

An Application of a Reflection Principle

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Abstract

We define a recursive theory which axiomatizes a class of models of $I\Delta_0 + \Omega_3 + \neg exp$ all of which share two features: firstly, the set of Δ_0 definable elements of the model is majorized by the set of elements definable by Δ_0 formulae of fixed complexity; secondly, Σ_1 truth about the model is recursively reducible to the set of true Σ_1 formulae of fixed complexity.

In the present paper, we define a consistent recursive theory T , implying $I\Delta_0$ and inconsistent with $I\Delta_0 + exp$, which has the following two properties:

- 1) in every model $\mathbf{M} \models T$ elements definable by Δ_0 formulae of fixed quantifier complexity are cofinal among all Δ_0 definable elements;
- 2) for every model $\mathbf{M} \models T$, the set of Σ_1 sentences true in \mathbf{M} is recursively reducible to the set of true Σ_1 sentences whose Δ_0 part has fixed quantifier complexity.

Thus, T axiomatizes to some extent the phenomenon of the cofinality of elements definable by Δ_0 formulae with fixed complexity among all Δ_0 definable elements, and of the reducibility of the set of true Σ_1 sentences to the set of true Σ_1 sentences whose complexity is fixed.

From the logical point of view, the idea behind the construction of T seems to be interesting in itself. The axioms of T reduce the validity of a Π_1 sentence ψ to the validity a sentence expressing (roughly) a form of “consistency” of ψ . To show the consistency of T , we have to be able to build a model in which all “consistent” Π_1 sentences are true.

We construct such a model by iterating the following procedure: given a model \mathbf{M} satisfying the “consistency” of the Π_1 sentence ψ_0 , we build another

model \mathbf{M}_0 satisfying ψ_0 , and still satisfying the “consistency” of ψ_0 . We then move on to the next Π_1 sentence, ψ_1 . To carry on the construction, we now must — if \mathbf{M}_0 satisfies the “consistency” of ψ_1 — be able to construct another model \mathbf{M}_1 satisfying ψ_1 , but still satisfying ψ_0 and the “consistency” of ψ_0 and ψ_1 . Etc.

Thus, we need our models have the property that “what is true is consistent”. Moreover, this property has to be preserved under the iteration. Therefore, what we need is in fact the “consistency” of the set of true Π_1 and Σ_1 sentences together with the “consistency” of the set of true Π_1 and Σ_1 sentences together with the “consistency” of the set of true Π_1 and Σ_1 sentences...”. To make this formal, we have to define a kind of “self-reproducing consistency statement”. This is subtle since we are very close to contradicting Gödel’s second incompleteness theorem.

The paper is organized as follows. Section 1 is preliminary. Section 2 discusses our basic technical tool: evaluations on sequences of terms. In section 3, we define our “self-reproducing consistency statement”, and we argue that it is a kind of reflection principle. Finally, in section 4 we introduce the theory T and prove our main results.

1 Preliminaries

Some notational conventions:

The symbol \log stands for the discrete-valued binary logarithm function; $\exp(x)$ is 2^x . Whenever f denotes a function, $f^{(k)}$ denotes f iterated k times. For a model \mathbf{M} , $\log^{(k)}(\mathbf{M})$ (the k -th logarithm of \mathbf{M}) consists of those elements of \mathbf{M} for which $\exp^{(k)}$ exists. The variable i , possibly with indices, always ranges over elements of $\log^{(3)}$, and the variable j , possibly with indices, ranges over elements of $\log^{(4)}$. A “bar” (as in, say, “ \bar{x} ”) always denotes a tuple — depending on the context, it may happen that tuples of nonstandard length are also allowed.

We adopt the coding of sets and sequences in bounded arithmetic developed in [HP]. Also the notion of length $lh(\Lambda)$ of a sequence Λ is the one defined in [HP] for bounded arithmetic. If $\Lambda = \langle t_1, \dots, t_l \rangle$ is a sequence of length $l \in \log(\mathbf{M})$, then functions from Λ into $\{0, 1\}$ may be coded as subsets of size $lh(\Lambda)$ of $\Lambda \times \{0, 1\}$ (see [S]). We use a somewhat different coding, letting $f : \Lambda \rightarrow \{0, 1\}$ be represented by the pair $\langle \Lambda, p \rangle$, where p is a function from $\{1, \dots, l\}$ into $\{0, 1\}$ — thus, an object of size $\exp(l)$ — with $p(i)$ intended to code $f(t_i)$. Whenever Λ is fixed, we may simply identify f with p .

Our base language L contains the individual constants 0, 1, and the

relational symbols $+$, \leq , \times , $|\cdot|$, $\#_2$, $\#_3$, and $\#_4$.

The intuitive meaning of $|x| = y$ is that y is the length of the binary representation of x (equal to $\lceil \log(x+1) \rceil$). The $\#_i$'s are to stand for the graphs of the first three smash functions: $x\#_2y = \exp(|x| \cdot |y|)$, $x\#_{n+1}y = \exp(|x|\#_n|y|)$ for $n \geq 2$. A hierarchy of functions related to the smash functions is defined by: $\omega_1(x) = x^{|x|}$; $\omega_{n+1}(x) = \exp(\omega_n(|x|))$. Note that for any $n \geq 1$, $\omega_n(x)$ is roughly $x\#_{n+1}x$.

We assume that some appropriate Gödel numbering of L -formulae has been fixed; we shall identify the formulae with their Gödel numbers.

An L -formula φ is in *negation normal form* if no quantifiers in φ occur in the scope of a negation. φ is Δ_0 if all the quantifiers in φ are bounded, i.e. of the form $\exists x \leq y$. Σ_1 and Π_1 formulae are defined in the natural way.

For any natural number r , the class E_r consists of Δ_0 formulae in prenex normal form which contain $(r-1)$ alternations of quantifier blocks, starting with an existential block, and *not* counting sharply bounded quantifiers¹. The class U_r is defined dually. The class \exists_r consists of Σ_1 formulae of the form $\exists x\psi$ where ψ is U_{r-1} . The class \forall_r is defined dually.

We take $I\Delta_0 + \Omega_3$ to be the theory which consists of: a finite number of basic axioms relating the interpretations of the L -symbols to each other; the induction scheme for all Δ_0 formulae; and an axiom stating that $\#_4$ is a total function (note that this is equivalent to the totality of ω_3). $I\Delta_0 + \Omega_n$, for $i = 1, 2$, is defined analogously. $I\Delta_0$ states only the totality of $+$ and \times . $I\Delta_0 + \exp$, on the other hand, additionally states the totality of the \exp function.

$I\Delta_0^*$ is an auxiliary system which contains the basic axioms and the Δ_0 induction scheme, but no axioms stating the totality of $+$, \times etc. Thus, a model of $I\Delta_0^*$ may have a greatest element. Note that (under a reasonable choice of the basic axioms), all axioms of $I\Delta_0^*$ are Π_1 .

One benefit of working with a relational language is that defining the relativization of a formula poses no difficulties. Namely, if φ is an L -formula, then φ^x is defined inductively, with only the quantifier step non-trivial: $(\exists y\psi)^x := \exists y \leq x \psi^x$.

The language L_T is an extension of L obtained by adding function symbols s^φ for all L -formulae φ in negation normal form which begin with an existential quantifier. The intention is that the symbol s^φ stands for a Skolem function for the first existential quantifier in φ . That is, given an L -formula $\varphi(\bar{x}) = \exists y\psi(\bar{x}, y)$ in negation normal form, s^φ is a function symbol of arity

¹The notion of *sharply bounded quantifier* is an obvious variant of the one known from functional languages for bounded arithmetic, e.g. in $\forall x\forall y \leq x \exists z \leq x((y = |x| \Rightarrow z = y) \wedge \dots)$ the quantifier $\forall z$ is sharply bounded.

$1 + lh(\bar{x})$, and $s^\varphi(\bar{t})$ is intended to be some y which satisfies $\psi(\bar{t}, y)$, if such a y exists.

Whenever we speak of a formula $\varphi(\bar{t})$, it is assumed that $\varphi(\bar{x})$ itself is an L -formula, although the terms \bar{t} do not have to be terms of L .

We have to encode the language L_T in arithmetic. We use numbers divisible by 3 to enumerate terms of the form $s^\varphi(\bar{t})$, numbers congruent to 1 (*mod* 3) for a special enumeration of numerals, and numbers congruent to 2 (*mod* 3) to enumerate some additional terms. In more detail: we let the number $3\langle\varphi(\bar{x}), \bar{t}\rangle$ correspond to $s^\varphi(\bar{t})$; we let $3k + 1$ correspond to a numeral for k ($3k + 1$ will be referred to as \underline{k}); finally, we let $3k + 2$ correspond to a special term s_k (the role of the s_k 's is explained by clause (v) of definition 2.3). We also code $\varphi(\bar{t})$ by the ordered pair $\langle\varphi(\bar{x}), \bar{t}\rangle$.

From now on, we identify the terms of L_T with their numbers.

The models \mathbf{M} we work with are — unless explicitly stated or obvious from the context that this is not the case — assumed to be nonstandard countable models of $I\Delta_0 + \Omega_3$.

We shall consider various sequences of closed terms. About such a sequence Λ we shall always assume that if a term of the form $s^\varphi(\bar{t})$ appears in Λ , then all terms in \bar{t} also do, and moreover, that they have smaller indices in Λ than $s^\varphi(\bar{t})$. Also, whenever dealing with a sequence Λ and a model \mathbf{M} , we shall assume that $lh(\Lambda)$ is in $log(\mathbf{M})$.

Given a sequence of terms Λ , let the collection $\mathcal{A}(\Lambda)$ of atomic sentences over Λ consist of all sentences obtained by substituting terms from Λ for variables in atomic formulae of L . Observe that there is a standard polynomial $\pi(n)$ such that $lh(\mathcal{A}(\Lambda)) \leq \pi(lh(\Lambda))$. Let us fix some such π .

Some more notation: if \mathcal{F} is a class of formulae, the symbol $\mathcal{F}(\mathbf{M})$ denotes the family of all \mathcal{F} -definable elements of \mathbf{M} , while $\mathbf{M}^{\mathcal{F}}$ denotes the set of \mathcal{F} -sentences true in \mathbf{M} .

Finally, let us recall some relevant facts about universal formulae. Firstly, in $I\Delta_0 + exp$ there is a Σ_1 universal formula Sat for Δ_0 . Thus, Sat is Σ_1 , and for any $\mathbf{M} \models I\Delta_0 + exp$, $\varphi \in \mathbf{M}$ a Δ_0 formula,

$$\mathbf{M} \models Sat(\varphi) \text{ iff } \mathbf{M} \models \varphi.$$

Secondly, in $I\Delta_0 + \Omega_3$ there is an \exists_r universal formula Sat_r for \exists_r , for each $r \in \omega$. Sat_r can obviously be also used as a universal formula for E_r , and additionally, if we limit our attention to the truth of E_r formulae smaller than some a with parameters smaller than some b , then the initial existential quantifier in Sat_r can also be bounded (thus giving an “ E_r formula with a parameter”: call this formula Sat_{E_r}).

2 Evaluations and evaluation models

Let $p : \mathcal{A}(\Lambda) \rightarrow \{0, 1\}$ map every axiom of equality in $\mathcal{A}(\Lambda)$ to 1. We call such a p an *evaluation* on Λ , since we may think of p as assigning a logical value to sentences in $\mathcal{A}(\Lambda)$ (see also [A1], [A2], [A3], [AZ1], [AZ2], [S]). Of course, p can be uniquely extended to all boolean combinations of sentences in $\mathcal{A}(\Lambda)$ in the routine way.

Note in passing that any evaluation on Λ is an object of size at most $\exp(\text{lh}(\mathcal{A}(\Lambda)))$ and thus at most $\exp(\pi(\text{lh}(\Lambda)))$.

For $\varphi(\bar{x})$ in negation normal form, $\bar{t} \in \Lambda$, we define the notion that Λ is *good enough* (g.e.) for $\langle \varphi, \bar{t} \rangle$ by induction on φ . Λ is always g.e. for $\langle \varphi, \bar{t} \rangle$ if φ is open. Λ is g.e. for $\langle \varphi_1 \vee \varphi_2, \bar{t} \rangle$ iff it is g.e. for $\langle \varphi_1, \bar{t} \rangle$ and $\langle \varphi_2, \bar{t} \rangle$, similarly for conjunctions. If φ is $\exists y \varphi'(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^\varphi(\bar{t}) \in \Lambda$ and Λ is g.e. for $\langle \varphi', \bar{t} \frown s^\varphi(\bar{t}) \rangle$. Finally, if φ is $\forall y \tilde{\varphi}(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^{\exists y \neg \tilde{\varphi}(\bar{x}, y)}(\bar{t}) \in \Lambda$ (where $\exists y \neg \tilde{\varphi}$ is the normal form of $\neg \varphi$) and Λ is g.e. for $\langle \tilde{\varphi}, \bar{t} \frown s^{\exists y \neg \tilde{\varphi}(\bar{x}, y)}(\bar{t}) \rangle$.

The idea is that Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if it contains enough appropriate Skolem terms so that assigning a logical value to $\varphi(\bar{t})$ based on an evaluation on Λ makes sense.

Definition 2.1 *Let $\bar{t} \in \Lambda$. We define the relation $p \models \varphi(\bar{t})$ for $\varphi(\bar{x})$ in negation normal form by induction:*

- (i) $p \models \varphi(\bar{t})$ iff $p(\varphi(\bar{t})) = 1$ for $\varphi(\bar{t})$ open;
- (ii) the relation $p \models \varphi$ behaves in the natural way with respect to conjunctions and disjunctions;
- (iii) if φ is $\exists y \varphi'(\bar{x}, y)$, then $p \models \varphi(\bar{t})$ iff Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ and $p \models \varphi'(\bar{t}, s^\varphi(\bar{t}))$,
- (iv) if φ is $\forall y \tilde{\varphi}(\bar{x}, y)$, then $p \models \varphi(\bar{t})$ iff for all $t \in \Lambda$ such that Λ is g.e. for $\langle \tilde{\varphi}, \bar{t} \frown t \rangle$, $p \models \tilde{\varphi}(\bar{t}, t)$.

We will be especially interested in the case where Λ is one of a number of canonical sequences of terms. To define these, let $K(i)$ be the unique function satisfying $K(0) = 1$ and $K(i+1) = c \cdot \exp(i) \cdot K(i)^i$, where c is an appropriately large standard integer. Note that for any i , $K(i) \in \text{log}$, as for almost all i , $K(i) \leq \exp(i)$, and i^i is always in $\text{log}^{(2)}$, since we have:

Remark 2.2 *In any model of $I\Delta_0 + \Omega_3$, log is closed under ω_2 , $\text{log}^{(2)}$ is closed under ω_1 , $\text{log}^{(3)}$ is closed under multiplication, and $\text{log}^{(4)}$ is closed under addition.*

The notion of *canonical sequence of rank i* , Λ_i , is now defined by induction. Λ_{i+1} is the smallest sequence Λ such that:

- for any $j \leq i + 1$, Λ contains the term s^j and is good for $\langle \text{exp}^{(3)}(x) = y, \hat{j} \hat{s}_j \rangle$;
- for any $a \leq K(i)$, Λ contains the numeral \underline{a} , and if $\text{exp}(a) \leq \text{exp}^{(3)}(i + 1)$, then Λ is g.e. for $\langle \exists \text{exp}(x), \underline{a} \rangle$;
- for any formula $\varphi < \text{exp}(i)$ of the form ψ^t or $\exists x \leq t \psi^x$ (where $t \in \Lambda_i$), and any $\bar{t} \in \Lambda_i$, Λ is g.e. for $\langle \varphi, \bar{t} \rangle$.

Observe that if c is chosen large enough, then $lh(\Lambda_i) \leq K(i)$ for all i (since a formula smaller than $\text{exp}(i)$ contains at most i quantifiers).

Some particularly well-behaved evaluations on Λ_i will be called evaluations of rank i (we let \mathcal{A}_i stand for $\mathcal{A}(\Lambda_i)$):

Definition 2.3 A function $p : \mathcal{A}_i \longrightarrow \{0, 1\}$ is called an evaluation of rank i if the following holds:

(i) for every $\varphi(\bar{x}) < \text{exp}(i)$ and every $\bar{t} \in \Lambda_i$ of appropriate length, if Λ_i is g.e. for $\langle \varphi, \bar{t} \rangle$, then for all $j \leq i$,

$$p \models \varphi(\bar{t})^{s_j} \text{ or } p \models \neg \varphi(\bar{t})^{s_j};$$

(ii) if $\varphi < \text{exp}(i)$ is an axiom of $I\Delta_0^*$, then assuming Λ_i is g.e. for $\langle \varphi, \emptyset \rangle$, $p \models \varphi$;

(iii) $p \models (\underline{0} = 0 \wedge \underline{1} = 1)$, and given any $\underline{a}, \underline{b} \in \Lambda_i$: if $\underline{a} + \underline{b} \in \Lambda_i$, then $p \models (\underline{a} + \underline{b} = \underline{a + b})$, and similarly for the other symbols of L ;

(iv) for all $\underline{a} \in \Lambda_i$ such that $\text{exp}(a) \leq \text{exp}^{(3)}(i + 1)$, $p \models \exists \text{exp}(\underline{a})$;

(v) for all $j \leq i$, $p \models s_j = \text{exp}^{(3)}(j)$.

We let “ $p \in \mathcal{E}_i$ ” stand for “ p is an evaluation of rank i ”. This is a slight abuse of notation, since the code for the set of evaluations of rank i might be too large to be an element of the model.

We claim that both “ $p \models \varphi$ ” (for p an evaluation on Λ_i) and “ $p \in \mathcal{E}_i$ ” are Δ_0 definable with an appropriately large parameter (and thus Δ_1 definable).

To see whether an evaluation p on Λ_i sets φ to “True” (i.e. whether $p \models \varphi$), we need to deal with sets V_0, \dots, V_r , where where V_l is the set of values given by p to the l -th subformula of φ under all relevant substitutions

of terms in Λ_i for the free variables in that subformula. Since there are at most $\log \varphi$ variables in any subformula of φ , the number of possible substitutions is not greater than $K(i)^{\log \varphi}$, and hence $V_i \leq \exp(K(i)^{\log \varphi})$. Again, there can be no more than $\log \varphi$ subformulae of φ . Thus, the sequence $\langle V_0, \dots, V_r \rangle$ is at most $\log \varphi$ -long, so its code is at most $\exp(\log \varphi \cdot K(i)^{\log \varphi})$. This is the largest object relevant to the truth value given to φ by ψ , which shows that “ $p \models \varphi$ ” is indeed Δ_0 definable with a parameter.

To see whether an evaluation p on Λ_i is in \mathcal{E}_i , we have to check what truth value it assigns to a number of formulae φ smaller than $\exp(i)$. With some additional work, one may verify that all objects we need to consider are smaller than $\omega_3(\exp^{(3)}i)$, which implies that also “ $p \in \mathcal{E}_i$ ” is Δ_0 definable with a parameter.

We let $True(p, i, \varphi)$ be a Δ_1 formula which says “ $p \in \mathcal{E}_i$ and $p \models \varphi$ ”.

Definition 2.4 *If $p_1 \in \mathcal{E}_{i_1}$ and $p_2 \in \mathcal{E}_{i_2}$ with $i_1 \leq i_2$, we say that p_2 extends p_1 iff $p_1 \subseteq p_2$.*

The following proposition lists some “conservativity” relationships between evaluations one of which extends the other. The proofs are simple inductive arguments.

Proposition 2.5 *Let $p_2 \in \mathcal{E}_{i_2}$ extend $p_1 \in \mathcal{E}_{i_1}$. Then:*

- (1) *if Λ_{i_1} is g.e. for $\langle \varphi, \bar{t} \rangle$ and $j \leq i_1$, then $p_1 \models \varphi(\bar{t})^{s_j}$ iff $p_2 \models \varphi(\bar{t})^{s_j}$;*
- (2) *if $\varphi(\bar{x})$ is an open formula and $\bar{t} \in \Lambda_{i_1}$, then $p_1 \models \varphi(\bar{t})$ iff $p_2 \models \varphi(\bar{t})$;*
- (3) *if Λ_{i_1} is g.e. for $\langle \varphi, \bar{t} \rangle$, then $p_2 \models \varphi(\bar{t})$ implies $p_1 \models \varphi(\bar{t})$.*

Another simple fact about evaluations is:

Proposition 2.6 *Let $p \in \mathcal{E}_i$ and let $i' \leq i$. Then $(p \upharpoonright \mathcal{A}_{i'}) \in \mathcal{E}_{i'}$.*

The importance of evaluations consists in the fact that they make possible the construction of models for $I\Delta_0 + \Omega_3$. More precisely, such a model is determined by an ascending chain of evaluations whose ranks are cofinal in $\log^{(3)}$ (note that by remark 2.2, in a model of $I\Delta_0 + \Omega_3$ the third logarithm has no last element).

Let $P = \langle p_n : n \in \omega \rangle$ be such a chain. If \bar{t} is a tuple of terms of L_T and $\varphi(\bar{t})$ is open, then almost all p_n 's agree on the logical value of $\varphi(\bar{t})$. So, we may define $P \models \varphi(\bar{t})$ by:

$$p_n \models \varphi(\bar{t}) \text{ for almost all } n.$$

We define the relation $=_P$ between terms in L_T by:

$$t_1 =_P t_2 \text{ iff } P \models (t_1 = t_2).$$

Since the p_n 's are evaluations, $=_P$ is an equivalence relation and a congruence with respect to the relations of L . Let $\mathbf{M}_0[P]$ be the model whose universe is the set of $=_P$ -equivalence classes and whose relations are defined by

$$[t_1] + [t_2] = [t_3] \text{ iff } P \models (t_1 + t_2 = t_3) \text{ etc.}$$

Clearly, we have:

$$\mathbf{M}_0[P] \models \varphi([\bar{t}]) \text{ iff } P \models \varphi(\bar{t})$$

for any open φ . If we introduce the more general relation $P \models \varphi(\bar{t})$, for φ not necessarily open, by the same clause as above, then induction yields

$$P \models \varphi(\bar{t}) \text{ implies } \mathbf{M}_0[P] \models \varphi([\bar{t}]).$$

The converse implication will not generally hold unless we accept a more restrictive definition of evaluation which is not needed here.

The next lemma and corollary show that the numeral \underline{a} may be treated as a name for the a -th element of $\mathbf{M}_0[P]$.

Lemma 2.7 *Let $p \in \mathcal{E}_i$. If for a term $t \in L_T$, $p \models (t \leq \underline{a})$, then there is $b \leq a$ such that $p \models (t = \underline{b})$.*

Moreover, if φ is an open formula and \bar{a} is a tuple of numerals for numbers less or equal to $K(i - 1)$, then $\varphi(\bar{a})$ implies $p \models \varphi(\bar{a})$.

Proof. We may assume that our formalization of $I\Delta_0^*$ contains axioms such as: $\forall x(x \leq 0 \Rightarrow x = 0)$, $\forall x \forall y(x \leq y + 1 \Rightarrow x = y + 1 \vee x \leq y)$, $\forall x(x + 0 = x)$, $\forall x \forall y((x + (y + 1) = (x + y) + 1)$ and similar axioms for the other symbols of L .

The first part of the lemma is proved by induction on $a \leq K(i - 1)$. For $a = 0$, $p \models (\underline{0} = \underline{0})$, so $p \models (t \leq \underline{0})$ implies $p \models (t \leq 0)$, hence $p \models (t = 0)$ by the appropriate axiom, hence $p \models (t = \underline{0})$. Assume that the thesis holds for a and that $p \models (t \leq \underline{a + 1})$. Then, since $p \models (\underline{a + 1} = \underline{a} + 1)$, we get either $p \models (t = \underline{a + 1})$ or $p \models (t \leq \underline{a})$, in which case we use the inductive assumption to get $p \models (t = \underline{b})$ for some $b \leq a$.

For the ‘‘moreover’’ part, first prove $p \models (\underline{a_1} + \underline{a_2} = \underline{a_1 + a_2})$ (assuming $a_1 + a_2 \leq K(i - 1)$) by induction, using appropriate axioms for the induction base and induction step. Then proceed similarly with

$p \models (\underline{a}_1 \cdot \underline{a}_2 = \underline{a}_1 \cdot \underline{a}_2)$ (again, assuming i is large enough) and the remaining symbols of L , and pass through boolean combinations to obtain the thesis. \square

Corollary 2.8 *The mapping $a \rightarrow [a]$ (for $a \in \log(\mathbf{M})$) is an isomorphism between $\log(\mathbf{M})$ and an initial segment I of $\mathbf{M}_0[P]$*

Proof. It suffices to observe that if $P = \langle P_n : n \in \omega \rangle$ where $p_n \in \mathcal{E}_{i_n}$, then for any tuple $\bar{a} \in \log$, the maximal element of \bar{a} is smaller than $K(i_n - 1)$ for almost all n , so we may apply lemma 2.7. \square

By clause (iv) of definition 2.3, $I \subseteq \log(\mathbf{M}_0[P])$. Let $\mathbf{M}[P]$ be the initial segment of $\mathbf{M}_0[P]$ generated by $\exp(I)$. If we identify I with $\log(\mathbf{M})$, we obtain:

Corollary 2.9 *$\log(\mathbf{M}) = \log(\mathbf{M}[P])$. Thus, more generally, $\log^{(n)}(\mathbf{M}) = \log^{(n)}(\mathbf{M}[P])$ for all $n \geq 1$.*

We also have:

Corollary 2.10 *If $\varphi(\bar{x})$ is a Π_1 formula and $P \models \varphi(\bar{t})$, then $\mathbf{M}[P] \models \varphi(\bar{t})$*

We close this section with a theorem on evaluation models (i.e. models of the form $\mathbf{M}[P]$) which will play a key role later on.

Theorem 2.11 *Let \mathbf{M} be a countable model of $I\Delta_0 + \Omega_3 + B\Sigma_1$. Assume that \mathcal{F} is a set of standard L -formulae,*

$$\mathcal{F} = \{\theta_n(x_1, \dots, x_r) : n \in \omega\},$$

and is a subset of a set

$$\{\theta_l(x_1, \dots, x_r) : l \in \log^{(3+k)} \mathbf{M}\}$$

(for some $k \in \omega$) which is Δ_1 -definable in \mathbf{M} and satisfies

$$(\#) \forall i \exists p \in \mathcal{E}_i \forall l, l_1, \dots, l_r < \log^{(k)} i \ p \models \theta_l(\underline{l}_1, \dots, \underline{l}_r).$$

Then there exists an increasing and cofinal sequence $P = \langle p_n : n \in \omega \rangle$ of evaluations such that $P \models \varphi(\underline{l}_1, \dots, \underline{l}_r)$ for each $\varphi \in \mathcal{F}$, $\underline{l}_1, \dots, \underline{l}_r \in \log^{(3+k)}(\mathbf{M})$, and the model $\mathbf{M}[P]$ satisfies $I\Delta_0 + \Omega_3$.

In particular, for any n such that θ_n is Π_1 , $\mathbf{M}[P] \models \theta_n(l_1, \dots, l_r)$, for each $l_1, \dots, l_r \in \log^{(3+k)}(\mathbf{M})$, $n \in \omega$.

Proof. Let us introduce the following convention: every evaluation p of rank i appearing in this proof satisfies $p \models \theta_l(\underline{l}_1, \dots, \underline{l}_r)$ for all $l, l_1, \dots, l_r < \log^{(k)} i$.

Let $i_1 < i_2 < \dots$ be cofinal in $\log^{(3)}(\mathbf{M})$. We shall define a sequence $P = \langle p_n : n \in \omega \rangle$ such that $p_n \in \mathcal{E}_{i_n}$.

P is defined by induction as follows. Suppose that at a given step n we already have evaluations $p_1 \subseteq \dots \subseteq p_n$ such that $p_1 \in \mathcal{E}_{i_1}, \dots, p_n \in \mathcal{E}_{i_n}$ satisfying the inductive condition

$$(*) \forall i > i_n \exists p \in \mathcal{E}_i [p_n \subseteq p].$$

Note that at the initial step the validity of the inductive condition is ensured by the assumption of the theorem.

We claim that it follows by $B\Sigma_1$ that:

$$(**) \exists p_{n+1} \in \mathcal{E}_{i_{n+1}} [\forall i > i_{n+1} \exists p \in \mathcal{E}_i p_n \subseteq p_{n+1} \subseteq p].$$

Indeed, assume $(**)$ fails. Then for any $\tilde{p} \in \mathcal{E}_{i_{n+1}}$ extending p_n there exists $i(\tilde{p}) > i_{n+1}$ for which there is no evaluation $p \in \mathcal{E}_i$ extending \tilde{p} . Now, all \tilde{p} 's are bounded by $\exp(\pi(K(i_{n+1})))$. Thus, we may use $B\Sigma_1$ to find a common bound i for all the $i(\tilde{p})$'s. It follows that there is no $p \in \mathcal{E}_i$ extending any of the $i(\tilde{p})$'s. On the other hand, by $(*)$ there is some $p \in \mathcal{E}_i$ extending p_n . But $(p \upharpoonright \mathcal{A}_{i_{n+1}}) \in \mathcal{E}_{i_{n+1}}$, and $p_n \subseteq (p \upharpoonright \mathcal{A}_{i_{n+1}}) \subseteq p$, a contradiction. Hence, $(**)$ must hold and the claim is proved.

Finally, the evaluation p_{n+1} given by $(**)$ satisfies the inductive condition at stage $n + 1$.

Now let $P = \langle p_n : n \in \omega \rangle$. Obviously P is increasing and cofinal.

Since all the axioms of $I\Delta_0^*$ are Π_1 we infer from corollary 2.10 that

$$\mathbf{M}[P] \models I\Delta_0^*.$$

On the other hand, the set $\{\exp^{(3)}i : i \in \log^{(3)} \mathbf{M}\}$ is cofinal in both \mathbf{M} and $\mathbf{M}[P]$ (cf. corollary 2.9). Since $\mathbf{M} \models \Omega_3$, we infer in view of corollary 2.9 that $\mathbf{M}[P] \models \Omega_3$. Consequently, $\mathbf{M}[P] \models I\Delta_0 + \Omega_3$ since obviously $I\Delta_0^* + \Omega_3$ implies $I\Delta_0 + \Omega_3$. This completes the proof of the theorem. \square

Remark 2.12 *To keep the enunciation of the above theorem reasonably concise, we have formulated its assumptions in a relatively simple way. It is clear, however, that appropriate variants of the theorem would also be true if the assumptions were modified in one or more of the following ways:*

- in $(\#)$, $\forall i \exists p \in \mathcal{E}_i(\dots)$ could be replaced by $\forall^\infty i \exists p \in \mathcal{E}_i(\dots)$;

- also in $(\#)$, $\forall l, l_1, \dots, l_r < \log^{(k)}i$ could be replaced by $\forall l, l_1, \dots, l_r < (\log^{(k)}i)/r$ (for any standard r), as long as $\log^{(3+k)}$ is closed under addition;
- \mathcal{F} could be extended by adding finitely many formulae of the form $\varphi(\bar{t})$ evaluated to “True” by almost all of the p 's given by $(\#)$.

In the sequel, we will sometimes speak of using “theorem 2.11” when some such variant is actually meant.

3 The principle τ

The present section introduces a consistent sentence τ which is a kind of reflection principle (mentioned in the title). We begin by formulating some preservation properties of evaluations.

For a Σ_1 sentence Φ of the form $\exists x\phi^x$ let $\Gamma_\Phi(p, i)$ be the formula

$$\begin{aligned} \forall j \leq i (\exists x \leq \text{exp}^{(3)}j \phi^x \Rightarrow \text{True}(p, i, \exists x \leq s_j \phi^x)) \\ \wedge (\forall x \leq \text{exp}^{(3)}i \neg \phi^x \Rightarrow \text{True}(p, i, \forall x \neg \phi^x), \end{aligned}$$

and, for a fixed sufficiently large m which depends on some further constructions but could be specified in advance, let $\Gamma_m(p, i)$ be the formula

$$\begin{aligned} \forall \psi < i, \psi \in \exists_m \forall j \leq i \forall \underline{a}_1, \dots, \underline{a}_r \in \Lambda_i \\ (\text{Sat}_m(\psi^{\text{exp}^{(3)}j}(a_1, \dots, a_r)) \Rightarrow \text{True}(p, i, \psi^{s_j}(\underline{a}_1, \dots, \underline{a}_r))). \end{aligned}$$

Intuitively, $\Gamma_\Phi(p, i)$ says “ p preserves the size of a witness for $\Phi = \exists x\phi^x$, and disallows witnesses of size greater than $\text{exp}^{(3)}i$ ”, while $\Gamma_m(p, i)$ says “ p preserves the restrictions $\psi^{\text{exp}^{(3)}j}$, for $j \leq i$, of all \exists_m sentences smaller than i ”.

Arguments similar to those in the previous section show that both Γ_Φ and Γ_m are Δ_0 with a parameter (and hence Δ_1), as they make no reference to objects greater than $\omega_3(\text{exp}^{(3)}i)$.

We will now define some (possibly non-standard) sentences τ_{j, j_1} for $j, j_1 \in \log^{(4)}$. The definition is by induction on j_1 . Let $\tau_{j, 0}$ be:

$$\left(\exists p \in \mathcal{E}_{\text{exp } 0} \{ \Gamma_m(p, \text{exp } 0) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, \text{exp } 0) \} \right)^{\text{exp}^{(4)}2 \cdot 0},$$

and let τ_{j, j_1+1} be:

$$\left(\exists p \in \mathcal{E}_{\exp(j_1+1)} \{ \Gamma_m(p, \exp(j_1+1)) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, \exp(j_1+1)) \right. \\ \left. \wedge \bigwedge_{l, l_1 < (j_1+1)/2} \text{True}(p, \exp(j_1+1), \underline{\tau_{l, l_1}}) \} \right)^{\exp^{(4)} 2(j_1+1)}.$$

If the definition of τ_{j, j_1} is to make sense, an evaluation of rank $\exp j_1$ should be able to decide the truth value of τ_{l, l_1} for $l, l_1 < j_1/2$. To check that this is so, let $\varphi_{j, j_1}(z, \bar{x})$ stand for

$$\left(\exists y \in \mathcal{E}_{\exp z} \{ \Gamma_m(y, \exp z) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(y, \exp z) \right. \\ \left. \wedge \bigwedge_{l, l_1 < (j_1/2)} \text{True}(y, \exp(z), x_{l, l_1}) \} \right)^{\exp^{(4)}(2z)},$$

and \bar{t}_{j_1} stand for $\langle \tau_{l, l_1} : l, l_1 < (j_1/2) \rangle$.

Observe that for τ_{j, j_1} is $\varphi_{j, j_1}(j_1, \bar{t}_{j_1})$. Therefore, it is enough to check that for any j_1 , $\varphi_{j_1/2, j_1/2}$ is smaller than $\exp(\exp(j_1) - 1)$ and that $\bar{t}_{j_1/2} \in \Lambda_{\exp(j_1) - 1}$.

To see the former, note that given any j_1 a code for φ_{j_1, j_1} is about j_1^2 , which is smaller than $\omega_1(\exp j_1)$ (a precise bound on φ_{j_1, j_1} depends on the details of how we code the syntax, esp. the variables, but the main ingredient of φ_{j_1, j_1} is a $(j_1^2/4)$ -long conjunction of formulae whose codes will not greatly exceed the code for the $(j_1^2/4)$ -th variable, which in turn may be around j_1^2). So for us it suffices if $\omega_1(\exp(j_1/2))$ is smaller than $\exp(\exp(j_1) - 1)$, which is clearly always the case.

To see the latter, we only need to check that for all j_1 , $\tau_{j_1/4, j_1/4}$ is smaller than $K(\exp(j_1) - 1)$. But for any j_1 , the size of τ_{j_1, j_1} can be bounded by roughly $\omega_1(\exp j_1)$ (the code for φ_{j_1, j_1}) times the code for the $(j_1^2/4)$ -long sequence of the τ_{l, l_1} 's (for $l, l_1 < j_1/2$). This sequence will have a code smaller than $(3 \cdot \tau_{j_1/2, j_1/2})^{j_1^2/4}$. Using the fact that $K(i+1) > cK(i)^i$ for some large standard c , it is easy to verify that $K(\exp(j_1) - 1)$ is more than τ_{j_1, j_1} (not to mention $\tau_{j_1/4, j_1/4}$).

In addition to the τ_{j, j_1} 's we also define, for any $j \in \log^{(4)}$, a formula $\tau_j(j_1)$ with j_1 as a free variable. $\tau_j(j_1)$ is:

$$\left(\exists p \in \mathcal{E}_{\exp j_1} \{ \Gamma_m(p, \exp j_1) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, \exp j_1) \right. \\ \left. \wedge \forall l, l_1 < (j_1/2) \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(p, \exp j_1, x)) \} \right)^{\exp^{(4)}(2j_1)},$$

where $x = \tau_{l, l_1}$ is an abbreviation for the inductive definition of τ_{l, l_1} with l and l_1 as parameters. Note that although the $\tau_j(\cdot)$'s are in general again non-standard, $\tau_n(\cdot)$ is a standard formula for any standard n .

Note also that $\exp^{(4)}(2j_1)$ is not less than $\omega_3(\exp^{(4)}j_1)$ — the greatest element we possibly need to access in order to check whether a given $p \in \mathcal{E}_{\exp j_1}$ satisfies all the conditions required in τ_{j,j_1} or $\tau_j(j_1)$ (as long as j is not unreasonably large in comparison to j_1). For this reason, the relativization to $\exp^{(4)}(2j_1)$, which is necessary for technical reasons, does not essentially influence the sense of τ_{j,j_1} or $\tau_j(j_1)$.

Let $\psi_j(z)$ stand for

$$\left(\exists y \in \mathcal{E}_{\exp z} \{ \Gamma_m(y, \exp z) \} \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(y, \exp z) \right) \wedge \forall l, l_1 < (z/2) \forall x (x = \tau_{l,l_1} \Rightarrow \text{True}(y, \exp z, x)) \Big\}^{\exp^{(4)}(2z)},$$

where $x = \tau_{l,l_1}$ is an abbreviation for the inductive definition of τ_{l,l_1} .

The following lemma establishes an important connection between τ_{j,j_1} and $\tau_j(j_1)$.

Lemma 3.1 *Let $j \leq j_1$ and let i be such that:*

- *The formulae $x = \tau_{l,l_1}$ (as a formula of x, l, l_1) and $\text{True}(y, \exp z, x)$ may be bounded by $\exp^{(3)}(i/2)$ for any choice of $l, l_1 < j_1/2$, $z < j_1$, $y < \exp(\pi(K(\exp j_1)))$, and $x < K(\exp j_1)^{\exp j_1}$;*
- *Λ_i is g.e. for $\langle \varphi_{j,j_1}, \underline{j_1} \hat{\sim} \bar{t}_{j_1} \rangle$ and for $\langle \psi_{j,j_1}, \underline{j_1} \rangle$.*

Let $p \in \mathcal{E}_i$ satisfy $\Gamma_m(p, i)$.

Then $\mathbf{M} \models \text{True}(p, i, \tau_{j,j_1})$ iff $\mathbf{M} \models \text{True}(p, i, \tau_j(\underline{j_1}))$

Remark 3.2 *Any $i \geq \exp(2j_1)$ satisfies the conditions of the lemma.*

Proof. We prove the left-to-right direction as the other direction is very similar.

Assume $\mathbf{M} \models \text{True}(p, i, \tau_{j,j_1})$.

As already noted, τ_{j,j_1} is $\varphi_{j,j_1}(\underline{j_1}, \bar{t}_{j_1})$. So, by the definition of τ_{j,j_1} and the meaning of the formula True , it follows that for all $l, l_1 < j_1/2$,

$$p \models \text{True}(s^{\varphi_{j,j_1}}(\underline{j_1} \hat{\sim} \bar{t}_{j_1}), \exp \underline{j_1}, \tau_{l,l_1}).$$

We may assume that m was chosen large enough so that the formula $x = \tau_{l,l_1}$ is \exists_m . Then, by our assumptions on the size of i , we may use the fact that p satisfies Γ_m to get $p \models (\tau_{l,l_1} = \tau_{\underline{l}, \underline{l}_1})$ for all $l, l_1 < j_1/2$. Thus, for every t such that $p \models (t = \tau_{l,l_1})$, we also have $p \models (t = \tau_{\underline{l}, \underline{l}_1})$. By definition 2.1, this means that

$$p \models \forall x (x = \tau_{\underline{l}, \underline{l}_1} \Rightarrow \text{True}(s^{\varphi_{j,j_1}}(\underline{j_1} \hat{\sim} \bar{t}_{j_1}), \exp \underline{j_1}, x))$$

for any choice of $l, l_1 < j_1/2$.

Similarly, for every t such that $p \models (t < \underline{j_1}/2)$, we also have $p \models (t = \underline{l})$ for some $l < j_1/2$. Therefore, we get

$$p \models \forall l, l_1 < \underline{j_1}/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(s^{\varphi_{j, j_1}}(\underline{j_1} \wedge \bar{t}_{j_1}), \text{exp } \underline{j_1}, x)).$$

Combining this with the original assumption that $\mathbf{M} \models \text{True}(p, i, \tau_{j, j_1})$, we obtain:

$$p \models \left(\{\Gamma_m(s^{\varphi_{j, j_1}}(\underline{j_1} \wedge \bar{t}_{j_1}), \text{exp } \underline{j_1}) \wedge \bigwedge_{\Phi \leq j} \Gamma_{\Phi}(s^{\varphi_{j, j_1}}(\underline{j_1} \wedge \bar{t}_{j_1}), \text{exp } \underline{j_1}) \wedge \forall l, l_1 < \underline{j_1}/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(s^{\varphi_{j, j_1}}(\underline{j_1} \wedge \bar{t}_j), \text{exp}(\underline{j_1}), x))\} \right)^{\text{exp}^{(4)}(2\underline{j_1})}.$$

To prove $\mathbf{M} \models \text{True}(p, i, \tau_j(\underline{j_1}))$, we only need to check that p also evaluates the above formula to “True” if we substitute the appropriate Skolem term for $s^{\varphi_{j, j_1}}(\underline{j_1} \wedge \bar{t}_{j_1})$. If that was not the case, we would have neither $p \models \tau_j(\underline{j_1})$ nor $p \models \neg \tau_j(\underline{j_1})$ (since we have a witness for the initial existential quantifier in $\tau_j(\underline{j_1})$). But $p \models (s_{\text{exp } 2\underline{j_1}} = \text{exp}^{(4)}(2\underline{j_1}))$, so p treats $\tau_j(\underline{j_1})$ as a formula relativized to $s_{\text{exp } 2\underline{j_1}}$. Now, $p \in \mathcal{E}_i$, and thus it follows from part (i) of definition 2.3 that at least one of $p \models \tau_j(\underline{j_1})$ and $p \models \neg \tau_j(\underline{j_1})$ must hold. \square

Corollary 3.3 *Let $j \leq j_1$. Then*

$$\mathbf{M} \models \forall i (\Lambda_i \text{ g.e. for } \langle \varphi_{j, j_1}, \underline{j_1} \wedge \bar{t}_{j_1} \rangle \Rightarrow \exists p \in \mathcal{E}_i \text{ True}(p, i, \tau_{j, j_1}))$$

iff

$$\mathbf{M} \models \forall i (\Lambda_i \text{ g.e. for } \langle \psi_j, \underline{j_1} \rangle \Rightarrow \exists p \in \mathcal{E}_i \text{ True}(p, i, \tau_j(\underline{j_1})).$$

Proof. Follows from the lemma via propositions 2.5 and 2.6. \square

We now let τ be $\forall j \forall j_1 \text{ Sat}(\tau_j(j_1))$.

In view of lemma 3.1, the sentence τ can be treated as a form of reflection principle (an observation due to A. Blass). Indeed, a “ Π_1 reflection principle” is usually understood to be a formalized version of the principle

$$(*) \psi \text{ is provable} \Rightarrow \psi \text{ is true,}$$

for $\psi \in \Pi_1$, in other words,

$$(**) \phi \text{ is true} \Rightarrow \phi \text{ is consistent,}$$

for $\phi \in \Sigma_1$. Now, the existence of evaluations which satisfy ϕ is a kind of consistency of ϕ . So, in any model in which *Sat* is well-behaved as a truth definition, τ says:

$$\phi \text{ is true} \Rightarrow \phi \text{ plus a restricted fragment of } \tau \text{ is consistent,}$$

for $\phi \in \Sigma_1$. Thus, τ expresses (**) and additionally has a limited “self-reproducing” property.

As remarked above, τ is a consistent sentence. Even more:

Theorem 3.4 *The theory $I\Delta_0 + exp$ proves τ .*

Proof. Let us work in a model of $I\Delta_0 + exp$. We prove $\forall j \leq j_1 Sat(\tau_j(j_1))$ by induction on j_1 .

Assume $\forall j \leq j_1 Sat(\tau_j(j_1))$. We want to show $\forall j \leq j_1 + 1 Sat(\tau_j(j_1))$. Thus, given any $j \leq j_1 + 1$, we need

$$\left(\exists p \in \mathcal{E}_{exp(j_1+1)} \{ \Gamma_m(p, exp(j_1 + 1)) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp(j_1 + 1)) \} \right) \wedge \forall l, l_1 < ((j_1 + 1)/2) \forall x (x = \tau_{l,l_1} \Rightarrow True(p, exp(j_1 + 1), x)) \Big)^{exp^{(4)}(2(j_1+1))}.$$

We will find an evaluation p of rank $j_1 + 1$ such that

$$(*) \Gamma_m(p, exp(j_1 + 1)) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp(j_1 + 1)) \\ \wedge \forall l, l_1 < ((j_1 + 1)/2) (True(p, exp(j_1 + 1), \tau_l(l_1))).$$

The fact that p is as required in $\tau_j(j_1 + 1)$ will then follow from lemma 3.1, since $exp(j_1 + 1)$ is a large enough rank for the lemma to ensure the equivalence of $p \models \tau_{l,l_1}$ and $p \models \tau_l(l_1)$ for $l \leq l_1 < (j_1 + 1)/2$ (see remark 3.2).

The way to obtain p is by constructing a *Skolem hull* on $\Lambda_{exp(j_1+1)}$. A Skolem hull on a given Λ is a sequence $H = \langle h_t : t \in \Lambda \rangle$ of elements of M , where the element h_t is thought of as an interpretation of the term t . One may define the satisfaction relation $H \models \varphi(\bar{t})$ in much the same way as $p \models \varphi(\bar{t})$, i.e. by postulating

- $H \models \varphi(\bar{t})$ iff $\mathbf{M} \models \varphi(\bar{h}_t)$

for φ open, and then proceeding as in definition 2.1, so that e.g.

- $H \models \exists y \varphi'(\bar{t}, y)$ iff Λ_i is g.e. for $\langle \exists y \varphi'(\bar{x}, y), \bar{t} \rangle$ and $H \models \varphi'(\bar{h}_t, h_{s\varphi(\bar{t})})$.

It is clear that any hull H on Λ determines an evaluation p_H such that $p_H \models \varphi$ iff $H \models \varphi$. If Λ is Λ_i , and H is a hull of rank i (defined analogously to “evaluation of rank i ”, cf. def. 2.3), then $p_H \in \mathcal{E}_i$.

The hull we want to construct on $\Lambda_{exp(j_1+1)}$ is to satisfy:

- (i) for any $h \in H$, $h \leq exp^{(4)}(j_1 + 1)$,
- (ii) for any $a \leq K(exp\ j_1)$, $h_{\underline{a}} = a$, and for any $j' \leq exp(j_1 + 1)$, $h_{s_{j'}} = exp^{(3)}\ j'$;
- (iii) for any $h_t \in H$ and for any formula of the form ψ^t smaller than $exp(exp(j_1 + 1) - 1)$,

$$(H \models \psi^t) \text{ iff } Sat(\psi^{h_t});$$

- (iv) for any $h_t \in H$ and for any formula of the form $\exists x \leq t \psi^x$ smaller than $exp(exp(j_1 + 1) - 1)$,

$$(H \models \exists x \leq t \psi^x) \text{ iff } Sat(\exists x \leq h_t \psi^x);$$

- (v) for every $\varphi < exp(j_1 + 1)$,

$$\text{if } Sat(\forall x \leq exp^{(4)}(j_1 + 1) \neg \varphi^x) \text{ then } H \models \forall x \neg \varphi^x.$$

The actual construction of H is based on a straightforward induction. Given that $\Lambda_{exp(j_1+1)}$ is ordered as $\langle t_1, \dots, t_k \rangle$, we assign interpretations to the t_r 's by induction on $r \leq k$. If t_{r+1} is \underline{a} , then $h_{t_{r+1}} = a$, if it is $s_{j'}$, then $h_{t_{r+1}} = exp^{(3)}\ j'$. If $t_{r+1} = s^\varphi(\bar{t})$, then we define $h_{t_{r+1}}$ to be the smallest witness below $exp^{(4)}(j_1 + 1)$ for $\varphi(\bar{h}_t)$ whenever it exists, and arbitrary (but smaller than $exp^{(4)}(j_1 + 1)$) if there is no such witness.

We take p to be p_H . It is again straightforward to check that $p \in \mathcal{E}_{exp(j_1+1)}$ and that p has all the properties required in (*). In particular, $\forall l, l_1 < ((j_1 + 1)/2)(True(p, exp(j_1 + 1), \tau_l(l_1)))$ follows by the construction of p from the inductive assumption $\forall j \leq j_1 Sat(\tau_j(j_1))$. \square

4 The main theorem

To define the theory T mentioned in the introduction, we will use “finite fragments” of the principle τ . Namely, let τ_n denote

$$\forall i \exists p \in \mathcal{E}_i \left(\{ \Gamma_m(p, i) \} \wedge \bigwedge_{\Phi \leq n} \Gamma_\Phi(p, i) \right) \\ \wedge \forall l, l_1 < (\log i)/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(p, i, x)) \} \Big)^{\text{exp}^{(3)}(2 \log i)}.$$

Thus, using the notation of the previous section, τ_n is (approximately) $\forall j_1 \tau_n(j_1)$. In particular, for $n \in \omega$, τ_n is a standard Π_1 sentence.

Lemma 4.1 *Any \exists_m sentence χ consistent with $I\Delta_0 + \Omega_3$ is consistent with all the τ_n 's.*

Proof. Let $\mathbf{M} \models I\Delta_0 + \Omega_3 + \tau_0 + \chi$. W.l.o.g. we may assume that $\mathbf{M} \models B\Sigma_1$, since (cf. e.g. [P]) \mathbf{M} has a 1-elementary extension \mathbf{M}' of the same height satisfying $B\Sigma_1$.

Let \mathcal{F} be the set

$$\{ \tau_n(x) : n \in \omega \} \cup \{ \chi \}.$$

This is a subset of

$$\{ \tau_l(x) : l \in \log^{(4)} \} \cup \{ \chi \}.$$

Using the fact that $\mathbf{M} \models \tau_0$, we infer from (a minor variant of) corollary 3.3 that:

$$\forall i \exists p \in \mathcal{E}_i \forall l, j_1 < (\log i)/2 \ p \models \tau_l(j_1),$$

since Λ_i is g.e. for $\langle \psi_l, j_1 \rangle$ whenever $l, j_1 < (\log i)/2$. Also, almost all the p 's evaluate χ to “True”, because all p 's given by τ_0 satisfy Γ_m , and there is an i such that a witness for χ exists below $\text{exp}^{(3)}i$. Since $\log^{(4)}$ is closed under addition, we may apply theorem 2.11 and obtain an increasing and cofinal sequence P_0 of evaluations such that: $\mathbf{M}[P_0] \models I\Delta_0 + \Omega_3$, $P_0 \models \chi$, and $P_0 \models \tau_n(\underline{l}_1)$ for any $n \in \omega, j_1 \in \log^{(4)}(\mathbf{M})$. Since the $\tau_n(\cdot)$'s are Π_1 , it holds that $\mathbf{M}[P_0] \models \tau_n(j_1)$ for any n and j_1 .

But this means that for any n, j_1 ,

$$\mathbf{M}[P_0] \models \exists p \in \mathcal{E}_{\text{exp } j_1} \{ \Gamma_m(p, \text{exp } j_1) \} \wedge \bigwedge_{\Phi \leq n} \Gamma_\Phi(p, \text{exp } j_1) \\ \wedge \forall l, l_1 < j_1/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(p, \text{exp } j_1, x)) \}.$$

We may obtain suitable p 's in \mathcal{E}_i for i not of the form $\text{exp } j_1$ by restricting the evaluations we have in $\mathcal{E}_{\text{exp } j_1}$ (use propositions 2.6 and 2.5 to ensure that these restrictions are indeed evaluations of the appropriate ranks and that they have the desired properties). Hence, for any n it holds in $\mathbf{M}[P_0]$ that

$$\forall i \exists p \in \mathcal{E}_i \{ \Gamma_m(p, i) \wedge \bigwedge_{\Phi \leq n} \Gamma_\Phi(p, i) \\ \wedge \forall l, l_1 < (\log i)/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(p, i, x)) \},$$

which implies that $\mathbf{M} \models \tau_n$ for all n . It remains to point out that $\mathbf{M}[P_0]$ also satisfies χ , since $P_0 \models \chi$ and we may in this context treat χ as a Δ_0 formula by considering its relativization to the smallest witness for χ . \square

Observe that the construction described in the proof of the lemma would have also worked if we started in a model of some higher τ_N , and not just τ_0 . In that case, we would be able to replace the set \mathcal{F} by a set which additionally contains $\neg\Phi$, for all $\Phi \leq N$ false in \mathbf{M} , and suitable true relativizations of Φ , for $\Phi \leq N$ true in \mathbf{M} . Theorem 2.11 would then give us a sequence P_N corresponding to that set. $\mathbf{M}[P_N]$ would satisfy all the τ_n 's and χ just as $\mathbf{M}[P_0]$ did, but it would additionally satisfy exactly those $\Phi \leq N$ which are true in \mathbf{M} .

Given any Π_1 consequence θ of $I\Delta_0 + \text{exp}$, there exists a purely existential sentence φ_θ consistent with $I\Delta_0 + \Omega_3 + B\Sigma_1 + \theta$, but inconsistent with $I\Delta_0 + \text{exp}$ (see [HP]). If we apply this result to $\theta := \tau_0$, we obtain an \exists_m sentence χ consistent with $I\Delta_0 + \Omega_3 + B\Sigma_1 + \{\tau_n : n \in \omega\}$, but inconsistent with $I\Delta_0 + \text{exp}$. Let us fix such a χ and define the theory T_0 by

$$T_0 := I\Delta_0 + \Omega_3 + B\Sigma_1 + \{\tau_n : n \in \omega\} + \chi.$$

In the sequel ϵ will denote a binary sequence, $\epsilon = \langle \epsilon_0, \dots, \epsilon_{lh(\epsilon)-1} \rangle$.

Given a fixed Δ_0 enumeration $\langle \varphi_n : n \in \omega \rangle$ of all Σ_1 sentences, let us introduce the sentences $\sigma_{n, \epsilon}$:

$$\sigma_{n, \epsilon} := (n = lh(\epsilon)) \\ \wedge \forall i \exists p \in \mathcal{E}_i \left(\Gamma_m(p, i) \wedge \forall l, l_1 < (\log i)/2 \forall x (x = \tau_{l, l_1} \Rightarrow \text{True}(p, i, x)) \right. \\ \left. \wedge \forall k < n (\epsilon_k = 0 \Rightarrow p \models \neg\varphi_k) \wedge p \models \neg\varphi_n \right).$$

Lemma 4.2 *Assuming m is sufficiently large, for any n and ϵ , $\sigma_{n, \epsilon}$ is (equivalent in $I\Delta_0 + \Omega_3$ to) a \forall_m sentence.*

Proof. The only difficulty is to show that $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p, i)$ can be equivalently written as a \forall_m formula. $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p, i)$ is:

$$\forall i \exists p \in \mathcal{E}_i \forall j \leq i \forall \psi < i, \psi \in \exists_m \forall \underline{a}_1, \dots, \underline{a}_r \in \Lambda_i \\ (\text{Sat}_m(\psi^{\text{exp}^{(3)j}}(\underline{a}_1, \dots, \underline{a}_r)) \Rightarrow \text{True}(p, i, \psi^{s_j}(\underline{a}_1, \dots, \underline{a}_r))),$$

so the main problem is that an implication with the \exists_m precedent Sat_m occurs in the scope of the bounded existential quantifier $\exists p$. Actually, we could clearly replace Sat_m by its E_m analogue Sat_{E_m} (see the Preliminaries), but this still does not solve our problem.

By the definition of \exists_m , an \exists_m formula $\psi(a_1, \dots, a_r)$ is $\exists x \psi'(a_1, \dots, a_r, x)$ for $\psi' \in U_{m-1}$. Thus $\psi^{exp^{(3)}j}(a_1, \dots, a_r)$ holds iff there is a witness $x \leq exp^{(3)}j$ such that $\psi'(a_1, \dots, a_r, x)$.

Consider the sentence ξ_m

$$\begin{aligned} \forall i \forall \langle x_{j,\psi,\bar{a}} : j \leq i, \psi < i \text{ in } \exists_m, \bar{a} \in \Lambda_i \text{ of appropriate length} \rangle \\ \text{such that each } x_{j,\psi,\bar{a}} \text{ is } \leq exp^{(3)}j \\ \exists p \in \mathcal{E}_i \forall j \leq i \forall \psi < i, \psi \in \exists_m \forall \underline{a}_1, \dots, \underline{a}_r \in \Lambda_i \\ (Sat_{U_{m-1}}(\psi'(a_1, \dots, a_r, x_{j,\psi,\bar{a}})) \Rightarrow True(p, i, \psi^{sj}(\underline{a}_1, \dots, \underline{a}_r))), \end{aligned}$$

where $\langle x_{j,\psi,\bar{a}} \rangle$ should be thought of as a sequence of “potential witnesses” smaller than $exp^{(3)}j$ for $\psi(\bar{a})$, and $Sat_{U_{m-1}}$ is dual to $Sat_{E_{m-1}}$.

ξ_m is easily seen to be equivalent to a \forall_m sentence. Indeed: $Sat_{U_{m-1}}$ is U_{m-1} with an appropriately large parameter, so it is E_{m-1} in the precedent of an implication; the universal quantifiers for j, ψ , and \bar{a} may be treated as sharply bounded (in particular, \bar{a} is an at most $\log \psi$ -long sequence of objects smaller than $3K(i-1)$, so it is $\leq (3K(i-1))^{log i} \in \log$); and the initial unbounded universal quantifiers may obviously be merged into one.

Moreover, ξ_m is also equivalent to $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p, i)$. The right-to-left direction is trivial: for any i the $p \in \mathcal{E}_i$ satisfying $\Gamma(p, i)$ will be good for all sequences of witnesses. For the other direction, given a fixed i , there is always an “optimal” sequence of witnesses $\langle x_{j,\psi,\bar{a}} \rangle$, i.e. one such that if there is any $x \leq exp^{(3)}j$ for which $\psi'(a_1, \dots, a_r, x)$ holds, then $x_{j,\psi,\bar{a}}$ is such an x . Now, ξ_m gives us a $p \in \mathcal{E}_i$ which works for this “optimal” sequence. One easily checks that p must satisfy $\Gamma_m(p, i)$. \square

We also introduce the sentences Ψ_n , for $n \in \omega$:

$$\Psi_n := \bigvee_{\epsilon \in \{0,1\}^{n+1}} \left(\bigwedge_{r \leq n, \epsilon_r=0} \neg \phi_r \wedge \bigwedge_{r \leq n, \epsilon_r=1} (\phi_r \wedge \neg \sigma_{r,(\epsilon \upharpoonright r)}) \right).$$

Finally, we define our theory T by:

$$T := T_0 + \{\Psi_n : n \in \omega\}.$$

Obviously, T is a recursive theory. We will now prove our main theorem, which shows, among others, that T axiomatizes a certain class of models of T_0 in which the set of elements definable by Δ_0 formulae of restricted complexity is cofinal in the set of all Δ_0 definable elements:

Theorem 4.3 (a) T is consistent.

(b) For any (not necessarily countable) $\mathbf{M} \models T$, \mathbf{M}^{Σ_1} is recursively reducible to \mathbf{M}^{\exists_m} .

(c) In any (not necessarily countable) $\mathbf{M} \models T$, $E_{m+1} \wedge U_{m+1}(\mathbf{M})$ is cofinal in $\Delta_0(\mathbf{M})$.

Proof. We first prove (a). The proof is an inductive construction based on repeated application of theorem 2.11.

In the initial step, take an arbitrary countable model \mathbf{M} of T_0 . Consider $\sigma_0 = \sigma_{0,\emptyset}$ and put

$$\mathbf{M}'_0 = \begin{cases} \mathbf{M}[P_0], & \text{if } \mathbf{M} \models \sigma_0; \\ \mathbf{M}, & \text{otherwise,} \end{cases}$$

where P_0 is as in theorem 2.11 for $k = 1$ and \mathcal{F}_0 defined as

$$\{\tau_n(x) : n \in \omega\} \cup \{\neg\phi_0\} \cup \{\chi\},$$

where χ should be treated as a relativization of the original χ to some $\text{exp}^{(3)}(\cdot)$ true in \mathbf{M} (note that the existence of P_0 follows from the fact that $\mathbf{M} \models \sigma_0$ via theorem 2.11 and lemma 3.1). Also let $\epsilon_0 = 0$ in the former and $\epsilon_0 = 1$ in the latter case.

\mathbf{M}'_0 clearly satisfies $I\Delta_0 + \Omega_3$ and $\{\tau_n : n \in \omega\} + \chi$ (either by our assumptions on \mathbf{M} or by the choice of \mathcal{F}_0). Furthermore, by lemma 2.10, $\mathbf{M} \models \sigma_0$ implies $\mathbf{M}[P_0] \models \neg\phi_0$. On the other hand, in all models of $\{\tau_n : n \in \omega\}$, $\neg\phi_0$ implies σ_0 , because of the validity of a suitable τ_N . Hence either

$$\epsilon_0 = 0 \text{ and } \mathbf{M}'_0 \models \neg\phi_0$$

or

$$\epsilon_0 = 1 \text{ and } \mathbf{M}'_0 \models \phi_0 \wedge \neg\sigma_0.$$

In other words, $\mathbf{M}'_0 \models \Psi_0$. Thus, we always have $\mathbf{M}'_0 \models (T_0 \setminus B\Sigma_1) + \Psi_0$. By passing to a 1-elementary extension of the same height if necessary (see the beginning of the proof of lemma 4.1), we may obtain a model \mathbf{M}_0 satisfying $T_0 + \Psi_0$.

Proceeding inductively, assume that we are given a model \mathbf{M}_n satisfying $T_0 + \Psi_n$. Similarly as in the initial step, consider $\sigma_{n+1} = \sigma_{n+1,\epsilon}$ for the sequence $\epsilon = \langle \epsilon_0, \dots, \epsilon_n \rangle$ determined uniquely in view of $\mathbf{M}_n \models \Psi_n$. Put

$$\mathbf{M}'_{n+1} = \begin{cases} \mathbf{M}_n[P_{n+1}], & \text{if } \mathbf{M}_n \models \sigma_{n+1}; \\ \mathbf{M}_n, & \text{otherwise,} \end{cases}$$

where P_{n+1} is as in theorem 2.11 for $k = 1$ and \mathcal{F}_{n+1} defined as

$$\begin{aligned} & \{\tau_n(x) : n \in \omega\} \cup \{\neg\phi_r : r \leq n, \epsilon_r = 0\} \\ & \cup \{\neg\sigma_{r,(\epsilon \upharpoonright r)} : r \leq n, \epsilon_r = 1\} \cup \{\neg\phi_{n+1}\} \cup \{\chi\}, \end{aligned}$$

where χ and the $\neg\sigma$'s should again be treated as true relativizations to some $exp^{(3)}(\cdot)$ (note as previously that the existence of P_{n+1} follows from $\mathbf{M}_n \models \sigma_{n+1}$ via theorem 2.11 and lemma 3.1). Define $\epsilon_{n+1} = 0$ in the former and $\epsilon_{n+1} = 1$ in the latter case.

Again, it is clear that \mathbf{M}'_{n+1} satisfies $I\Delta_0 + \Omega_3$ and $\{\tau_n : n \in \omega\} + \chi$. As in the initial step, we get either

$$\epsilon_{n+1} = 0 \text{ and } \mathbf{M}'_{n+1} \models \neg\phi_{n+1}$$

or

$$\epsilon_{n+1} = 1 \text{ and } \mathbf{M}'_{n+1} \models \phi_{n+1} \wedge \neg\sigma_{n+1}.$$

We now check that $\mathbf{M}'_{n+1} \models \Psi_{n+1}$. This is obvious if $\epsilon_{n+1} = 1$, so assume $\epsilon_{n+1} = 0$ and thus $\mathbf{M}'_{n+1} = \mathbf{M}_n[P_{n+1}]$. For a given $r \leq n$, if $\epsilon_r = 0$, then $\mathbf{M}'_{n+1} \models \neg\phi_r$ as required, since $P_{n+1} \models \neg\phi_k$. On the other hand, if $\epsilon_r = 1$, then $\mathbf{M}'_{n+1} \models \neg\sigma_{r,(\epsilon \upharpoonright r)}$, since P_{n+1} sets a suitable relativization of $\neg\sigma_{r,(\epsilon \upharpoonright r)}$ to "True". But this also means $\mathbf{M}'_{n+1} \models \phi_r$, as $\neg\phi_r$ would imply $\sigma_{r,(\epsilon \upharpoonright r)}$ in view of a suitable τ_N . Thus, in either case, $\mathbf{M}'_{n+1} \models \Psi_{n+1}$.

As before, we may pass to a 1-elementary extension if necessary to get a model \mathbf{M}_{n+1} satisfying $T_0 + \Psi_n$. Since Ψ_n clearly implies Ψ_k for $k < n$, this shows that every finite subtheory of T is consistent. By compactness, T itself is also consistent, which ends the proof of (a).

To prove (b), let \mathbf{M} be an arbitrary model of T . Let the infinite binary sequence ϵ be the unique extension of the sequences given by the Ψ_n 's. Then for each $n \in \omega$ we have

$$(*) \mathbf{M} \models \phi_n \equiv \neg\sigma_{n,(\epsilon \upharpoonright n)}.$$

For, just as in the proof of (a), $\neg\phi_n$ implies $\sigma_{n,(\epsilon \upharpoonright n)}$ since $\mathbf{M} \models \{\tau_n : n \in \omega\}$, while ϕ_n yields $\epsilon_n = 1$, whence we have $\neg\sigma_{n,(\epsilon \upharpoonright n)}$ because of Ψ_n .

From (*) we obtain a recursive reduction of Σ_1 truth about \mathbf{M} to \exists_m truth about \mathbf{M} . Indeed, knowing $(\epsilon \upharpoonright n)$ and knowing whether $\sigma_{n,(\epsilon \upharpoonright n)}$ is true we deduce whether ϕ_n is true, whence we deduce $(\epsilon \upharpoonright n + 1)$ and so on: we recover the Σ_1 truth from the \forall_m truth step by step.

For a proof of (c), suppose that $a \in \Delta_0(\mathbf{M})$. In other words, $\mathbf{M} \models \varphi(a)$, where $\varphi(x) \in \Delta_0$ and $\mathbf{M} \models \exists!x\varphi^x(x)$. Thus, $\exists x\varphi^x(x)$ is a Σ_1 sentence, say ϕ_n , true in \mathbf{M} . Let i be such that $exp^{(3)}i < a$, so that we have $\mathbf{M} \models \neg\exists x < exp^{(3)}i \varphi^x(x)$. Using an appropriate τ_N (recall that $\mathbf{M} \models \{\tau_n : n \in \omega\}$), we can find a $p \in \mathcal{E}_i$ such that in \mathbf{M} we have:

$$\begin{aligned} & \Gamma_m(p, i) \wedge \forall l, l_1 < (\log i)/2 \forall x (x = \tau_{l, l_1} \Rightarrow True(p, i, x)) \\ & \wedge \bigwedge_{r < n} (\epsilon_r = 0 \Rightarrow p \models \neg\phi_r) \wedge p \models \neg\phi_n, \end{aligned}$$

where ϵ is the sequence given by Ψ_n .

Hence

$$\mathbf{M} \models \bar{\sigma}_{n,\epsilon}(i),$$

where $\bar{\sigma}_{n,\epsilon}$ is obtained from $\sigma_{n,\epsilon}$ (the standard version, not necessarily the one discussed in lemma 4.2) by deleting the universal quantifier $\forall i$.

We have proved that for any i , $\neg\exists x < \exp^{(3)}i \varphi^x(x)$ implies $\bar{\sigma}_{n,\epsilon}(i)$. However, we have $\mathbf{M} \models \neg\sigma_{n,\epsilon}$ since $\mathbf{M} \models \phi_n$. It follows from $\neg\sigma_{n,\epsilon}$ that there exists a number i_0 such that $\mathbf{M} \models \neg\bar{\sigma}_{n,\epsilon}(i_0)$.

Let b_0 be $\exp^{(4)}(2 \cdot \log i_0)$ (thus, b_0 is large enough to be a bound for all the quantifiers in $\bar{\sigma}_{n,\epsilon}(i_0)$), and let $b > b_0$ be the smallest element of \mathbf{M} which is large enough to be a bound for all the quantifiers in $b_0 = \exp^{(4)}(2 \cdot \log i_0)$. Now, b is the smallest element satisfying the E_{m+1} formula

$$\exists i_0 < b \exists b_0 < b ((b_0 = \exp^{(4)}(2 \cdot \log i_0))^b \wedge \neg(\bar{\sigma}_{n,\epsilon}(i_0))^{b_0}),$$

so it is definable in \mathbf{M} by an $E_{m+1} \wedge U_{m+1}$ formula. Furthermore, $b > a$. This proves that (c) holds. \square

We conclude this paper with a remark on Σ_1 -definability of \mathbb{N} in models of $I\Delta_0 + \Omega_1$ — more precisely, on its relation to the question whether elements definable by Σ_1 formulae of some fixed complexity are cofinal in a given model.

Let $\mathbf{M} \models I\Delta_0 + \Omega_1$ and assume that the set $\exists_r(\mathbf{M})$ is cofinal in \mathbf{M} . We claim that if \mathbf{M}^{\exists_r} has a code $a \in \mathbf{M}$, then \mathbb{N} is Σ_1 definable in \mathbf{M} with a as a parameter. For, given an enumeration $\langle \varphi_n : n \in \mathbb{N} \rangle$ of \exists_r sentences, let $\psi(x)$ be the formula

$$\exists y \forall z < x (\varphi_z \in a \Rightarrow \text{Sat}_r^y(\varphi_z)).$$

Clearly, it follows from the cofinality of $\exists_r(\mathbf{M})$ that $\psi(x)$ defines \mathbb{N} in \mathbf{M} .

We have thus proved one half of the following proposition (the other follows easily by a standard argument):

Proposition 4.4 *Assume that \mathbb{N} is not Σ_1 definable (with parameters) in \mathbf{M} . Then for any r : $\exists_r(\mathbf{M})$ is cofinal in \mathbf{M} iff \exists_r truth is not codable in \mathbf{M} .*

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