

# Well-behaved Principles Alternative to Bounded Induction

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## Abstract

We introduce some  $\Pi_1$ -expressible combinatorial principles which may be treated as axioms for some bounded arithmetic theories. The principles, denoted  $Sk(\Sigma_n^b, \text{length } \log^k)$  and  $Sk(\Sigma_n^b, \text{depth } \log^k)$  (where ‘ $Sk$ ’ stands for ‘Skolem’), are related to the consistency of  $\Sigma_n^b$ -induction: for instance, they provide models for  $\Sigma_n^b$ -induction. However, the consistency is expressed indirectly, via the existence of evaluations for sequences of terms. The evaluations do not have to satisfy  $\Sigma_n^b$ -induction, but must determine the truth value of  $\Sigma_n^b$  statements.

Our principles have the property that  $Sk(\Sigma_n^b, \text{depth } \log^k)$  proves  $Sk(\Sigma_{n+1}^b, \text{length } \log^k)$ . Additionally,  $Sk(\Sigma_n^b, \text{length } \log^{k-2})$  proves  $Sk(\Sigma_{n+1}^b, \text{length } \log^k)$ . Thus, some provability is involved where conservativity is known in the case of  $\Sigma_n^b$  induction on an initial segment and induction for higher  $\Sigma_m^b$  classes on smaller segments.

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# 1 Introduction

Bounded arithmetic theories are normally axiomatized using induction principles for various classes of bounded formulae, such as Buss'  $\Sigma_n^b$  classes (see e.g. [HP]). Some of these these principles are additionally restricted to proper initial segments of models. For example, Buss' theory  $S_2^n$  is  $I\Sigma_n^b|log$ , induction for  $\Sigma_n^b$  formulae which is restricted to the logarithmic part of a model. In general, it is not known whether induction principles restricted in this way can be derived from full induction for slightly smaller classes of formulae. In particular, the question of whether  $I\Sigma_{n+1}^b|log$  can be derived from  $I\Sigma_n^b$  is an outstanding open problem.

On the other hand, what is known about these induction principles is that there are interesting conservativity relationships. The most famous result here is that  $I\Sigma_{n+1}^b|log$  is  $\forall\Sigma_{n+1}^b$  conservative over  $I\Sigma_n^b$ . This has been generalized by A. Beckmann ([B]) and C. Pollett ([P]) to the case of  $I\Sigma_{n+1}^b|log^{k+1}$  in place of  $I\Sigma_{n+1}^b|log$  and  $I\Sigma_n^b|log^k$  in place of  $I\Sigma_n^b$  (where  $log^k$  is the  $k$ -th iteration of logarithm). Some changes in the assumptions were needed to obtain the generalization: in particular, functions of slightly higher growth rate than the standard  $\omega_1$  had to be allowed, and standard  $\Sigma_n^b$  classes had to be replaced by their prenex versions.

In the present paper, we propose and discuss a different class of principles. Our theories will contain only some fixed small amount of induction; their most important component will be a certain combinatorial principle, denoted  $Sk$ .  $Sk(\Sigma_n^b, depth\ log^k)$  will stand for the version of  $Sk$  restricted to  $\Sigma_n^b$  formulae and sequences of terms of depth in the  $log^k$  part of a model, and  $Sk(\Sigma_n^b, length\ log^k)$  for the version of  $Sk$  restricted to  $\Sigma_n^b$  formulae and sequences of terms whose length is in the  $log^k$  part of a model (see section 3 and the beginning of section 4 for precise definitions).

After introducing some basic definitions and constructions, we try to explain the link between the  $Sk$  principles and bounded induction (section 4). We then go on to prove our main result, which states that the  $Sk$  principles are, in a sense, better-behaved than induction principles:  $Sk(\Sigma_n^b, depth\ log^k)$  suffices to prove  $Sk(\Sigma_{n+1}^b, length\ log^k)$ , and furthermore, also  $Sk(\Sigma_n^b, length\ log^{k-2})$  proves  $Sk(\Sigma_{n+1}^b, length\ log^k)$ .  $Sk(\Sigma_n^b, length\ log^{k-2})$  and  $Sk(\Sigma_{n+1}^b, length\ log^k)$  are related to  $I\Sigma_n^b|log^{k-2}$  and  $I\Sigma_{n+1}^b|log^k$  respectively (theorem 4.3 below). Via this relationship it follows that a counterpart of less induction restricted to a larger cut, expressed by  $Sk(\Sigma_n^b, length\ log^{k-2})$ , implies a counterpart of more induction restricted to a smaller cut, expressed by  $Sk(\Sigma_{n+1}^b, length\ log^k)$ .

For the theories  $I\Sigma_n^b \log^{k-2}$  and  $I\Sigma_{n+1}^b \log^k$  themselves this is an open question — see the diagram following corollary 5.2. As in the case of the Beckmann–Pollett results, we need to restrict ourselves to prenex  $\Sigma_n^b$  classes, and to allow some functions which grow slightly faster than  $\omega_1$  (more specifically, we have to allow  $\omega_K$  for some  $K$  which depends on  $k$ .)

Our main notion is the notion of a  $\Sigma_n^b$  evaluation. Given a sequence of closed terms  $\Lambda$ , an evaluation on  $\Lambda$  is a function which assigns logical values to some atomic sentences with terms from  $\Lambda$ . An evaluation is  $\Sigma_n^b$  if the information it provides makes it possible to decide which  $\Sigma_n^b$  sentences with terms in  $\Lambda$  are to be considered true and which false.

## 2 Preliminaries

Some notational conventions:

The symbol  $\log$  stands for the discrete-valued binary logarithm function;  $\exp(m)$  is  $2^m$ . A superscript over a function symbol (say,  $\log^k$ ) denotes iteration. For a model  $\mathbf{M}$ ,  $\log^k(\mathbf{M})$  consists of those elements of  $\mathbf{M}$  for which  $\exp^k$  exists. A “bar” always denotes a tuple, and if  $\bar{t}$  is  $\langle t_1, \dots, t_l \rangle$ , then  $\bar{h}_{\bar{t}}$  is  $\langle h_{t_1}, \dots, h_{t_l} \rangle$ . If  $\Lambda$  is a sequence of terms,  $\bar{t} \in \Lambda$  means all of  $t_1, \dots, t_l$  appear in  $\Lambda$ .

We adopt the coding of sets and sequences in bounded arithmetic developed in [HP]. Also the notion of length  $lh(\Lambda)$  of a sequence  $\Lambda$  is the one defined in [HP] for bounded arithmetic. If  $lh(\Lambda)$  is in  $\log(\mathbf{M})$  for a model  $\mathbf{M}$  of bounded arithmetic, then functions from  $\Lambda$  into  $\{0, 1\}$  can be coded in  $\mathbf{M}$  as subsets of size  $lh(\Lambda)$  of  $\Lambda \times \{0, 1\}$  (see [S]). Here we shall use a different coding of such functions. If  $\Lambda = \langle t_1, \dots, t_l \rangle$ , then a function  $f$  from  $\Lambda$  into  $\{0, 1\}$  is given by the pair  $\langle \Lambda, p \rangle$ , where  $p$  is a function from  $\{1, \dots, l\}$  into  $\{0, 1\}$  (an object of size  $2^l$ ), with  $p(i)$  intended to code  $f(t_i)$ . Whenever  $\Lambda$  is fixed, we may simply identify  $f$  with  $p$ .

Our base language,  $L_K$  (for some natural number  $K$ ), contains  $0, 1, +, <, \times, |\cdot|$  (the length function symbol), and, for  $i \leq K$ , the symbols  $\#_i$  for the smash functions<sup>1</sup>. We assume that some appropriate Gödel numbering of  $L_K$  formulae has been fixed; we shall identify the formulae with their Gödel numbers.

<sup>1</sup>The length  $|x|$  of  $x$  is  $\lceil \log(x+1) \rceil$ . The smash functions are defined by:  $x\#_2y = \exp(|x| \cdot |y|)$ ;  $x\#_{m+1}y = \exp(|x|\#_m|y|)$ . A related family of functions is defined by:  $\omega_1(x) = x^{|x|}$ ;  $\omega_{m+1}(x) = \exp(\omega_m(|x|))$ . Note that  $\omega_m(x)$  is roughly  $x\#_{m+1}x$ .

To this language we add function symbols  $s^\varphi$  for all  $L_K$  formulae  $\varphi$  in prenex normal form which begin with an existential quantifier. The symbol  $s^\varphi$  is intended to stand for a Skolem function for the first existential quantifier in  $\varphi$ . That is, given an  $L_K$  formula  $\varphi(\bar{x})$  in normal form, if  $\varphi(\bar{x})$  is  $\exists y\psi(\bar{x}, y)$ , then  $s^\varphi$  is a function symbol of arity  $1 + lh(\bar{x})$ , and  $s^\varphi(\bar{t})$  is intended to be some  $y$  which satisfies  $\varphi'(\bar{t}, y)$ , if such a  $y$  exists.

We include the symbols of  $L_K$  among the  $s^\varphi$ 's: for example,  $t_1 + t_2$  may be treated as  $s^{\exists z(z=x+y)}(t_1, t_2)$ .

Whenever we speak of a formula  $\varphi(\bar{t})$ , it is assumed that  $\varphi(\bar{x})$  itself is an  $L_K$  formula, although the terms  $\bar{t}$  do not have to be terms of  $L_K$ .

We have to encode our extended language in arithmetic. We use even numbers to enumerate terms of the form  $s^\varphi(\bar{t})$ , and odd numbers for a special enumeration of numerals. More precisely, we let the number  $2\langle\varphi(\bar{x}), \bar{t}\rangle$  correspond to  $s^\varphi(\bar{t})$  (it is assumed that some Gödel numbering of the formulae of  $L_K$  has already been fixed), and we let  $2k + 1$  correspond to a numeral for  $k$  ( $2k + 1$  will be referred to as  $\underline{k}$ ). From now on, we identify terms with their numbers.

We fix a standard natural number  $N$ , which will play the role of a parameter. Many of our definitions depend on  $N$ , and often we will consider only formulae  $< N$  (more precisely, formulae of the form  $\varphi(\bar{t})$ , where  $\bar{t}$  is a tuple of terms and (the Gödel number of)  $\varphi(\bar{x})$  is smaller than  $N$ ).<sup>2</sup> We also fix the numbers  $k \geq 1$  (this will determine which iteration of the logarithm function we work with),  $K$  (in order to fix  $L_K$ ), and  $n$  (in order to fix  $\Sigma_n^b$ ). Our definition of  $\Sigma_n^b$  differs slightly from the one most commonly used. For one thing, we allow quantifiers bounded by any terms of the language  $L_K$ , and thus also by  $\#_i$ , even if  $i$  is not equal to 2. For another, we work with prenex  $\Sigma_n^b$  classes, instead of  $\Sigma_n^b$  in the more usual, broader sense (see [HP]). In the standard model, every  $\Sigma_n^b$  formula in the broader sense is equivalent to a prenex  $\Sigma_n^b$  formula, but some theories we consider might not be able to prove this equivalence.

A term is  $\Sigma_n^b$  if it is  $s^\varphi(\bar{t})$  for  $\varphi \in \Sigma_n^b$ . Numerals are considered  $\Sigma_n^b$  terms for any  $n$ . The depth of a term is defined in the natural inductive way, with the exception that all numerals are considered to have depth 0.

Some of our constructions and definitions require a limited amount of

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<sup>2</sup>We make this restriction for the sake of technical simplicity. We could work without it, and use appropriate universal formulae for  $\Sigma_n^b$  where necessary. The restriction is related to the fact that we may axiomatize  $I\Sigma_n^b$  by instances of induction referring to formulae  $< N$  — we make use of this fact in theorem 4.3.

induction. Therefore, we assume that all models we deal with satisfy  $I\Sigma_{n_0}^b$  for some appropriate fixed small number  $n_0$ . We write  $T_0$  to denote the theory  $I\Sigma_{n_0}^b$  in the language  $L_K$ . Thus, whenever we speak of a model  $\mathbf{M}$ , it is assumed that  $\mathbf{M} \models T_0$  — and that  $\mathbf{M}$  is nonstandard. The universe of  $\mathbf{M}$  will be denoted by  $M$ .

The results of sections 4 and 5 which involve the parameter  $n$  hold for  $n$  “sufficiently large with respect to  $n_0$ ”.

We shall consider various sequences of closed terms. About such a sequence  $\Lambda$  we shall always assume that if a term of the form  $s^\varphi(\bar{t})$  appears in  $\Lambda$ , then all terms in  $\bar{t}$  also do, and moreover, that they have smaller indices in  $\Lambda$  than  $s^\varphi(\bar{t})$ . Given a  $\Lambda$ , we denote by  $top(\Lambda)$  the largest number  $h$  such that the numeral  $\underline{h}$  is in  $\Lambda$ .

From now on, whenever we deal with a sequence of terms  $\Lambda$  and a model  $\mathbf{M}$  of bounded arithmetic, we shall assume that  $lh(\Lambda)$  is in  $log(\mathbf{M})$ .

Given a tuple of variables  $\langle x_1, \dots, x_m \rangle$ , the collection of *simple atomic formulae* over  $\langle x_1, \dots, x_m \rangle$  consists of  $x_i = x_j, x_i < x_j, x_i = 0, x_i + x_j = x_l, x_i = |x_j|$  etc. for other symbols of  $L_K$  ( $1 \leq i, j, l \leq m$ ; basically, simple atomic formulae are those which would still be considered atomic if the vocabulary was relational). Any open formula over  $\langle x_1, \dots, x_m \rangle$  which does not contain nested terms (such as  $(x_i + x_j) \times x_l$ ) is a boolean combination of simple atomic formulae. For a sequence of closed terms  $\Lambda$ , let the collection  $\mathcal{E}(\Lambda)$  of simple atomic sentences over  $\Lambda$  consist of all sentences obtained by substituting terms from  $\Lambda$  for the  $x_i$ 's in simple atomic formulae. Note that  $lh(\mathcal{E}(\Lambda))$  is polynomial in  $lh(\Lambda)$ .

### 3 Evaluations

Suppose a sequence of closed terms  $\Lambda$  is given. For  $\varphi(\bar{x})$  in normal form,  $\bar{t} \in \Lambda$ , we define the notion that  $\Lambda$  is *good enough* (g.e.) for  $\langle \varphi, \bar{t} \rangle$  by induction on  $\varphi$ .  $\Lambda$  is always g.e. for  $\langle \varphi, \bar{t} \rangle$  if  $\varphi$  is simple atomic. If  $\varphi(\bar{x})$  is  $f_1(\bar{x}) = f_2(\bar{x})$  where  $f_1$  and/or  $f_2$  are nested terms, then  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  if  $\Lambda$  contains  $s^{\exists y(y=f_i(\bar{x}))}(\bar{t})$  for the appropriate  $i$ 's (similarly for ‘<’ in place of ‘=’). If  $\varphi$  is  $\exists y\varphi'(\bar{x}, y)$ , then  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  if  $s^\varphi(\bar{t}) \in \Lambda$  and  $\Lambda$  is g.e. for  $\langle \varphi', \bar{t} \frown s^\varphi(\bar{t}) \rangle$ . Finally, if  $\varphi$  is  $\forall y\tilde{\varphi}(\bar{x}, y)$ , then  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  if  $s^{\exists y\tilde{\varphi}(\bar{x}, y)}(\bar{t}) \in \Lambda$  (where  $\exists y\tilde{\varphi}$  is the normal form of  $\neg\varphi$ ) and  $\Lambda$  is g.e. for  $\langle \tilde{\varphi}, \bar{t} \frown s^{\exists y\tilde{\varphi}(\bar{x}, y)}(\bar{t}) \rangle$ .

The idea is that  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  if it contains enough appropriate Skolem terms so that assigning a logical value to  $\varphi(\bar{t})$  based on an evaluation

on  $\Lambda$  (defined below) makes sense.

**Definition 3.1** <sup>3</sup>

Let  $p : \mathcal{E}(\Lambda) \longrightarrow \{0, 1\}$  map every axiom of equality in  $\mathcal{E}(\Lambda)$  to 1. We think of  $p$  as assigning a logical value to sentences in  $\mathcal{E}(\Lambda)$ .

Let  $\bar{t} \in \Lambda$ . We define the relation  $p \models \varphi(\bar{t})$  for  $\varphi(\bar{x})$  in normal form by induction:

- (i)  $p \models \varphi(\bar{t})$  iff  $p(\varphi(\bar{t})) = 1$  for  $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$ , and the relation  $p \models \varphi$  behaves in the natural way with respect to boolean combinations of formulae in  $\mathcal{E}(\Lambda)$ ;
- (ii) if  $\varphi(\bar{t})$  is atomic but contains nested terms, then  $p \models \varphi(\bar{t})$  iff:  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$ , and if  $\psi(\bar{t}, \bar{s}(\bar{t}))$  is the formula obtained by substituting the Skolem terms for the nested terms in  $\varphi(\bar{t})$ , then  $p \models \psi(\bar{t}, \bar{s}(\bar{t}))$ ,
- (iii) if  $\varphi$  is  $\exists y \varphi'(\bar{x}, y)$ , then  $p \models \varphi(\bar{t})$  iff  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  and  $p \models \varphi'(\bar{t}, s^{\varphi}(\bar{t}))$ ,
- (iv) if  $\varphi$  is  $\forall y \tilde{\varphi}(\bar{x}, y)$ , then  $p \models \varphi(\bar{t})$  iff for all  $t \in \Lambda$  such that  $\Lambda$  is g.e. for  $\langle \tilde{\varphi}, \bar{t} \smallfrown t \rangle$ ,  $p \models \tilde{\varphi}(\bar{t}, t)$ .

**Definition 3.2** Let  $\Lambda$  be given. A function  $p : \mathcal{E}(\Lambda) \longrightarrow \{0, 1\}$  is called a  $\Sigma_n^b$  evaluation on  $\Lambda$  if the following holds:

- (1) For every  $\Sigma_n^b$  formula  $\varphi(\bar{x})$ ,  $\varphi < N$ , and every  $\bar{t} \in \Lambda$  of the appropriate length, if  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$ , then

$$p \models \varphi(\bar{t}) \text{ or } p \models \neg \varphi(\bar{t});$$

- (2) if  $\varphi(\bar{t})$ ,  $\varphi < N$ ,  $\bar{t} \in \Lambda$ , is an instance of an axiom of  $T_0$  or if  $\bar{t}$  are numerals and  $\varphi(\bar{t})$  is a true  $\Sigma_{n_0}^b$  or  $\Pi_{n_0}^b$  sentence, then assuming  $\Lambda$  is g.e. for  $\langle \varphi, \bar{t} \rangle$ ,  $p \models \varphi(\bar{t})$ .

An “evaluation on  $\Lambda$ ” is simply a  $\Sigma_n^b$  evaluation on  $\Lambda$  for some  $n \geq n_0$ .

**Definition 3.3** Let  $p, p'$  be evaluations on  $\Lambda, \Lambda'$  respectively. We say that  $p'$  extends  $p$  if  $\Lambda \subseteq \Lambda'$  and  $p \subseteq p'$ .

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<sup>3</sup>See also [A1], [A2], [A3], [AZ1], [AZ2], [S].

**Proposition 3.4** *If  $p, p'$  are  $\Sigma_n^b$  evaluations on  $\Lambda, \Lambda'$  respectively and  $p'$  extends  $p$ , then and for any  $\varphi \in \Sigma_n^b$ ,  $\varphi < N$ ,  $\bar{t} \in \Lambda$ , if  $\Lambda$  is good enough for  $\langle \varphi, \bar{t} \rangle$  then*

$$p \models \varphi(\bar{t}) \text{ iff } p' \models \varphi(\bar{t}).$$

**Proof.** A simple inductive argument.  $\square$

Let  $\mathbf{M}$  be a model and let a sequence of closed terms  $\Lambda$  be given. Denote by  $TERM(\Lambda)$  the set of all terms of those terms of standard depth whose subterms of depth 0 (i.e. numerals) are in  $\Lambda$ . Assume that  $TERM(\Lambda) \subseteq \Lambda$ . In that case, every  $\Sigma_n^b$  evaluation  $p$  on  $\Lambda$  determines a structure  $\mathbf{M}(p)$  which “agrees” with  $p$  about which  $\Sigma_n^b$  formulae smaller than  $N$  are satisfied.

$\mathbf{M}(p)$  is constructed as follows. Let the relation  $\sim$  on  $TERM(\Lambda)$  be defined by  $t \sim t' \iff p \models (t = t')$ . Since  $p$  is an evaluation,  $\sim$  is an equivalence relation and a congruence with respect to the arithmetical operations. Thus, we can define the universe of  $\mathbf{M}(p)$  as  $TERM(\Lambda)/\sim$ ; the operations of  $\mathbf{M}(p)$  are defined in the obvious way. It now follows from the definition of evaluation that for any  $\Sigma_n^b \cup \Pi_n^b$  formula  $\varphi < N$  and any tuple  $\bar{t} = \langle t_0, \dots, t_m \rangle \in \Lambda$ ,

$$\mathbf{M}(p) \models \varphi([t_0], \dots, [t_m]) \text{ iff } p \models \varphi(\bar{t}),$$

where  $[t_i]$  denotes the  $\sim$ -equivalence class of  $t_i$ .

A convenient way to obtain evaluations on a sequence  $\Lambda$  is to use *Skolem hulls*. A hull on  $\Lambda$  is a sequence  $H = \langle h_t : t \in \Lambda \rangle$  of elements of  $M$ , where the element  $h_t$  is thought of as an interpretation of the term  $t$ . It is assumed that for every numeral  $\underline{k} \in \Lambda$ ,  $h_{\underline{k}} = k$ . The satisfaction relation  $H \models \varphi(\bar{t})$  is defined similarly to  $p \models \varphi(\bar{t})$ . We take

$$H \models \varphi(\bar{t}) \text{ iff } \mathbf{M} \models \varphi(\bar{h}_t)$$

for  $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$ , and later proceed inductively just as in definition 3.1. A *Skolem  $\Sigma_n^b$  hull* is then defined analogously to a  $\Sigma_n^b$  evaluation.

Observe that if  $H$  is a Skolem  $\Sigma_n^b$  hull on  $\Lambda$ , then the function  $p_H$  defined for  $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$  by the clause

$$p_H(\varphi(\bar{t})) = 1 \text{ iff } H \models \varphi(\bar{t})$$

is a  $\Sigma_n^b$  evaluation on  $\Lambda$ . We say that  $p_H$  is *isomorphic* with  $H$ .

A *true Skolem  $\Sigma_n^b$  hull* on  $\Lambda$  is a Skolem  $\Sigma_n^b$  hull  $H$  on  $\Lambda$  which additionally satisfies the following: for every formula  $\varphi(\bar{x}) < N$  which is at most  $\Pi_n^b$  and starts with a universal quantifier, and every  $\bar{t} \in \Lambda$ , if  $H \models \varphi(\bar{t})$ , then  $\varphi(\bar{h}_{\bar{t}})$  is true (in  $\mathbf{M}$ ).

## 4 The $Sk$ principles

In this section, we introduce the  $Sk$  principles and the theories they axiomatize.

**Definition 4.1** *Let  $\Lambda$  be a sequence of closed terms. We say that  $\Lambda$  is of depth  $i$  if all terms in  $\Lambda$  have depth  $\leq i$ .*

Let  $Sk(\Sigma_n^b, \text{depth } \log^k)$  be the theory axiomatized by  $T_0$  and the following sentence:

“For every  $i \in \log^k$  and every  $\Lambda$  of depth at most  $i$  there exists a  $\Sigma_n^b$  evaluation on  $\Lambda$ .”

Also let  $Sk(\Sigma_n^b, \text{length } \log^k)$  be the theory axiomatized by  $T_0$  and the following sentence:

“For every  $i \in \log^k$  and every  $\Lambda$  of length at most  $i$  and depth at most  $\log(i)$  there exists a  $\Sigma_n^b$  evaluation on  $\Lambda$ .”

Observe that both  $Sk(\Sigma_n^b, \text{length } \log^k)$  and  $Sk(\Sigma_n^b, \text{depth } \log^k)$  are  $\Pi_1$ -axiomatizable theories. It is clear that  $T_0$  is  $\Pi_1$ -axiomatizable, but perhaps less obvious that the additional principles can also be formulated as  $\Pi_1$  statements. Let us argue the case of the depth principle (the other is quite similar). We may express this principle by a formula which begins with universal quantifiers for  $y = \text{exp}^k(i)$ , for  $\Lambda$ , and for  $z = 2^{\pi(\text{lh}(\Lambda))}$  where  $\pi$  is some standard polynomial to be specified below. We claim that the rest of the formula may then be bounded. Being of depth  $i$  is certainly definable by a bounded formula, so the main question is whether the existential quantifier for evaluations can be bounded. Any evaluation  $p$  on  $\Lambda$  is a pair  $\langle \mathcal{E}(\Lambda), p' \rangle$ , where  $p'$  is a function from  $\{1, \dots, \text{lh}(\mathcal{E}(\Lambda))\}$  into  $\{0, 1\}$ . Since  $\text{lh}(\mathcal{E}(\Lambda))$  is polynomial in  $\text{lh}(\Lambda)$ , we may take  $\pi$  to be a polynomial such that  $\mathcal{E}(\Lambda)$  and  $p'$  are both bounded by  $2^{\pi(\text{lh}(\Lambda))}$ . As  $\langle a, b \rangle \leq 2(a+b)^2$ , the claim now follows.

Actually, we may assume that both  $Sk(\Sigma_n^b, \text{length } \log^k)$  and  $Sk(\Sigma_n^b, \text{depth } \log^k)$  are even  $\forall \Sigma_{n_0+1}^b$ -axiomatizable. To see this, we only need to check that our principles are  $\forall \Sigma_{n_0+1}^b$ . An examination of definition 3.1 reveals that the relation “ $p \models \varphi$ ” is definable by a fixed bounded formula, so we may assume that it is  $\Sigma_{n_0}^b$ -definable. It follows that the property of being a  $\Sigma_n^b$  evaluation is

also definable by a bounded formula of fixed (i.e. independent of  $n$ ) complexity. Here we are not allowed to assume that this is a  $\Sigma_{n_0}^b$  property, as part (2) of the definition of a  $\Sigma_n^b$  evaluation (def. 3.2) contains some implications with  $\Sigma_{n_0}^b$  antecedents. However, there are no obstacles to assuming that being a  $\Sigma_n^b$  evaluation is  $\Sigma_{n_0+1}^b$ . Hence, the statement that a  $\Sigma_n^b$  evaluation exists on every appropriate  $\Lambda$  is, as required,  $\forall \Sigma_{n_0+1}^b$ .

The next two theorems show that there is some connection between  $Sk(\Sigma_n^b, \text{length } \log^k)$  and induction.

**Theorem 4.2** *Assume  $T_0$  and  $I\Sigma_n^b | \log^k$ . Let  $\Lambda$  of length  $i \in \log^k$  consist of  $\Sigma_n^b$  terms. Then there exists a true  $\Sigma_{n-1}^b$  hull on  $\Lambda$ .*

**Proof.** Let  $\Lambda = \langle t_0, \dots, t_l \rangle$ . We want to apply  $I\Sigma_n^b | \log^k$  to the formula “there exists a true  $\Sigma_{n-1}^b$  hull on  $\langle t_0, \dots, t_m \rangle$ ” for  $m \leq l$ . The inductive step is quite straightforward, the only difficulty is to check that our formula is indeed  $\Sigma_n^b$ .

The initial existential quantifier can be bounded, since, by our restriction to formulae  $< N$ , elements of the required hulls can be bounded by  $f(\text{top}(\Lambda))$  for some fixed  $L_K$ -term  $f$ . So, it suffices to verify that being a true Skolem  $\Sigma_n^b$  hull is, for sufficiently large  $n$ , a  $\Pi_n^b$  property.

Being a Skolem  $\Sigma_n^b$  hull is, just as being a  $\Sigma_n^b$  evaluation (see above),  $\Sigma_{n_0+1}^b$ -definable. To say that  $H$  is a true Skolem  $\Sigma_n^b$  hull, we need to state that  $H$  is a Skolem  $\Sigma_n^b$  hull and additionally that it satisfies

$$\forall \bar{t} \in \Lambda ((H \models \varphi(\bar{t})) \Rightarrow \varphi(\bar{h}_{\bar{t}})),$$

for a fixed finite number of  $\Pi_n^b$  formulae.  $\square$

Thus,  $T_0 + I\Sigma_n^b | \log^k$  implies  $Sk(\Sigma_{n-1}^b, \text{length } \log^k)$  (and hence, if  $n \geq n_0 + k$ ,  $I\Sigma_n^b | \log^k$  itself implies  $Sk(\Sigma_{n-1}^b, \text{length } \log^k)$ ).

Since  $Sk(\Sigma_{n-1}^b, \text{length } \log^k)$  is  $\forall \Sigma_{n_0+1}^b$ , we may additionally infer

$$I\Sigma_{n-k}^b \vdash Sk(\Sigma_{n-1}^b, \text{length } \log^k)$$

provided  $n \geq n_0 + \max(k, 2)$  in the case when  $K > k$  (cf [P]).

The relation in the other direction is somewhat more difficult to express. In general terms, we may say that  $Sk(\Sigma_n^b, \text{length } \log^k)$  allows us to build a model for  $I\Sigma_n^b | \log^k$  with an appropriately large  $k$ -th logarithm.

In the following theorem, we assume that  $N$  is so large that induction axioms for  $\Sigma_n^b$  formulae smaller than  $N$  axiomatize  $I\Sigma_n^b$ . Note that this is always possible, as  $I\Sigma_n^b$  is finitely axiomatizable for  $n \geq 1$ .

**Theorem 4.3** *Let  $\mathbf{M} \models \text{Sk}(\Sigma_n^b, \text{length } \log^k)$ .*

*Let  $l_0, l_1 \in \log^k(\mathbf{M})$  satisfy  $\omega_{K-1}^{\mathbb{N}}(\text{exp}^k(l_0)) < \text{exp}^k(l_1)$ .*

*Let  $\Lambda \in M$  be such that:  $\Lambda$  is of length  $i$  for some  $i \in \log^k$ ,  $\text{TERM}(\Lambda) \subseteq \Lambda$ , and  $\text{TERM}(\Lambda)$  contains numerals for:  $0, \dots, l_1$ ,  $\text{exp}^k(j)$  for any  $j \leq l_1$ , and all standard iterations of the smash functions  $\#_i$  ( $i \leq K$ ) applied to  $\text{exp}^k(j)$  for  $j \leq l_1$ .*

*Let  $p$  be a  $\Sigma_n^b$  evaluation on  $\Lambda$  given by  $\text{Sk}(\Sigma_n^b, \text{length } \log^k)$ . Then there exists an initial segment  $\mathfrak{M}$  of  $\mathbf{M}(p)$  satisfying  $I\Sigma_n^b | \log^k$  and such that  $l_0 \in \log^k(\mathfrak{M})$ .*

**Proof.** We first show that  $\mathbf{M}(p)$  satisfies  $I\Sigma_n^b | l_1$ . Consider a  $\Sigma_n^b$  formula  $\varphi < N$  (we may restrict ourselves to  $\varphi < N$  without loss of generality). Assume that  $\mathbf{M}(p) \models \varphi(0)$  and  $\mathbf{M}(p) \models \varphi(l) \Rightarrow \varphi(l+1)$  for all  $l < l_1$ . We thus have  $\mathbf{M} \models (p \models \varphi(0))$  and

$$\mathbf{M} \models (p \models \varphi(l) \Rightarrow p \models \varphi(l+1))$$

for  $l < l_1$ . By  $\Sigma_{n_0}^b$  induction in  $\mathbf{M}$ , it follows that  $\mathbf{M} \models (p \models \varphi(l_1))$ , whence  $\mathbf{M}(p) \models \varphi(l_1)$ .

We may now take  $\mathfrak{M}$  to be the initial segment  $\omega_{K-1}^{-\mathbb{N}}(\text{exp}^k(l_1))$  of  $\mathbf{M}$  (i.e.  $\mathfrak{M}$  consists of those elements  $l \in \mathbf{M}(p)$  which satisfy  $\omega_{K-1}^n(l) < \text{exp}^k(l_1)$  for all  $n \in \mathbb{N}$ ). Clearly, the operations of  $L_K$  are well-defined in  $\mathfrak{M}$ , and  $l_0 \in \log^k(\mathfrak{M})$  by the assumption that  $\omega_{K-1}^{\mathbb{N}}(\text{exp}^k(l_0)) < \text{exp}^k(l_1)$ . Moreover, since  $\log^k(\mathfrak{M})$  is contained in the segment  $[0, l_1)$ , we also have  $\mathfrak{M} \models I\Sigma_n^b | \log^k$ .  $\square$

## 5 The main theorem

Our next aim is the proof of our main theorem. All the results of this section require  $k$  to be at least 3, since sequences of terms whose length is in  $\log^{k-2}$  are involved. Recall that our base language  $L_K$  contains the symbols  $\#_i$  for  $i \leq K$ .

**Theorem 5.1** *Assume  $K > k+1$ . Then  $\text{Sk}(\Sigma_n^b, \text{depth } \log^k) \vdash \text{Sk}(\Sigma_{n+1}^b, \text{length } \log^k)$ .*

In the proof, we make the notational convention that whenever  $\exists y\psi(\bar{x}, y)$  is a formula in normal form, then this  $\psi$  is denoted by  $\varphi'$ .

**Proof.** <sup>4</sup>

Assume  $Sk(\Sigma_n^b, \text{depth } \log^k)$ .

Let  $i_0 \in \log^{k+1}$ ,  $l_0 = \text{exp}(i_0)$  and let  $\hat{\Lambda}$  be of length at most  $l_0$  and contain terms of depth  $\leq i_0$ . We may assume that  $\hat{\Lambda}$  has length exactly  $l_0$  (so that  $\hat{\Lambda} = \langle t_0, \dots, t_{l_0-1} \rangle$ ) and consists of  $\Sigma_{n+1}^b$  terms. Present  $\hat{\Lambda}$  as  $\hat{\Lambda}_0 \cup \dots \cup \hat{\Lambda}_{i_0}$ , where  $\hat{\Lambda}_m$  consists of those terms in  $\hat{\Lambda}$  which have depth  $m$ .

Let  $j = (l_0)^{i_0}$ . Observe that  $i_0 \leq |l_0|$ , so  $j \leq \omega_1(l_0)$ . Since  $K > k + 1$ ,  $\omega_{k+1}$  is a total function, so  $\log^k$  is closed under  $\omega_1$ . Hence,  $j \in \log^k$ .

Let  $\Lambda$  contain  $\hat{\Lambda}_0$ , consist of  $\Sigma_n^b$  terms of depth  $\leq j$ , and be such that for any  $\Sigma_n^b$  formula  $\varphi \leq N$  and any  $\bar{t} \in \Lambda$  of depth  $i < j$  and appropriate length, it holds that  $s^\varphi(\bar{t}) \in \Lambda$ . We additionally assume that  $0 \in \Lambda$ . Let  $p$  be a  $\Sigma_n^b$  evaluation on  $\Lambda$  given by  $Sk(\Sigma_n^b, \text{depth } \log^k)$ .

Let  $u_1, \dots, u_l$  be an enumeration of all pairs  $\langle \varphi, \bar{t} \rangle$ ,  $u_{l'} = \langle \varphi_{l'}, \bar{t}_{l'} \rangle$ , where  $\bar{t}$  is a tuple of terms from  $\hat{\Lambda}_0$  of length at most  $(N - 1)$  and  $\varphi < N$  is a  $\Sigma_{n+1}^b$  formula such that  $s^\varphi(\bar{t}) \in \hat{\Lambda}_1$ . Note that there are at most  $l_0 - 1$  such pairs.

We define by induction a function  $f_1 : \{u_1, \dots, u_l\} \rightarrow [0, j]$  (along with a sequence  $\langle s_1(u_{l'}) : l' \leq l \rangle$  of terms) as follows:

If  $\varphi_{l'}$  is a  $\Sigma_n^b$  formula, then  $f_1(u_{l'}) = 1$  and  $s_1(u_{l'}) = s^{\varphi_{l'}}(\bar{t}_{l'})$ .

Otherwise,  $f_1(u_{l'})$  is: either the least  $1 \leq i < j$  for which there is a  $\Sigma_n^b$  term  $s \in \Lambda$  of depth  $\leq i$  such that  $p \models \varphi'_{l'}(\bar{t}_{l'}, s)$  (in that case,  $s_1(u_{l'})$  is some such  $s$ ); or, if no such  $i$  exists,  $f_1(u_{l'}) = 0$  and  $s_1(u_{l'}) = 0$ .

In more detail:

If  $\varphi_1$  is a  $\Sigma_n^b$  formula, then  $f_1(u_1) = 1$  and  $s_1(u_1) = s^{\varphi_1}(\bar{t}_1)$  (note that in this case  $f_1(u_1)$  is the depth of  $s_1(u_1)$ ). Otherwise,  $f_1(u_1)$  is: either the least  $1 \leq i < j$  for which there is a  $\Sigma_n^b$  term  $s \in \Lambda$  of depth  $\leq i$  such that  $p \models \varphi'_1(\bar{t}_1, s)$  (in that case,  $s_1(u_1)$  is some such  $s$ ); or, if no such  $i$  exists,  $f_1(u_1) = 0$  and  $s_1(u_1) = 0$  (it then holds that  $p \models \neg\varphi_1(\bar{t}_1)$ ).

Similarly, if  $\varphi_2$  is a  $\Sigma_n^b$  formula, then  $f_1(u_2) = 1$  and  $s_1(u_2) = s^{\varphi_2}(\bar{t}_2)$ . Otherwise,  $f_1(u_2)$  is: either the least  $1 \leq i < j$  for which there is a  $\Sigma_n^b$  term  $s \in \Lambda$  of depth  $\leq i$  such that  $p \models \varphi'_2(\bar{t}_2, s)$  (in that case,  $s_1(u_1)$  is some such  $s$ ); or, if no such  $i$  exists,  $f_1(u_2) = 0$  and  $s_1(u_2) = 0$  (it then holds that  $p \models \neg\varphi_2(\bar{t}_2)$ ). Etc.

Note that all the notions required in the definition of  $f_1$ , in particular the relation “ $p \models \varphi'(\bar{t}, s)$ ”, are definable by bounded (possibly with an extra

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<sup>4</sup>The proof has some ideas in common with [BR], in particular the use of the pigeon hole principle.

parameter) formulae of fixed complexity. By choosing a large enough  $n_0$  we may assume that this complexity is suitably less than  $\Sigma_{n_0}^b$ . Also,  $f_1$  can be coded as described in the preliminaries, i.e. a tuple  $\bar{t}$  such that  $\langle \varphi, \bar{t} \rangle$  is in the domain of  $f_1$  can be identified with the tuple of indices of the terms  $\bar{t}$  in the enumeration of  $\hat{\Lambda}_0$ . Thus,  $T_0$  will suffice to prove the existence of a code for a function  $f_1$  with the required properties.

We have

$$[0, j) = [0, (l_0)^{i_0-1}) \cup [(l_0)^{i_0-1}, 2(l_0)^{i_0-1}) \cup \dots \cup [(l_0 - 1)(l_0)^{i_0-1}, l_0(l_0)^{i_0-1}).$$

As  $l_0$  is quite small (it is certainly in *log*), we may apply the pigeon hole principle to find  $r < l_0$  such that the interval  $[r(l_0)^{i_0-1}, (r+1)(l_0)^{i_0-1})$  does not contain any value of the function  $f_1$ . This is because if all  $l_0$  of the above intervals contained a value of  $f_1$ , we could use the code of  $f_1$  to obtain a coded function  $f$  from  $l_0 - 1$  onto  $l_0$ . But the pigeon hole principle of the form

$$\forall f, x \text{ (} f \text{ is not a function from } x - 1 \text{ onto } x)$$

is provable in (a finite fragment of)  $I\Delta_0$  — hence we may assume that it is provable in  $T_0$ .

So, let  $r < l_0$  be such that the interval  $[r(l_0)^{i_0-1}, (r+1)(l_0)^{i_0-1})$  does not contain any value of the function  $f_1$ .

Let  $r_1 = r(l_0)^{i_0-1}$ ,  $r'_1 = (r+1)(l_0)^{i_0-1}$ . Let  $\Lambda_1$  be  $\hat{\Lambda}_0 \cup \{s_1(u_{l'}) : l' \leq l, f_1(u_{l'}) < r_1\}$ . Also define  $\tilde{g}_1 : \hat{\Lambda}_1 \longrightarrow \Lambda_1$  by:

$$\tilde{g}_1(s^\varphi(\bar{t})) = \begin{cases} s_1(\bar{t}, \varphi) & \text{if } f_1(\bar{t}, \varphi) < r_1 \\ 0 & \text{otherwise} \end{cases}$$

and  $g_1 : \hat{\Lambda}_0 \cup \hat{\Lambda}_1 \longrightarrow \Lambda_1$  as  $\tilde{g}_1 \cup id \upharpoonright \hat{\Lambda}_0$ .

Note that for  $\varphi \in \Sigma_n^b$ ,  $g_1(s^\varphi(\bar{t})) = s^\varphi(\bar{t})$ . For in this case,  $f_1(\bar{t}, \varphi) = 1$ , whence  $f_1(\bar{t}, \varphi) \in [0, (l_0)^{i_0-1})$  and consequently,  $f_1(\bar{t}, \varphi) < r_1$ .

Now let  $u_1, \dots, u_l$  be an enumeration of all pairs  $\langle \varphi, g_1(\bar{t}) \rangle$ ,  $u_{l'} = \langle \varphi_{l'}, g_1(\bar{t}_{l'}) \rangle$ , where  $\bar{t}$  is a tuple of terms from  $\hat{\Lambda}_0 \cup \hat{\Lambda}_1$  of length at most  $(N-1)$  and  $\varphi < N$  is a  $\Sigma_{n+1}^b$  formula such that  $s^\varphi(\bar{t}) \in \hat{\Lambda}_2$ . Again, there are at most  $l_0 - 1$  such pairs.

Let  $f_2 : \{u_1, \dots, u_l\} \longrightarrow [0, j)$  and  $\langle s_2(u_{l'}) : l' \leq l \rangle$  be defined by:

If  $\varphi_{l'}$  is a  $\Sigma_n^b$  formula, then  $f_2(u_{l'})$  is the depth of  $s^{\varphi_{l'}}(g_1(\bar{t}_{l'}))$  and  $s_2(u_{l'}) = s^{\varphi_{l'}}(g_1(\bar{t}_{l'}))$ . Note that the depth of  $g_1(\bar{t}_{l'})$  is  $< r_1 < j$ , whence  $s^{\varphi_{l'}}(\bar{t}_{l'}) \in \Lambda$ , by our assumption on  $\Lambda$ . Otherwise,  $f_2(u_{l'})$  is: either the least  $2 \leq i < j$

for which there is a  $\Sigma_n^b$  term  $s \in \Lambda$  of depth  $\leq i$  such that  $p \models \varphi'_{l'}(g_1(\bar{t}_{l'}), s)$  (in that case,  $s_2(u_{l'})$  is some such  $s$ ); or, if no such  $i$  exists,  $f_2(u_{l'}) = 0$  and  $s_2(u_{l'}) = 0$ .

We now have

$$[r_1, r'_1] = [r_1, r_1 + (l_0)^{i_0-2}] \cup [r_1 + (l_0)^{i_0-2}, r_1 + 2(l_0)^{i_0-2}] \\ \cup \dots \cup [r_1 + (l_0 - 1)(l_0)^{i_0-2}, r_1 + l_0(l_0)^{i_0-2}].$$

Let  $r < l_0$  be such that the interval  $[r_1 + r(l_0)^{i_0-2}, r_1 + (r+1)(l_0)^{i_0-2}]$  does not contain any value of the function  $f_2$ .

Let  $r_2 = r_1 + r(l_0)^{i_0-2}$ ,  $r'_2 = r_1 + (r+1)(l_0)^{i_0-2}$ . Let  $\Lambda_2$  be  $\Lambda_1 \cup \{s_2(u_{l'}) : l' \leq l, f_2(u_{l'}) < r_2\}$ . Define  $\tilde{g}_2 : \hat{\Lambda}_2 \rightarrow \Lambda_2$  by:

$$\tilde{g}_2(s^\varphi(\bar{t})) = \begin{cases} s_2(g_1(\bar{t}), \varphi) & \text{if } f_2(g_1(\bar{t}), \varphi) < r_2 \\ 0 & \text{otherwise} \end{cases}$$

and  $g_2 : \hat{\Lambda}_0 \cup \hat{\Lambda}_1 \cup \hat{\Lambda}_2 \rightarrow \Lambda_2$  as  $\tilde{g}_2 \cup g_1$ .

Again for  $\varphi \in \Sigma_n^b$ ,  $g_2(s^\varphi(\bar{t})) = s^\varphi(g_1(\bar{t}))$ . For in this case,  $f_2(g_1(\bar{t}), \varphi)$  is the depth of  $g_1(\bar{t})$  plus 1, whence  $f_2(g_1(\bar{t}), \varphi) \leq r_1 < r_2$

For  $2 < m \leq i_0$ , we construct  $f_m, r_m, r'_m, \Lambda_m, g_m$ , in a similar way. Finally, we take  $g : \hat{\Lambda} \rightarrow \Lambda$  to be  $\bigcup_{m \leq i_0} g_m$  and let  $\hat{p}$  be the evaluation on  $\hat{\Lambda}$  defined by:

$$(*) \quad \hat{p}(\varphi(\bar{t})) = p(\varphi(g(\bar{t})))$$

(for  $\bar{t} \in \hat{\Lambda}$  and  $\varphi$  simple atomic). It remains to show that  $\hat{p}$  is a  $\Sigma_{n+1}^b$  evaluation on  $\hat{\Lambda}$ .

Note that for any  $\Sigma_n^b$  formula  $\varphi < N$ , if  $s^\varphi(\bar{t}) \in \hat{\Lambda}$ , then  $s^\varphi(g(\bar{t})) \in \Lambda$ , and moreover,  $g(s^\varphi(\bar{t})) = s^\varphi(g(\bar{t}))$ . This makes it possible to prove by induction on formula complexity that  $(*)$  holds also if  $\varphi < N$  is a  $\Sigma_n^b$  formula and  $\hat{\Lambda}$  is g.e. for  $\langle \varphi, \bar{t} \rangle$  (use the fact that  $p$  is a  $\Sigma_n^b$  evaluation in the step for the universal quantifier).

It follows that  $\hat{p}$  satisfies part (2) of the definition of a  $\Sigma_{n+1}^b$  evaluation (since  $p$  is a  $\Sigma_n^b$  evaluation). As for (1), the most interesting case is when  $\varphi \in \Sigma_{n+1}^b \setminus (\Sigma_n^b \cup \Pi_n^b)$ . So let  $\varphi < N$  be such a formula and assume that  $\hat{\Lambda}$  is g.e. for  $\langle \varphi, \bar{t} \rangle$ . We thus know that in, say, the  $m$ -th step of the construction  $\langle \varphi, g_{m-1}(\bar{t}) \rangle$  appeared as some  $u_{l'}$ .

If there was at that point no term  $s \in \Lambda$  for which  $p \models \varphi'(g_{m-1}(\bar{t}), s)$ , then no such term could have appeared later on in the construction, so for any  $s \in \Lambda$ ,  $p \models \neg \varphi'(g(\bar{t}), s)$ . Then by  $(*)$  and definition 3.1,  $\hat{p} \models \neg \varphi$ .

Otherwise, either  $f_m(u_\nu) < r_m$  or the contrary. In the former case, clearly  $p \models \varphi'(g(\bar{t}), g(s^\varphi(\bar{t})))$  and hence  $\hat{p} \models \varphi(\bar{t})$ . In the latter case, since no term  $s \in \Lambda$  of depth  $< f_m(u_\nu)$  satisfies  $p \models \varphi'(g(\bar{t}), s)$ , and no term of depth  $\geq f_m(u_\nu)$  is in the range of  $g$ , we have  $\hat{p} \models \neg\varphi(\bar{t})$ . This completes the proof of the theorem.  $\square$

Note that the assumption  $K > k + 1$  was only needed to assure that the number  $j = (l_0)^{i_0}$  appearing in the proof is an element of  $\log^k$ . It is quite possible that this assumption is not optimal; we have not made a serious effort to improve it.

From the proof of theorem 5.1 we obtain the following corollary:

**Corollary 5.2** *Assume  $K > k+1$ . Then  $Sk(\Sigma_n^b, \text{length } \log^{k-2}) \vdash Sk(\Sigma_{n+1}^b, \text{length } \log^k)$ .*

**Proof.** The corollary follows immediately from the following observation.

Let  $\hat{\Lambda}, l_0, j$  be as in the proof of theorem 5.1. Then there is a  $\Lambda$  with the properties required in the proof of theorem 5.1 such that  $lh(\Lambda) \in \log^{k-2}$ .

Let us prove this observation.

For  $i \leq j$ , let  $L_i$  denote the number of terms of depth at most  $i$  which have to be included in  $\Lambda$ . Then  $L_0 \leq l_0 + 1$  and  $L_{i+1} \leq L_i + L_i^{N-1} \cdot N \leq L_i^{N+1}$ . Hence  $L_j \leq (l_0 + 1)^{(N+1)^j}$ . Since  $j \in \log^k$ ,  $(N+1)^j \in \log^{k-1}$  and  $L_j \in \log^{k-2}$ .  $\square$

There is no direct connection between  $Sk(\Sigma_n^b, \text{depth } \log^k)$  and  $Sk(\Sigma_n^b, \text{length } \log^{k-2})$ .

Based on the above results, we may summarize the known relationships between the theories  $I\Sigma_n^b | \log^k$  and  $Sk(\Sigma_n^b, \text{length } \log^k)$ , for various  $n$  and  $k$ , in the following diagram:

$$\begin{array}{ccccc}
Sk(\Sigma_n^b, \text{length } \log^k) & \Leftarrow & I\Sigma_{n+1}^b | \log^k & \Leftarrow & Sk(\Sigma_{n+1}^b, \text{length } \log^k) \\
& & \Downarrow (*) & & \\
\Uparrow & & I\Sigma_n^b | \log^{k-1} & & \Uparrow \\
& & \Uparrow & & \\
Sk(\Sigma_{n-1}^b, \text{length } \log^{k-2}) & \Leftarrow & I\Sigma_n^b | \log^{k-2} & \Leftarrow & Sk(\Sigma_n^b, \text{length } \log^{k-2})
\end{array}$$

It is assumed in the diagram that  $k \geq 3$  and that  $n \geq n_0 + k$ ,  $K > k + 1$  (the dependence of  $n$  on  $k$  is to have  $T_0$  implied by all the theories in question). Thick arrows denote provability, thin arrows denote inducing a model in the sense of theorem 4.3. The arrow marked with an asterisk is the one whose reversibility is a famous open problem.

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