Investigation of L^2-stability – a few examples

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1 Diffusion vs autocatalysis

Let us first consider following (affine) problem for a function $u\colon \Omega \times [0,T] \to \mathbb{R}$ (with Ω – bounded domain in $\mathbb{R}^n$) satisfying, in every time instance $t \in [0,T]$,

$$u_t = \kappa \Delta u + au \quad \text{in } \Omega, \tag{1}$$

$$u = f \quad \text{at } \partial\Omega. \tag{2}$$

The function u may represent temperature of a (solid) medium filling a container whose walls coincide with $\partial\Omega$. Then $\kappa > 0$ is thermal diffusivity. The medium undergoes an exothermal reaction whose speed depends on temperature in such a way that net change of temperature in a unit of time due to reaction is equal to au for some constant $a > 0$. Further, the walls of container are kept at a constant distribution of temperature $f\colon \partial\Omega \to \mathbb{R}$.

We are interested in investigating stability of a stationary solution to (1, 2) u^* that satisfies

$$0 = \kappa \Delta u^* + au^* \quad \text{in } \Omega, \tag{3}$$

$$u^* = f \quad \text{at } \partial\Omega \tag{4}$$

in a class of classical solutions to (1, 2) with any initial datum. [1]

To this purpose let us subtract (3, 4) from (1, 2) and denote $w = u - u^*$. Then w satisfies

$$w_t = \kappa \Delta w + aw \quad \text{in } \Omega, \tag{5}$$

$$u = 0 \quad \text{at } \partial\Omega. \tag{6}$$

We may notice that there is a competition in (5) between diffusion and autocatalysis. In the case $a = 0$, w would vanish exponentially due to "transport of thermal energy outside Ω" (diffusion + Dirichlet boundary condition). We would like to investigate the behaviour of perturbation w in time depending on the coefficients κ and a. To this point,

[1] A unique solution to the problem (3, 4) with any given smooth initial datum may be shown to exist whenever κ, a satisfy the same condition as for stability (provided that f and $\partial\Omega$ are sufficiently smooth). In other case a solution may or may not exist or be unique.

we introduce quantity $\|w\|_2 = (\int_\Omega |w|^2)^{\frac{1}{2}}$ i. e. the norm of w in $L_2(\Omega)$.[2] which may be referred to as *energy* of the perturbation w. Multiplying (5) by w and integrating over Ω we obtain

$$\frac{1}{2}\int_\Omega (w^2)_t = \kappa \int_\Omega w\Delta w + a\int_\Omega w^2.$$

Integrating by parts using (6) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 = -\kappa\|\nabla w\|_2^2 + a\|w\|_2^2. \tag{7}$$

If we were able to show that

$$-\kappa\|\nabla w\|_2^2 + a\|w\|_2^2 \le -\epsilon\|w\|_2^2 \tag{8}$$

for any w for some $\varepsilon > 0$, we'd obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 \le -\varepsilon\|w\|_2^2 \tag{9}$$

which would lead to exponential decay of $\|w\|_2$, i. e. *(exponential) asymptotic L^2-stability* of u^*. Let's assume that we were able to show that there exists a constant λ_L such that

$$\frac{\|\nabla w\|_2^2}{\|w\|_2^2} \ge \lambda_L \tag{10}$$

for any w. This would imply that (8) is satisfied whenever $\lambda_L \ge \frac{a+\varepsilon}{\kappa}$ and consequently that we have exponential stability whenever $\lambda_L > \frac{a}{\kappa}$. It turns out that we have

Proposition 1 (Poincaré inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. For any $w \in C_0^2(\Omega)$ there holds*

$$\frac{\|\nabla w\|_2^2}{\|w\|_2^2} \ge \lambda_L.$$

λ_L, the least eigenvalue of Laplace operator with Dirichlet boundary conditions in Ω, is strictly positive and is the optimal constant in this inequality.

Sketch of proof. Let's consider a functional $w \mapsto \frac{\|\nabla w\|_2^2}{\|w\|_2^2}$. For this functional to attain a minimum at some w_0, a function $\varepsilon \mapsto \frac{\|\nabla(w_0+\varepsilon w)\|_2^2}{\|(w_0+\varepsilon w)\|_2^2}$ should have a minimum at $\epsilon = 0$ for any given w. This condition leads to integral equation

$$\int_\Omega \nabla w_0 \cdot \nabla w = \frac{\|\nabla w_0\|_2^2}{\|w_0\|_2^2}\int_\Omega w_0 w \tag{11}$$

satisfied by w_0 for any w. Integrating by parts,

$$\int_\Omega (-\Delta w_0 - \frac{\|\nabla w_0\|_2^2}{\|w_0\|_2^2} w_0) w = 0 \tag{12}$$

[2]This quantity may be as well defined for vector (or tensor) valued functions. $|w|^2$ then denotes the sum of squares of all components of w (the Euclidean length squared).

for any w. Due to fundamental lemma of calculus of variation,

$$-\Delta w_0 = \frac{\|\nabla w_0\|_2^2}{\|w_0\|_2^2} w_0 \tag{13}$$

follows. This means that w_0 is an eigenvector of $-\Delta$ with eigenvalue $\frac{\|\nabla w_0\|_2^2}{\|w_0\|_2^2}$. As it is known that the eigenvalues of $-\Delta$ with Dirichlet boundary conditions form a monotone sequence of positive numbers converging to ∞,[3] denoting the least one by λ_L ends proof. □

We have now proved the stability of any stationary solution to (1, 2) whenever $\lambda_L > \frac{a}{\kappa}$. On the other hand, consider a situation where $\frac{a}{\kappa} = \lambda$ for some $\lambda \geq \lambda_L$ – eigenvalue of $-\Delta$. Then for w, eigenvector of $-\Delta$ corresponding to λ, we have

$$w_t = \kappa \Delta w + aw = -\kappa\lambda + a = 0$$

and so, asymptotic stability does not hold. This consideration and the above proof of Poincaré inequality show that for linear problems, "linear-anzac" and "energetic" methods of investigation of stability are more or less the same thing and it is therefore no surprise that they lead to the same results.

2 Navier-Stokes system – the role of constraints

Let's now consider the following problem for functions $\mathbf{v}\colon \Omega \times [0,T] \to \mathbb{R}^n$ and $p\colon \Omega \times [0,T] \to \mathbb{R}$ on some domain Ω in $\mathbb{R}^n$ ($n = 2$ or 3) representing velocity and pressure of incompressible flow

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = \mu \Delta \mathbf{v} - \nabla p + a\mathbf{v} \quad \text{in } \Omega, \tag{14}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \tag{15}$$

$$\mathbf{v} = 0 \quad \text{at } \partial\Omega \tag{16}$$

in every time instance $t \in [0,T]$.

The differences from the previous example are:

- Vector character of this problem (which does not really introduce any additional difficulty).
- Nonlinearity of the convection term. We will deal with it easily, however, due to nonlinearity, it is not possible to reduce investigation of stability of this problem to the "linear-anzac" analysis.
- The incompressibility constraint (15). As we will see, its presence will allow us to obtain slightly better results than in unconstrained case.

[3] This follows from the fact that $(-\Delta)^{-1}\colon L^2(\Omega) \to L^2(\Omega)$ is a positive compact operator.

To avoid technical difficulties (and to disregard the matters of existence) we now chose homogeneous boundary condition (16) and we will only investigate stability of the solution $\mathbf{v}^* = 0$, $p^* = 0$. To this purpose we take a scalar product of (14) with $\mathbf{v}$ and integrate over Ω. We have, due to (15) and (16),

$$\int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{v} = \frac{1}{2}\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}^2 = -\frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{v})\mathbf{v}^2 = 0,$$

$$\int \nabla p\cdot\mathbf{v} = \int p\nabla\cdot\mathbf{v} = 0$$

and therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_2^2 = -\mu\|\nabla\mathbf{v}\|_2^2 + a\|\mathbf{v}\|_2^2. \tag{17}$$

This is basically the same equation as (7), so we could follow the same analysis as before and obtain stability whenever $\lambda_L > \frac{a}{\mu}$. However, due to (15) we may do better. Let's, as before, consider a functional $\mathbf{v}\mapsto\frac{\|\nabla\mathbf{v}\|_2^2}{\|\mathbf{v}\|_2^2}$. However, this time we restrict ourselves to functions $\mathbf{v}$ such that $\nabla\cdot\mathbf{v} = 0$. For a function $\mathbf{v}_0$ to be a minimizer in this class, it is necessary that

$$\int_\Omega \left(-\Delta\mathbf{v}_0 - \frac{\|\nabla\mathbf{v}_0\|_2^2}{\|\mathbf{v}_0\|_2^2}\mathbf{v}_0\right)\cdot\mathbf{v} = 0 \tag{18}$$

for every $\mathbf{v}$ such that $\nabla\cdot\mathbf{v} = 0$. As the set of such vector fields is substantially smaller than the set of all differentiable vector fields vanishing on $\partial\Omega$, we may not simply pass to a differential equation from (18). However, we may take as $\mathbf{v}$ any vector field of form $\nabla\times\mathbf{w}$. Then, denoting $\mathbf{h} = -\Delta\mathbf{v}_0 - \frac{\|\nabla\mathbf{v}_0\|_2^2}{\|\mathbf{v}_0\|_2^2}\mathbf{v}_0$ and integrating by parts in (18) by virtue of equality

$$\nabla\cdot(\mathbf{h}\times\mathbf{w}) = -\mathbf{h}\cdot(\nabla\times\mathbf{w}) + (\nabla\times\mathbf{h})\cdot\mathbf{w}$$

leads to

$$\int_\Omega (\nabla\times\mathbf{h})\cdot\mathbf{w} = 0. \tag{19}$$

This implies that $\nabla\times\mathbf{h} = 0$ and, consequently, that $\mathbf{h} = -\nabla p$ for some function $p\colon \Omega\to\mathbb{R}$. Denoting $\lambda = \frac{\|\nabla\mathbf{v}_0\|_2^2}{\|\mathbf{v}_0\|_2^2}$ we obtain that $(\mathbf{v}, p)$ satisfies system

$$-\Delta\mathbf{v} + \nabla p = \lambda\mathbf{v} \quad \text{in } \Omega, \tag{20}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{21}$$

$$\mathbf{v} = 0 \quad \text{at } \partial\Omega \tag{22}$$

i. e. $\mathbf{v}$ is an eigenvector of Stokes operator with eigenvalue λ. [4] The Stokes operator with Dirichlet boundary conditions is, similarly as $-\Delta$, a positive unbounded operator. Therefore, we obtain, similarly as in the first example, that 0 is a stable solution to the system (14-16) whenever $\lambda_S > \frac{a}{\mu}$ where λ_S is the least eigenvalue of the Stokes operator. To appreciate this result we may notice that for $\Omega = [0,1]^2$, $\lambda_S \approx 5,3\pi^2$ [3] while $\lambda_L = 2\pi^2$.

[4] To understand the meaning of this phrase better, we might act on (20) with the Helmholtz projection P onto the subspace of divergenceless functions in $L^2(\Omega)$ to obtain $-P\Delta\mathbf{v} = \lambda\mathbf{v}$.

3 Nonlinear forcing – conditional stability

Now, let's consider following problem for $u\colon \Omega \times [0,T] \to \mathbb{R}$:

$$u_t = \kappa\Delta u + au^2 \quad \text{in } \Omega, \tag{23}$$

$$u = 0 \quad \text{at } \partial\Omega. \tag{24}$$

This time we restrict ourselves to $\Omega \subset \mathbb{R}^3$ and investigate stability of stationary solution $u^* = 0$, which as before, leads to the same system (23, 24). Multiplying by u and integrating over Ω as usual we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -\kappa\|\nabla u\|_2^2 + a\int u^3. \tag{25}$$

To deal with the expression $\int u^3$ we use a simple case of Sobolev inequality

Proposition 2 (Sobolev inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$. Let $u \in C_0^1(\Omega)$. Then*

$$(\int_\Omega |u|^{\frac{3}{2}})^{\frac{2}{3}} \le \int_\Omega |\nabla u|.$$

Proof. Let us extend u by 0 to whole $\mathbb{R}^3$. Due to fundamental theorem of calculus,

$$u(x,y,z) = \int_{-\infty}^{x} u_x(r,y,z)\mathrm{d}r$$

and consequently

$$|u| \le \int_{-\infty}^{\infty} |u_x|\mathrm{d}x.$$

Analogous inequalities may be written for u_y and u_z. Multiplying the three inequalities obtained this way and taking square root of both sides yields

$$|u|^{\frac{3}{2}} \le \left(\int_{-\infty}^{\infty} |u_x|\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |u_y|\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |u_z|\mathrm{d}z\right)^{\frac{1}{2}}.$$

Integrating both sides of this inequality with respect to x over $]-\infty,\infty[$ leads to

$$\begin{aligned}\int_{-\infty}^{\infty} |u|^{\frac{3}{2}}\mathrm{d}x &\le \left(\int_{-\infty}^{\infty} |u_x|\mathrm{d}x\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(\left(\int_{-\infty}^{\infty} |u_y|\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |u_z|\mathrm{d}z\right)^{\frac{1}{2}}\right)\mathrm{d}x \\ &\le \left(\int_{-\infty}^{\infty} |u_x|\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u_y|\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u_z|\mathrm{d}z\mathrm{d}x\right)^{\frac{1}{2}}\end{aligned}$$

using Schwarz inequality. Integrating this inequality with respect to y and z likewise yields

$$\int_{\mathbb{R}^3} |u|^{\frac{3}{2}} \le \left(\int_{\mathbb{R}^3} |u_x|\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u_y|\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u_z|\right)^{\frac{1}{2}}.$$

Estimating $|u_x|$, $|u_y|$, $|u_z|$ by $|\nabla u|$ finishes the proof. □

The constant 1 appearing in proven inequality is not optimal. According to [2], the optimal constant in this case is $C_S = \left(\frac{1}{36\pi}\right)^{\frac{1}{3}} \approx 0,21$. Whatever the exact value of the constant, it is remarkable that it does not depend on (the size of) Ω (unlike the constant in the Poincaré inequality).

Using Sobolev inequality we may estimate

$$\int u^3 \leq \int (u^2)^{\frac{3}{2}} \leq \left(\int |\nabla u^2|\right)^{\frac{3}{2}} \leq \left(\int |\nabla u||u|\right)^{\frac{3}{2}} \leq 2^{\frac{3}{2}} \|\nabla u\|_2^{\frac{3}{2}} \|u\|_2^{\frac{3}{2}}. \tag{26}$$

The last estimate is Schwarz inequality. Due to (25), we obtain stability if

$$-\kappa \|\nabla u\|_2^2 + a 2^{\frac{3}{2}} \|\nabla u\|_2^{\frac{3}{2}} \|u\|_2^{\frac{3}{2}} < 0. \tag{27}$$

Dividing by $\|\nabla u\|_2^2$ and using Poincaré inequality yields following condition for stability

$$-\kappa + a 2^{\frac{3}{2}} \lambda_L^{-\frac{1}{4}} \|u\|_2 < 0. \tag{28}$$

If this condition is satisfied in $t = 0$ then, due to (25), it is satisfied in any positive time. Thus, it is a sufficient condition for stability that

$$\|u_0\|_2 < 2^{-\frac{3}{2}} \lambda_L^{\frac{1}{4}} \frac{\kappa}{a} \tag{29}$$

where u_0 is the initial perturbation. Note that we did not show anything about the case when this inequality is not satisfied. However it is known that there is no stability whenever

$$\int_\Omega u_0 w_L > \lambda_L \frac{\kappa}{a} \tag{30}$$

for some perturbation $u_0 \geq 0$, where w_L is the eigenvector of $-\Delta$ corresponding to λ_L normalized so that $\int_\Omega w_L = 1$. [5] In fact, the L^2 norm of any perturbation $u_0 \geq 0$ satisfying (30) grows to ∞ in finite time. [1]

References

[1] L.C. Evans. *Partial differential equations.* Graduate Studies in Mathematics Series. American Mathematical Society, 2010.

[2] Giorgio Talenti. Best constant in sobolev inequality. *Annali di Matematica pura ed Applicata*, 110(1):353–372, 1976.

[3] G. I. Taylor. The buckling load for a rectangular plate with four clamped edges. *Zeitschrift für Angewandte Mathematik und Mechanik*, 13(2):147–152, 1933.

[5] Due to this normalization the powers of λ_L in (29) and (30) *are* consistent.