \partial F\cup\operatorname{supp} f),$$

whence (32) yields a contradiction, *unless* $\widetilde{Q}_\lambda$ is not simply connected in such a way that there is a connected component Γ of $\partial\widetilde{Q}_\lambda$ such that

- $\widetilde{Q}_\lambda$ is inside of Γ (Γ is the exterior boundary of $\widetilde{Q}_\lambda$),
- Γ does not intersect $\partial F\cup\operatorname{supp} f$,
- and Γ intersects all four sides of $B_\infty(R)$.

In this case, let us denote by $\widehat{Q}_\lambda$ the union of $\widetilde{Q}_\lambda$ and the region between Γ and $\partial B_\infty(R)$. We have $\int_{\Gamma\setminus\partial B_\infty(R)} |\boldsymbol{\nu}^{\widetilde{Q}_\lambda}|_1\,\mathrm{d}\mathcal{H}^1 \ge \mathcal{H}^1(B_\infty(R)\setminus\Gamma)$ and consequently (as $-c(R) < 1$)

$$\begin{aligned}\mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) - c(R)\mathcal{H}^1(\partial\widetilde{Q}_\lambda\cap\partial B_\infty(R)) \ge \mathrm{Per}_1(\widehat{Q}_\lambda, \operatorname{int} F\cap B_\infty(R))\\ - c(R)\mathcal{H}^1(\partial\widehat{Q}_\lambda\cap\partial B_\infty(R)) = \mathrm{Per}_1(\widehat{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) + \mathcal{H}^1(\partial F) + \int_F f\,\mathrm{d}\mathcal{L}^2,\end{aligned}$$

a contradiction with (35) which implies that $\operatorname{div} \boldsymbol{\xi}_R \equiv 0$. Now, we define $\boldsymbol{\xi}_F \in X_F$ by

$$\boldsymbol{\xi}_F(x_1,x_2) = \begin{cases} \boldsymbol{\xi}_R(x_1,x_2) & \text{in } F \cap B_\infty(R), \\ (c(R)\operatorname{sgn} x_1, 0) & \text{in } \{|x_1| > R, |x_2| < R\}, \\ (0, c(R)\operatorname{sgn} x_2) & \text{in } \{|x_1| < R, |x_2| > R\}, \\ (0,0) & \text{in } \{|x_1| > R, |x_2| > R\}. \end{cases} \tag{37}$$ ∎

Now given $\lambda > 0$, $u_0 \in PCR_+(\mathbb{R}^2)$, grid $G = G_f$, and an unbounded rectilinear polygon F subordinate to G with signature $(\partial F^+, \partial F^-) = (\emptyset, \partial F)$, let us denote by R_0 the smallest rectangle containing the support of u_0 (clearly, R_0 is subordinate to G and $\partial F \subset R_0$). Next, suppose that there is a set of finite perimeter $E \subset F$ such that $\mathcal{J}_{F,\partial F^+,\partial F^-,\frac{u_0}{\lambda}}(E) < 0$. Then there holds $\mathcal{J}_{F,\partial F^+,\partial F^-,\frac{u_0}{\lambda}}(R_0 \cap E) \le \mathcal{J}_{F,\partial F^+,\partial F^-,\frac{u_0}{\lambda}}(E)$. Indeed, we only need to argue that $\mathrm{Per}_1(R_0 \cap E) \le \mathrm{Per}_1(E)$, which follows easily by the approximation of E with smooth sets. This way, we obtain the following alternative:

- either $\mathcal{J}_{F,\partial F^+,\partial F^-,\frac{u_0}{\lambda}}$ is minimized by a bounded rectilinear polygon subordinate to G
- or $\mathrm{Per}_1(E, \operatorname{int} F) - \mathcal{H}^1(\partial^* E \cap \partial F) - \int_E \frac{u_0}{\lambda}\, \mathrm{d}\mathcal{L}^2 \ge 0$ for each $E \subset F$ of finite perimeter.

By virtue of Lemma 4, in the second case there exists a vector field $\boldsymbol{\xi}_F \in X_F$ such that (30) is satisfied. Supplementing the proof of Theorem 5 with this reasoning we obtain that it holds for $\Omega = \mathbb{R}^2$, provided that $u_0 \in PCR_+(\mathbb{R}^2)$. By a similar modification in section 5, Theorem 7 also holds for $\Omega = \mathbb{R}^2$ with the same provision on u_0. In place of (29) we get the following estimate on the extinction time t_n after which $u = 0$:

$$t_n \le \frac{\mathcal{L}^2(R_0)}{\mathrm{Per}_1(R_0)} \cdot \operatorname{ess\,max} u_0, \tag{38}$$

where we denoted by R_0 the minimal rectangle containing the support of u_0. Its particularly simple form as compared to (29) is due to the fact that R_0 clearly minimizes $\frac{\mathrm{Per}_1(F,\mathbb{R}^2)}{\mathcal{L}^2(F)}$ among bounded rectilinear polygons subordinate to G.

7. Preservation of continuity in rectangles. We start with a lemma concerning PCR functions on a rectangle, which says, roughly speaking, that the maximal oscillation on horizontal (or vertical) lines, on any given length scale, is not increased by the solution to (4) with respect to initial datum $u_0 \in PCR(\Omega)$. To make a precise statement, we fix a rectangle Ω and let G be any grid such that Ω is subordinate to G. Further, let m_1 (m_2) be the number of horizontal (vertical) lines of G. For any given integer $0 \le m \le m_1 - 3$ ($m_2 - 3$) we denote by $\mathcal{R}^2_{1,m}(G)$ ($\mathcal{R}^2_{2,m}(G)$) the set of pairs of rectangles that lay in the strip of Ω between any two successive horizontal (vertical) lines of G and are separated by at most m rectangles in $\mathcal{R}(G)$.

Lemma 5. *Let Ω be a rectangle and let u be the solution to (4) with $u_0 \in PCR(\Omega)$, $\lambda > 0$. Let G be a grid such that $\mathcal{Q}_{u_0}$ is subordinate to G. For $i = 1, 2$ there holds*

$$\max_{(R_1,R_2)\in\mathcal{R}^2_{i,m}(G)} |u|_{R_1} - u|_{R_2}| \le \max_{(R_1,R_2)\in\mathcal{R}^2_{i,m}(G)} |u_0|_{R_1} - u_0|_{R_2}|\,. \tag{39}$$

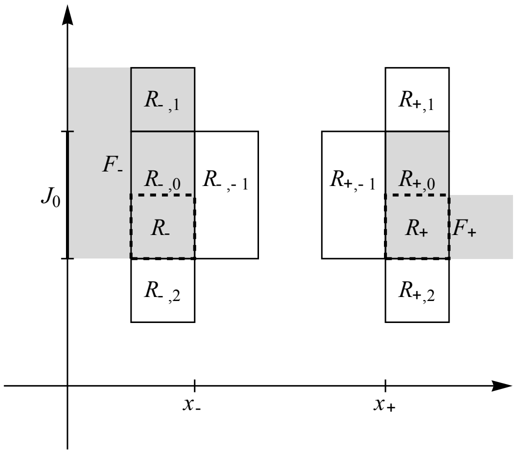


Figure 4. *Map key for the notation in the proof of Lemma 5.*

Remark. Taking $m = 0$ in Lemma 5 we obtain that

$$\mathcal{H}^1 - \text{ess}\max_{J_u}(u_+ - u_-) \leq \mathcal{H}^1 - \text{ess}\max_{J_{u_0}}(u_{0,+} - u_{0,-}).$$

Proof. In the course of the proof we will need to introduce an amount of notation (see Figure 4 for reference). For a given $0 \leq k \leq m_i$ assume we have already proved that (39) holds for each $0 \leq m < k$. Take any pair of rectangles $(R_+, R_-) \in \mathcal{R}^2_{i,k}(G)$ that realizes the maximum in $|u|_{R_1} - u|_{R_2}|$. Let us take rectilinear polygons F_+, F_- in Q_u such that $R_\pm \subset F_\pm$.

Now, we assume that $i = 1$ (i.e., rectangles $R_\pm$ are in the same row of $\mathcal{R}(G)$), $u|_{R_+} > u|_{R_-}$, and R_- is to the left of R_+. Let us denote by x_- the maximal value of the x coordinate of points in R_- and by x_+ the minimal value of the x coordinate of points in R_+. Further, let us denote by

- J_0 the maximal interval such that $\{x_\pm\} \times J_0 \cap \partial R_\pm \neq \emptyset$ and $\{x_\pm\} \times J_0 \subset \partial F_\pm$;
- $R_{\pm,0}$ minimal *rectangles* in $\mathcal{F}(G)$ that have $\{x_\pm\} \times J_0$ as one of their sides and contained $R_\pm$;
- $R_{+,-1}$ (resp., $R_{-,-1}$) the minimal rectangle in $\mathcal{F}(G)$ that has $\{x_+\} \times J_0$ (resp., $\{x_-\} \times J_0$) as one of its sides and does not contain R_+ (resp., R_-);
- K the number of endpoints of J_0 that do not intersect $\partial\Omega$ ($K \in \{0, 1, 2\}$);
- $R_{\pm,j}$, $j \in \mathbb{N}$, $j \leq K$, the K pairs of rectangles in $\mathcal{R}(G)$ such that
  - all of $R_{+,j}$, have a common side with $R_{+,0}$ and belong to the same column in $\mathcal{R}(G)$ as R_+,
  - all of $R_{-,j}$, have a common side with $R_{-,0}$ and belong to the same column in $\mathcal{R}(G)$ as R_-,
  - for a fixed j, both $R_{\pm,j}$ belong to the same row in $\mathcal{R}(G)$.

Due to the way these are defined, fixing $j \leq K$, at least one of the two rectangles $R_{\pm,j}$ is not contained in $F_+ \cup F_-$, and

(40) at least one of the inequalities $u|_{R_{+,j}} < u|_{R_{+,0}}$, $u|_{R_{-,j}} > u|_{R_{-,0}}$ holds.

If there is a pair of rectangles $R'_\pm$ in $\mathcal{R}^2_{1,m}(G)$, $m<k$, such that $R'_\pm \subset F_\pm$, then we have already proved that

$$\left|u|_{R_+} - u|_{R_-}\right| = \left|u|_{R'_+} - u|_{R'_-}\right| \le \max_{(R_1,R_2)\in\mathcal{R}^2_{i,k}} |u_0|_{R_1} - u_0|_{R_2}| .$$

Therefore, we can assume that

$$u|_{R_{+,-1}} < u|_{R_{+,0}} \text{ and } u|_{R_{-,-1}} > u|_{R_{-,0}} \text{ hold.} \tag{41}$$

We have

$$F_- \in \arg\min \mathcal{J}_{F_-,\partial F_-^+,\partial F_-^-,\frac{u_0}{\lambda}} \quad \text{and} \quad F_+ \in \arg\max \check{\mathcal{J}}_{F_+,\partial F_+^+,\partial F_+^-,\frac{u_0}{\lambda}}$$

(see the remarks before the statement of Theorem 5 and after its proof). Therefore, taking into account (40, 41),

$$\begin{aligned}
& u|_{F_+} - u|_{F_-} \\
&= -\lambda\left(\check{\mathcal{J}}_{F_+,\partial F_+^+,\partial F_+^-,\frac{u_0}{\lambda}}(F_+) - \mathcal{J}_{F_-,\partial F_-^+,\partial F_-^-,\frac{u_0}{\lambda}}(F_-)\right) \\
&\le -\lambda\left(\check{\mathcal{J}}_{F_+,\partial F_+^+,\partial F_+^-,\frac{u_0}{\lambda}}(R_+) + \mathcal{J}_{F_-,\partial F_-^+,\partial F_-^-,\frac{u_0}{\lambda}}(R_-)\right) \\
&\le \frac{1}{\mathcal{L}^2(R_+)}\int_{R_+} u_0 \,\mathrm{d}\mathcal{L}^2 - \frac{1}{\mathcal{L}^2(R_-)}\int_{R_-} u_0\,\mathrm{d}\mathcal{L}^2 \le \max_{(R_1,R_2)\in\mathcal{R}^2_{i,k}} |u_0|_{R_1} - u_0|_{R_2}| . \qquad \blacksquare
\end{aligned} \tag{42}$$

Theorem 8. *Let Ω be a rectangle and let u be the solution to (4) with $u_0 \in C(\Omega)$, $\lambda > 0$. Then $u \in C(\Omega)$. In fact, if $\omega_1, \omega_2 \colon [0,\infty[\to [0,\infty[$ are continuous functions such that*

$$|u_0(x_1,x_2) - u_0(y_1,y_2)| \le \omega_1(|x_1-y_1|) + \omega_2(|x_2-y_2|)$$

for each (x_1,x_2), (y_1,y_2) in Ω then we have

$$|u(x_1,x_2) - u(y_1,y_2)| \le \omega_1(|x_1-y_1|) + \omega_2(|x_2-y_2|)$$

for each $t>0$,(x_1,x_2), (y_1,y_2) in Ω.

Remark. Note that if ω is a concave modulus of continuity for u_0 with respect to norm $|\cdot|_1$, then ω_1,ω_2 defined by $\omega_1=\omega_2=\omega$ satisfy the assumptions of Theorem 8. On the other hand, given ω_1,ω_2 as in the Theorem, $\omega' = \omega_1+\omega_2$ is a modulus of continuity for u_0 (as well as u). Theorem 8 implies, for instance, that if L is the Lipschitz constant for u_0 with respect to norm $|\cdot|_1$, then the Lipschitz constant of u with respect to norm $|\cdot|_1$ is not greater than L.

Proof. We denote $\Omega = (s_1,s_2) + [0,l_1]\times[0,l_2]$. For $k=1,2,\dots$ let

$$G_k = \left(\left\{s_1 + \frac{jl_1}{k}, j=0,1,\dots,k\right\}\times\mathbb{R}\right) \cup \left(\mathbb{R}\times\left\{s_2+\frac{jl_2}{k}, j=0,1,\dots,k\right\}\right)$$

and let $u_{0,k}\in PCR(\Omega)$ be defined by

$$u_{0,k}(\boldsymbol{x}) = u_0(\boldsymbol{x}_R) \quad \text{for } \boldsymbol{x}\in R\in\mathcal{R}(G_k),$$

where $\boldsymbol{x}_R$ is the center of R.

For any $k = 1, 2, \dots, i = 1, 2, m = 0, \dots, k-1$, let $(R, R') \in \mathcal{R}^2_{i,m}(G_k)$ with $(x_1, x_2) \in R, (y_1, y_2) \in R'$ we have

$$|u_{0,k}|_R - u_{0,k}|_{R'}| \le \omega_i \left(|x_i - y_i| + \tfrac{1}{k}\right). \tag{43}$$

Let us denote by u_k the solution to (4) with datum $u_{0,k}$. Due to inequality (43) and Lemma 5 we have for any $(x_1, x_2), (y_1, y_2) \in \Omega$,

$$\begin{aligned} |u_k(x_1, x_2) - u_k(y_1, y_2)| &\le |u_k(x_1, x_2) - u_k(y_1, x_2)| + |u_k(y_1, x_2) - u_k(y_1, y_2)| \\ &\le \omega_1 \left(|x_1 - y_1| + \tfrac{1}{k}\right) + \omega_2 \left(|x_2 - y_2| + \tfrac{1}{k}\right). \end{aligned}$$

Now, due to monotonicity of $-\partial TV_{1,\Omega}$, we have $\|u_k - u\|_{L^2(\Omega)} \le \|u_{0,k} - u_0\|_{L^2(\Omega)}$. Therefore, there exists a set $N \subset \Omega$ of zero $\mathcal{L}^2$ measure and a subsequence (k_j) such that $u_{k_j}(\boldsymbol{x}) \to u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega \setminus N$. Now, for each pair $(x_1, x_2), (y_1, y_2) \in \Omega$ take any pair of sequences $((x_{1,n}, x_{2,n})), ((y_{1,n}, y_{2,n})) \subset \Omega \setminus N$ such that $x_{i,n} \to x_i$ and $y_{i,n} \to y_i$. Passing to the limit $j \to \infty$ and then $n \to \infty$ in

$$|u_{k_j}(x_{1,n}, x_{2,n}) - u_{k_j}(y_{1,n}, y_{2,n})| \le \omega_1 \left(|x_{1,n} - y_{1,n}| + \tfrac{1}{k_j}\right) + \omega_2 \left(|x_{2,n} - y_{2,n}| + \tfrac{1}{k_j}\right),$$

we conclude the proof. ∎

Analogous results can be obtained for the solutions to (5).

Lemma 6. *Let Ω be a rectangle, let u be the solution to (5) with $u_0 \in PCR(\Omega)$, and let G be a grid such that $\mathcal{Q}_{u_0}$ is subordinate to G. Then for $i = 1, 2$ there holds*

$$\max_{(R_1, R_2) \in \mathcal{R}^2_{i,m}(G)} |u(t, \cdot)|_{R_1} - u(t, \cdot)|_{R_2}| \le \max_{(R_1, R_2) \in \mathcal{R}^2_{i,m}} |u_0|_{R_1} - u_0|_{R_2}| \tag{44}$$

in any time instance $t > 0$.

Proof. The form of solution obtained in Theorem 7 implies that the function

$$t \mapsto \max_{(R_1, R_2) \in \mathcal{R}^2_{i,m}(G)} |u(t, \cdot)|_{R_1} - u(t, \cdot)|_{R_2}|$$

is piecewise linear and continuous, in particular, it does not have jumps. Having this observation in mind, let us consider time instance $\tau > 0$ that does not belong to the set of merging times $\{t_1, \dots, t_n\}$.

For a given $0 \le k \le m_i$ assume we have already proved that the slope of

$$t \mapsto \max_{(R_1, R_2) \in \mathcal{R}^2_{i,m}(G)} |u(t, \cdot)|_{R_1} - u(t, \cdot)|_{R_2}|$$

is nonpositive in $t = \tau$ for each $0 \le m < k$. Take any pair of rectangles $(R_+, R_-) \in \mathcal{R}^2_{i,k}(G)$ that realizes the maximum in $|u(t, \cdot)|_{R_1} - u(t, \cdot)|_{R_2}|$. Let us take rectilinear polygons F_+, F_- in $\mathcal{Q}_{u(t,\cdot)}$ such that $R_\pm \subset F_\pm$.

Then, reasoning as in the proof of Lemma 5, we obtain

$$\begin{aligned} u_t(\tau, \cdot)|_{F_+} - u_t(\tau, \cdot)|_{F_-} &= -\check{\mathcal{J}}_{F_+, \partial F_+^+, \partial F_+^-}(F_+) + \mathcal{J}_{F_-, \partial F_-^+, \partial F_-^-}(F_-) \\ &\le -\check{\mathcal{J}}_{F_+, \partial F_+^+, \partial F_+^-}(R_+) + \mathcal{J}_{F_-, \partial F_-^+, \partial F_-^-}(R_-) \le 0 \end{aligned} \tag{45}$$

which concludes the proof. ∎

Now, note that if u and v are two solutions to (15) with $L^2(\Omega)$ initial data u_0 and v_0, respectively, we have for each time instance $t > 0$ (see [28, Theorem 11])

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2(\Omega)} \le \|u_0 - v_0\|_{L^2(\Omega)}.$$

Using this fact and Lemma 6, we can obtain the following analog of Theorem 8 for solutions of (15). The proof is almost identical and we omit it.

Theorem 9. *Let Ω be a rectangle and let u be the solution to (15) with initial datum $u_0 \in C(\Omega)$. Then $u(t,\cdot) \in C(\Omega)$ in every $t > 0$. In fact, if $\omega_1, \omega_2 \colon [0,\infty[\to [0,\infty[$ are continuous functions such that*

$$|u_0(x_1,x_2) - u_0(y_1,y_2)| \le \omega_1(|x_1 - y_1|) + \omega_2(|x_2 - y_2|)$$

for each (x_1,x_2), (y_1,y_2) in Ω, then we have

$$|u(t,(x_1,x_2)) - u(t,(y_1,y_2))| \le \omega_1(|x_1 - y_1|) + \omega_2(|x_2 - y_2|)$$

for each $t > 0$,(x_1,x_2), (y_1,y_2) in Ω.

Finally, we note that all the results in this section carry over in a straightforward way to the case $\Omega = \mathbb{R}^2$, provided that in the statements of Theorems 8 and 9 $C(\Omega)$ is replaced with $C_{c,+}(\mathbb{R}^2)$ (meaning nonnegative, compactly supported continuous functions on $\mathbb{R}^2$). On the other hand, if Ω is a rectilinear polygon different from a rectangle, the continuity is not necessarily preserved as Example 3 shows.

8. Examples. We start with the following general fact showing that minimizing (2) and solving (15) is equivalent in some cases.

Theorem 10. *Let Ω be a bounded domain or $\Omega = \mathbb{R}^2$ and let u be the strong solution to (15) in $[0,T[$ with initial datum $u_0 \in L^2(\Omega)$. If there exists $\boldsymbol{z} \in L^\infty(]0,T[\times\Omega,\mathbb{R}^2)$ satisfying conditions (16)–(19) such that for some $0 < \lambda < T$ and almost all $0 < t < \lambda$*

$$\int_\Omega (\boldsymbol{z}(t,\cdot), Du(\lambda,\cdot)) = \int_\Omega |Du(\lambda,\cdot)|_\varphi, \tag{46}$$

then the minimizer of (2) with $\lambda > 0$ is given by $u_\lambda = u(\lambda,\cdot)$.

Proof. Let $\boldsymbol{z}_\lambda = \frac{1}{\lambda}\int_0^\lambda \boldsymbol{z}(t,\cdot)\,\mathrm{d}t$. Clearly, $\boldsymbol{z}_\lambda \in X_\Omega(u_\lambda)$ satisfies $[\boldsymbol{z}_\lambda, \boldsymbol{\nu}^\Omega] = 0$ and $|\boldsymbol{z}_\lambda|_\varphi^* \le 1$. Furthermore, by virtue of (46),

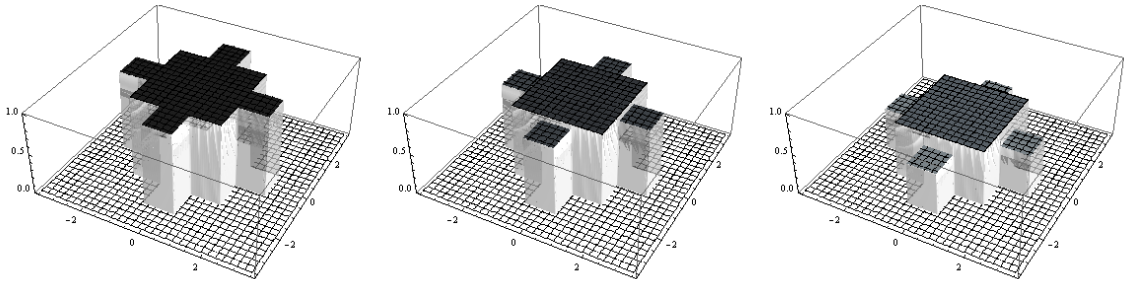
$$\int_\Omega (\boldsymbol{z}_\lambda, Du_\lambda) = \int_\Omega |Du_\lambda|_\varphi, \tag{47}$$

so $\boldsymbol{z}_\lambda \in X_{\varphi,\Omega}(u_\lambda)$. Finally, (16) implies

$$u_\lambda - u_0 = \lambda \operatorname{div} \boldsymbol{z}_\lambda \quad \text{in } \mathcal{D}'(\Omega). \tag{48}$$

■

One class of solutions to (15) such that (46) holds, are PCR solutions constructed in Theorem (7) up to the first (positive) breaking time, as defined in the following.



(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 5. *Plots of $u(t,\cdot)$ from Example 1 at certain time instances t.*

Definition 2. *Let $u \in W^{1,\infty}([0,\infty[, BV(\Omega))$ be the global strong solution to (5) with $u_0 \in PCR(\Omega)$ and let $0 < t_1 < \dots t_n$ be the sequence of time instances obtained in Theorem 7. We call each of $t_1, \dots, t_n$ a* merging time. *We say that t_i, $i = 1, \dots, n$, is a* (positive) breaking time *if $\mathcal{H}^1(J_{u(t,\cdot)} \setminus J_{u(t_i,\cdot)}) > 0$ for $t \in]t_i, t_{i+1}[$.*

Indeed, let $t_k > 0$ be the first breaking time and $0 < t < \lambda \le t_k$. Then $J_{u(\lambda,\cdot)} \subset J_{u(t,\cdot)}$ up to a $\mathcal{H}^1$-null set and $\frac{Du(\lambda,\cdot)}{|Du(\lambda,\cdot)|} = \frac{Du(t,\cdot)}{|Du(t,\cdot)|}$ holds $|Du(\lambda,\cdot)|$-a.e., which implies (46).

Now we provide several examples illustrating the strength of our results. Note that even though they are formulated in the language of the flow, in all of them $]0,\lambda[\ni t \mapsto \boldsymbol{z}(t,\boldsymbol{x})$ is constant a.e. for $|Du(\lambda,\cdot)|$-almost every $\boldsymbol{x} \in \Omega$ which implies (46), and therefore they are also solutions to (4).

Theorem 7 predicts that the jump set of a function PCR may expand under the TV_1 flow, i.e., facet breaking may occur. Many explicit examples of this kind can be constructed. Here we present a simple one, for which the procedure described in the proof of Theorem 6 is concise enough to be presented in detail.

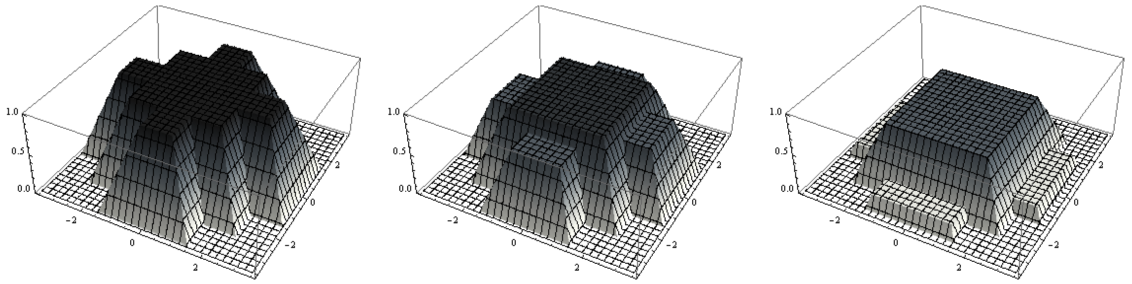
Example 1. Let

$$u(t,\cdot) = \left(1 - \tfrac{4}{3}t\right)_+ \chi_B + (1-2t)_+ \chi_C,$$

where we denoted

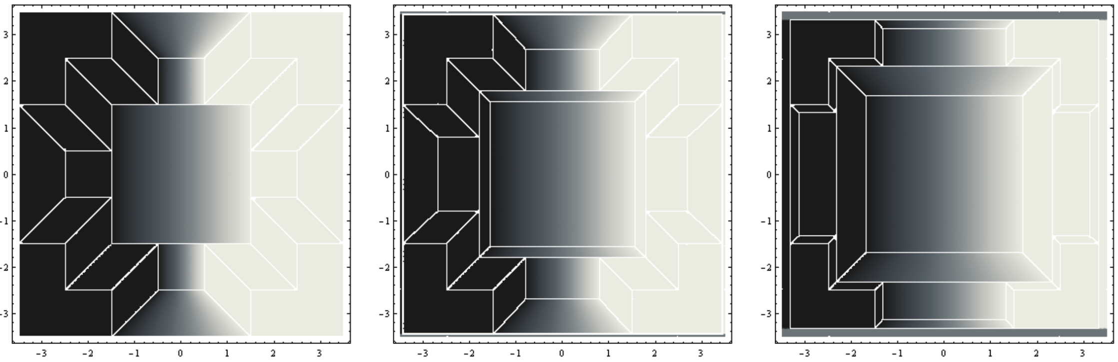
$$B = B_\infty\left((0,0), \tfrac{3}{2}\right), \quad C = B_\infty\left((2,0), \tfrac{1}{2}\right) \cup B_\infty\left((-2,0), \tfrac{1}{2}\right) \cup B_\infty\left((0,2), \tfrac{1}{2}\right) \cup B_\infty\left((0,-2), \tfrac{1}{2}\right).$$

Plots of $u(t,\cdot)$ at certain time instances are depicted in Figure 5. For each $t \ge 0$, $u(t,\cdot) \in PCR_+(\mathbb{R}^2)$ and u solves (5) with initial datum $u_0 = \chi_{B\cup C}$. To see this, we execute the algorithm described in Theorem 6. Let $Q_1 = u_0^{-1}(1) = B \cup C$. Due to symmetry, the only plausible largest minimizers of $\mathcal{J}_{Q_1,\partial Q_1,\emptyset,0}$ are B, C, and $B \cup C$ (we only need to consider elements of $\mathcal{F}_{u_0}$ and no subset of square B can produce a lower value of the functional than B). We check that the values of $\mathcal{J}_{Q_1,\partial Q_1,\emptyset,0}$ on these sets are, respectively, $\frac{4}{3}$, 4, and $\frac{20}{13}$, hence B is the minimizer and the initial velocity on B is $-\frac{4}{3}$. Next, we have to find the largest minimizer of $\mathcal{J}_{C,\partial Q_1,\partial B,0}$. There is only one competitor, C. To find the initial velocity on C, we calculate $-\mathcal{J}_{C,\partial Q_1,\partial B,0}(C) = -2$. Finally, as explained in section 6, we need to find the largest minimizer of $\mathcal{J}_{Q_0,\partial R_0,\partial Q_1,0}$, where we denoted R_0 to be the smallest rectangle (square)



(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 6. *Plots of $u(t, \cdot)$ from Example 2 at certain time instances t.*



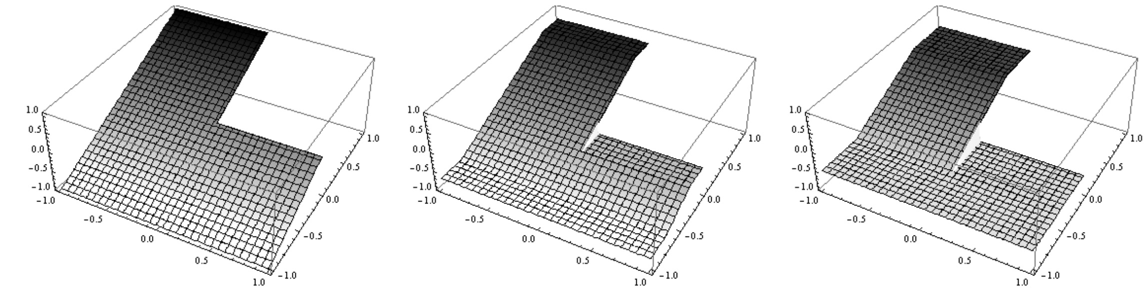
(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 7. *Density plots of $z_1(t, \cdot)$ corresponding to $u(t, \cdot)$ from Example 2 at certain time instances t. Black corresponds to value 1, ivory to -1; note the value 0 outside the minimal strip containing the support of $u(t, \cdot)$.*

containing the support of u_0 and $Q_0 = R_0 \cap u_0^{-1}(0)$. We check that the minimizer is Q_0 itself with $\mathcal{J}_{Q_0, \partial R_0, \partial Q_1, 0}(Q_0) = 0$.

On the other hand, Theorem 8 asserts that if u_0 is (Lipschitz) continuous, the solution u starting with u_0 is (Lipschitz) continuous in every time instance $t > 0$. For instance, if one extends the characteristic function form Example 1 continuously outside its support, no jumps will appear in the evolution—another manifestation of nonlocality of the equation.

Example 2. Here we present Figure 6, depicting the evolution u of a piecewise linear continuous function u_0 obtained by extending the initial datum from Example 1 outside its support up to 0 in such a way that $\nabla u_0 \in \{(0, 0), (1, 0), (0, 1)\}$. The evolution is obtained by explicit identification of the corresponding field $\boldsymbol{z} = (z_1, z_2)$ under an ansatz that in each of a finite number of evolving regions either $z_i = \pm 1$ or a z_i is a linear interpolation of boundary values, $i = 1, 2$ (see Figure 7). This reduces the problem to a decoupled infinite system of ODEs. The evolution obtained this way is the strong solution starting with u_0 as it satisfies all the requirements in Definition 1. Figures 6 and 7 are obtained by solving numerically the system of ODEs using the Mathematica `NDSolve` function. We omit the quite lengthy details.



(a) $t = 0$. (b) $t = 0.04$. (c) $t = 0.12$.

Figure 8. *Plots of $u(t, \cdot)$ from Example 3 at certain time instances t.*

Next we provide an example showing that in nonconvex rectilinear polygons (i.e., other than a rectangle), evolution starting with a continuous initial datum may develop discontinuities.

Example 3. Let

$$\Omega = \{(x_1, x_2) \colon |(x_1, x_2)|_\infty \leq 1, x_1 \leq 0, x_2 \leq 0\}, \quad u_0(x_1, x_2) = x_2,$$

and so $\nabla u(0, \cdot) \equiv (0, 1)$, $\boldsymbol{z}(0, \cdot) \equiv (0, 1)$. The solution can be written explicitly, for $t \leq \frac{1}{8}$ we have

$$u(t, x_1, x_2) = \begin{cases} -1 + \sqrt{2t} & \text{if } x_2 \leq -1 + \sqrt{2t}, \\ -\sqrt{2t} & \text{if } x_1 \geq 0 \text{ and } x_2 \geq -\sqrt{2t}, \\ 1 - \sqrt{2t} & \text{if } x_1 < 0 \text{ and } x_2 \geq 1 - \sqrt{2t}, \\ x_2 & \text{otherwise.} \end{cases}$$

Plots of $u(t, \cdot)$ at certain time instances are depicted in Figure 8.

We see that regions where $\nabla u = 0$ appear near the boundary and expand with speed $\frac{1}{\sqrt{2t}}$. In these regions, z_2 is a linear interpolation between 0 and 1. Also a jump in the x_2 direction appears near $\boldsymbol{x} = 0$ and grows with the same speed.

Finally, let us present an exact calculation for an example of the phenomenon of bending, which shows the effectiveness of approximation with PCR functions.

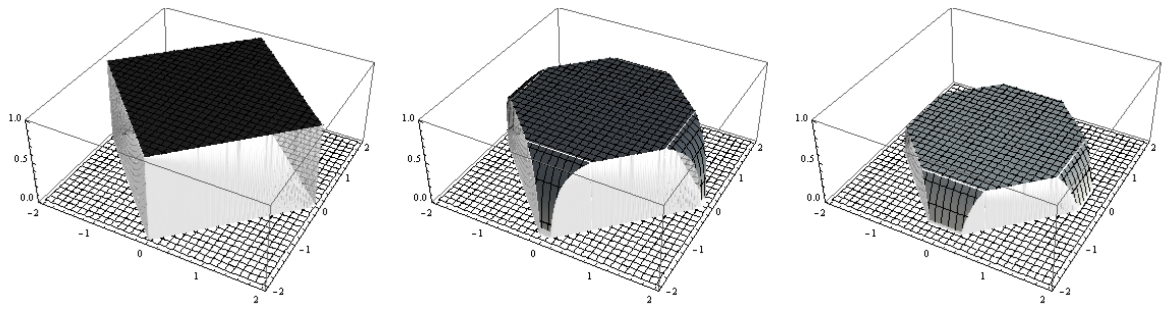
Example 4. Let $\Omega = \mathbb{R}^2$, $u_0 = \chi_{B_1(2)}$. We will show that the solution to (5) is given by

$$u(t, \boldsymbol{x}) = (1 - v(\boldsymbol{x})t)_+$$

with $v = \frac{1}{2-\sqrt{2}} \chi_{B_\infty(\sqrt{2}) \cap B_1(2)} + \frac{1}{2-|\boldsymbol{x}|_\infty} \chi_{B_1(2) \setminus B_\infty(\sqrt{2})}$. Plots of $u(t, \cdot)$ at certain time instances are depicted in Figure 9.

In order to prove the claim, we approximate $B_1(2)$ by a family of rectilinear polygons as follows. Given $n \in \mathbb{N}$, we define inductively

$$\begin{cases} A_{1,1} := B_\infty(1), \\ A_{k,1} := B_\infty((\frac{2(k-1)}{n}, 0), 1 - \frac{k-1}{n}) \setminus A_{k-1,1} \quad \text{for } k = 2, \dots, n. \end{cases}$$



(a) $t = 0$. (b) $t = 0.1$. (c) $t = 0.3$.

Figure 9. *Plots of $u(t,\cdot)$ from Example 4 at certain time instances t.*

We observe that $\bigcup_{k=1}^n A_{k,1}$ is an increasing sequence with respect to n and that

$$\lim_{n\to\infty} \bigcup_{k=1}^n A_{k,1} = B_1(2) \cap \{x_1 > 0, -1 \leq x_2 \leq 1\}.$$

By symmetry, we construct $A_{k,2}$, $A_{k,3}$, and $A_{k,4}$ for $k = 1, \dots, n$ such that

$$\lim_{n\to\infty} \bigcup_{k=1}^n A_{k,2} = B_1(2) \cap \{x_2 > 0, -1 \leq x_1 \leq 1\},$$
$$\lim_{n\to\infty} \bigcup_{k=1}^n A_{k,3} = B_1(2) \cap \{x_1 < 0, -1 \leq x_2 \leq 1\},$$
$$\lim_{n\to\infty} \bigcup_{k=1}^n A_{k,4} = B_1(2) \cap \{x_2 < 0, -1 \leq x_1 \leq 1\}.$$

Therefore, $B_1(2) = \lim_{n\to\infty} A_n := \bigcup_{k=1}^n \bigcup_{j=1}^4 A_{k,j}$.

We let $C_k := \bigcup_{j=1}^4 A_{k,1}$ for $k = 1, \dots, n$. Observe that the inequality

$$\frac{\mathrm{Per}_1(C_1 \cup \dots \cup C_{k+1})}{|C_1 \cup \dots \cup C_{k+1})|} > \frac{\mathrm{Per}_1(C_1 \cup \dots \cup C_k)}{|C_1 \cup \dots \cup C_k)|}$$

holds (and, therefore, the facet C_{k+1} breaks from $C_1 \cup \dots \cup C_k$) if and only if

$$\frac{1}{1-\frac{k}{n}} > \frac{\mathrm{Per}_1(C_1 \cup \dots \cup C_k)}{|C_1 \cup \dots \cup C_k)|} = 2\frac{1+\frac{k-1}{n}}{1+\frac{2(k-1)}{n}(1-\frac{k}{2n})} \leftrightarrow \frac{k}{n} \geq \sqrt{\left(1-\frac{1}{2n}\right)^2 + 1} - \left(1-\frac{1}{2n}\right).$$

Since the speed of C_j is given by $\frac{1}{1-j/n}$ (which increases with respect to $j = k+1, \dots, n$), once C_{k+1} breaks from $C_1 \cup \dots \cup C_k$), so do C_j from C_{j-1} for $j = k+1, \dots, n$. Therefore, the solution for $u_{0,n} = \chi_{A_n}$ is given by

$$u_n(t, \boldsymbol{x}) = \left(1 - \frac{\mathrm{Per}_1(C_1 \cup \dots \cup C_k)}{|C_1 \cup \dots \cup C_k)|} t\right)_+ \chi_{(C_1 \cup \dots \cup C_k)} + \sum_{j=k+1}^n \left(1 - \frac{1}{1-\frac{j}{n}} t\right)_+ \chi_{C_j} \tag{49}$$

with $k \in \mathbb{N}$ satisfying

$$[k-1] < n\left(\sqrt{\left(1-\frac{1}{2n}\right)^2+1}-\left(1-\frac{1}{2n}\right)\right) \leq [k].$$

Letting $n \to \infty$ in (49), we finish the proof.

The evolution of a bounded convex domain C satisfying an interior ball condition was explicitly given in [13, section 8.3]. There, the authors defined a notion of anisotropic variational mean curvature, denoted by $H_C\colon \mathbb{R}^N \to]-\infty, 0]$, based on the solvability of some auxiliary minimizing problems and on the existence of a Cheeger set in C. Then, the solution to (15) with data $u_0 = \chi_C$ was given by

$$u(t, \boldsymbol{x}) = (1 + H_C(\boldsymbol{x})t)_+ \chi_C(\boldsymbol{x}).$$

In general, it is not obvious how to compute this anisotropic variational mean curvature. However, Example 4 shows that one can compute it by approximation of the set C with rectilinear polygons, even in the case that C does not satisfy the interior ball condition. Note that the solution, starting with the initial datum as in Example 4, was calculated with an approximate numerical procedure in [29]. Here, we provided the exact evolution in this case.

9. Conclusions.

The core of our results is the explicit construction of *tetris-like solutions*, i.e., solutions in the PCR class. This class can be viewed as a natural generalization of mono-dimensional step functions, whose finite dimensional structure allows us to effectively reduce the original nonlinear problem. The directional diffusion allows us to analyze the solutions only in a grid given by a suitably chosen initial datum. We treat them as generic objects in the set of all weak solutions, hence PCR solutions are indeed smooth functions in the new analytical language exclusively dedicated to our variational problem. The detailed prescription of solutions allows us to even prove conservation of moduli of continuity for continuous initial data. It, unexpectedly, removes this classical viewpoint on the issue of solvability out of our interests. Just information obtained for PCR functions is much more complete than any knowledge of regularity in the classical setting.

At the end we would like to say a few words about the weakness of the approach. The procedure works due to the possibility of introducing a grid. It is the consequence of symmetry given by the $|\cdot|_1$ norm, the grid is just determined by directions $\hat{e}_{x_1}$, $\hat{e}_{x_2}$ for the initial datum. Here we have a natural shift symmetry, and the same structure at each vertex of the grid. It seems that it would be possible to attempt to repeat at least some of our analysis for anisotropic norms that generate a tiling of the plane. Here we think of $|\cdot|_\phi$ determined by the hexagon, and tiling given by a honeycomb structure. We are highly limited by regular tiling (triangular, rectangular, and hexagonal), however it seems to be possible to introduce a more complex structure for a different anisotropy. Such problems will definitely require a new framework not linked to the classical analysis.

Acknowledgments. The authors want to thank the anonymous referees whose comments really helped improve the quality of the paper. Thanks are also due to Michał Miśkiewicz for careful reading of the manuscript and pointing out some shortcomings.

All figures were prepared using Wolfram Mathematica.

REFERENCES

[1] M. Amar and G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11 (1994), pp. 91–133.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Oxford University Press, New York, 2000.

[3] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4), 135 (1983), pp. 293–318 (1984), https://doi.org/10.1007/BF01781073.

[4] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, Springer, New York, 2010, https://doi.org/10.1007/978-1-4419-5542-5.

[5] G. Bellettini, V. Caselles, and M. Novaga, *The total variation flow in* $\mathbb{R}^N$, J. Differential Equations, 184 (2002), pp. 475–525, https://doi.org/10.1006/jdeq.2001.4150.

[6] G. Bellettini, M. Novaga, and M. Paolini, *On a crystalline variational problem.* I. *First variation and global* L^∞ *regularity*, Arch. Ration. Mech. Anal., 157 (2001), pp. 165–191, https://doi.org/10.1007/s002050010127.

[7] G. Bellettini, M. Novaga, and M. Paolini, *On a crystalline variational problem.* II. *BV regularity and structure of minimizers on facets*, Arch. Ration. Mech. Anal., 157 (2001), pp. 193–217, https://doi.org/10.1007/s002050100126.

[8] B. Berkels, M. Burger, M. Droske, O. Nemitz, and M. Rumpf, *Cartoon extraction based on anisotropic image classification vision, modeling, and visualization*, in Vision, Modeling, and Visualization 2006: Proceedings, 2006, IOS Press, Aachen, Germany, 2006, pp. 293–300.

[9] M. Bonforte and A. Figalli, *Total variation flow and sign fast diffusion in one dimension*, J. Differential Equations, 252 (2012), pp. 4455–4480, https://doi.org/10.1016/j.jde.2012.01.003.

[10] A. Briani, A. Chambolle, M. Novaga, and G. Orlandi, *On the gradient flow of a one-homogeneous functional*, Confluentes Math., 3 (2011), pp. 617–635, https://doi.org/10.1142/S1793744211000461.

[11] M. Burger, G. Gilboa, M. Moeller, L. Eckardt, and D. Cremers, *Spectral decompositions using one-homogeneous functionals*, SIAM J. Imaging Sci., 9 (2016), pp. 1374–1408, https://doi.org/10.1137/15M1054687.

[12] E. Casas, K. Kunisch, and C. Pola, *Regularization by functions of bounded variation and applications to image enhancement*, Appl. Math. Optim., 40 (1999), pp. 229–257, https://doi.org/10.1007/s002459900124.

[13] V. Caselles, A. Chambolle, S. Moll, and M. Novaga, *A characterization of convex calibrable sets in* $\mathbb{R}^N$ *with respect to anisotropic norms*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 803–832, http://dx.doi.org/10.1016/j.anihpc.2008.04.003.

[14] V. Caselles, A. Chambolle, and M. Novaga, *The discontinuity set of solutions of the TV denoising problem and some extensions*, Multiscale Model. Simul., 6 (2007), pp. 879–894, https://doi.org/10.1137/070683003.

[15] V. Caselles, A. Chambolle, and M. Novaga, *Regularity for solutions of the total variation denoising problem*, Rev. Mat. Iberoam., 27 (2011), pp. 233–252, https://doi.org/10.4171/RMI/634.

[16] A. Chambolle and J. Darbon, *On total variation minimization and surface evolution using parametric maximum flows*, Int. J. Comput. Vis., 84 (2009), 288.

[17] R. H. Chan, S. Setzer, and G. Steidl, *Inpainting by flexible Haar-wavelet shrinkage*, SIAM J. Imaging Sci., 1 (2008), pp. 273–293, https://doi.org/10.1137/070711499.

[18] H. Chen, C. Wang, Y. Song, and Z. Li, *Split Bregmanized anisotropic total variation model for image deblurring*, J. Vis. Comun. Image Represent., 31 (2015), pp. 282–293, https://doi.org/10.1016/j.jvcir.2015.07.004.

[19] R. Choksi, Y. van Gennip, and A. Oberman, *Anisotropic total variation regularized* L^1 *approximation and denoising/deblurring of 2D bar codes*, Inverse Probl. Imaging, 5 (2011), pp. 591–617, https://doi.org/10.3934/ipi.2011.5.591.

[20] S. Esedoḡlu and S. J. Osher, *Decomposition of images by the anisotropic Rudin-Osher-Fatemi model*, Comm. Pure Appl. Math., 57 (2004), pp. 1609–1626, https://doi.org/10.1002/cpa.20045.

[21] B. G. Fitzpatrick and S. L. Keeling, *On approximation in total variation penalization for image reconstruction and inverse problems*, Numer. Funct. Anal. Optim., 18 (1997), pp. 941–958.

[22] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monogr. Math. 80, Birkhäuser, Basel, 1984, https://doi.org/10.1007/978-1-4684-9486-0.

[23] T. Goldstein and S. Osher, *The split Bregman method for $L1$-regularized problems*, SIAM J. Imaging Sci., 2 (2009), pp. 323–343, https://doi.org/10.1137/080725891.

[24] M. Grasmair and F. Lenzen, *Anisotropic total variation filtering*, Appl. Math. Optim., 62 (2010), pp. 323–339, https://doi.org/10.1007/s00245-010-9105-x.

[25] D. S. Hochbaum, *Multi-label Markov random fields as an efficient and effective tool for image segmentation, total variations and regularization*, Numer. Math. Theory Methods Appl., 6 (2013), pp. 169–198.

[26] K. Kielak, P. B. Mucha, and P. Rybka, *Almost classical solutions to the total variation flow*, J. Evol. Equ., 13 (2013), pp. 21–49, https://doi.org/10.1007/s00028-012-0167-x.

[27] G. Mercier, *Continuity results for TV-minimizers*, Indiana Math. J.. to appear.

[28] S. Moll, *The anisotropic total variation flow*, Math. Ann., 332 (2005), pp. 177–218, https://doi.org/10.1007/s00208-004-0624-0.

[29] P. B. Mucha, M. Muszkieta, and P. Rybka, *Two cases of squares evolving by anisotropic diffusion*, Adv. Differential Equations, 20 (2015), pp. 773–800.

[30] W. Ring, *Structural properties of solutions to total variation regularization problems*, ESAIM Math. Model. Numer. Anal., 34 (2000), pp. 799–810.

[31] L. I. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D, 60 (1992), pp. 259–268.

[32] S. Setzer, G. Steidl, and T. Teuber, *Restoration of images with rotated shapes*, Numer. Algorithms, 48 (2008), pp. 49–66, https://doi.org/10.1007/s11075-008-9182-y.