Subregularity, Hypercontractivity and Reliability

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Notations

R_+ denotes the set of all positive real numbers and $\bar{R}_+ =: R_+ \cup \{0, \infty\}$.
Given $a, b, c \in \bar{R}_+$ we put $a \vee b =: \max\{a, b\}$, $a \wedge b =: \min\{a, b\}$ and for $a \le b$ we put $a \vee c \wedge b =: a \vee (c \wedge b) = (a \vee c) \wedge b$.
For a random variable ξ we define $F_\xi(t) =: P(\xi \le t)$, $T_\xi(t) = P(\xi \ge t)$,
$M(\xi) =: \{t : F_\xi(t), T\xi(t) \ge 1/2\}$.
$||\xi||_p =: (E|\xi|^p)^{\frac{1}{p}}$ for $p \ne 0$ and $||\xi||_0 =: \exp\{E \ln |\xi|\}$.
For each $\alpha > 0$ and it is $M(\alpha\xi) = \alpha M(\xi)$.
$||\xi||_p$ is a nondecreasing function of p and of $|\xi|$.

Subregularity and Hypercontractivity

Throughout this section ξ will be a fixed positive random variable. Here we will abbreviate F_ξ, T_ξ and $M(\xi)$ to F, T and M.

Definition 1. Let $a, b > 0$. We say that ξ has (a,b)-subregular distribution, with constants A,B (we will write then $\xi \in V\Lambda(a, A; b, B)$) if for each $m \in M$ and $0 < s < t$

$$F(s) \le A(\tfrac{s}{t})^a F(t) \text{ whenever } t \le m \text{ and} \tag{1.1}$$

$$T(t) \le B(\tfrac{s}{t})^b T(s) \text{ whenever } m \le s \tag{1.2}$$

Let us observe that for each $\alpha > 0$
(1.3) $\qquad \xi \in V\Lambda(a, A; b, B)$ if and only if $\alpha\xi \in V\Lambda(a, A; b, B)$
and
(1.4) $\qquad \xi \in V\Lambda(a, A; b, B)$ if and only if $\xi^{-1} \in V\Lambda(b, B; a, A)$.

Definition 2 Let $p < q$ and $\sigma > 0$. We say that ξ is (p,q)-hypercontractive with parameter σ, which will be denoted by $\xi \in HV\Lambda(p, q, \sigma)$, if for all
$0 \le s \le t \le \infty$ it holds

$$(E(s \vee \sigma\xi \wedge t)^q)^{\frac{1}{q}} \le (E(s \vee \xi \wedge t)^p)^{\frac{1}{p}}. \tag{1.5}$$

A simple consequence of the definition is the following fact
(1.6) $\qquad \xi \in HV\Lambda(p, q, \sigma)$ if and only if $\xi^{-1} \in HV\Lambda(-q, -p, \sigma)$.

Theorem 1. If $\xi \in V\Lambda(a, A; b, B)$ then for each $-a < p < 0 < q < b$

$\xi \in HV\Lambda(p, q, \sigma)$ where $w = (1 + \frac{qB}{2(b-q)})^{\frac{1}{q}}/(1 - \frac{pA}{2(a+p)})^{\frac{1}{p}}$ and $\sigma = w^{-1}\{(\frac{(1-w^{-q})(p+a)}{Aq})^{\frac{1}{a}} \wedge (\frac{(w^p-1)(b-q)}{Bp})^{\frac{1}{b}} \wedge (\frac{w^{p-q}}{AB})^{\frac{1}{a\wedge b}}\}$.

Proof. We check easily that for each nonnegatitive random variable η, $0 < s < t$, $r \neq 0$ it is

$$E(s \vee \eta \wedge t)^r =$$

$$s^r(1 + r\int_1^{t/s} u^{r-1}T_\eta(us)du) = t^r(1 - r\int_{s/t}^1 u^{r-1}F_\eta(ut)du).$$

Therefore to prove (1.3) it is enough to prove one of the inequalities

$$(1.7) \qquad (1 + q\int_1^{t/s} u^{q-1}T(\tfrac{us}{\sigma})du)^{1/q} \leq (1 + p\int_1^{t/s} u^{p-1}T(us)du)^{1/p},$$

$$(1.8) \qquad (1 - q\int_{s/t}^1 u^{q-1}F(\tfrac{ut}{\sigma})du)^{1/q} \leq (1 - p\int_{s/t}^1 u^{p-1}F(ut)du)^{1/p}.$$

Since the function $(1 + x)^{q/p}$ for $x > -1$ is convex it fulfills $(1 + x)^{q/p} \geq 1 + (q/p)x$. Hence the inequality (1.7) is implied by

$$(1.9) \qquad \int_1^{t/s} u^{q-1}T(\tfrac{us}{\sigma})du \leq \int_1^{t/s} u^{p-1}T(us)du$$

and the inequality (1.8) by

$$(1.10) \qquad \int_{s/t}^1 u^{p-1}F(ut)du \leq \int_{s/t}^1 u^{q-1}F(\tfrac{ut}{\sigma})du.$$

We will divide the proof into four cases. The first case is $t/s > w, s > m/w$. Since $\frac{us}{\sigma} \geq ws \geq m$ for $u \geq 1$ by (1.2) the left side of the inequality (1.9) is estmated by $\int_1^{t/s} B(w\sigma)^b u^{q-1-b}T(ws)du = \frac{B}{b-q}(\sigma w)^b T(ws)$ and the right side of (1.9) is estimated from below by $T(ws)\int_1^w u^{p-1}du = T(ws)\frac{w^p-1}{p}$. Since $\sigma w \leq (\frac{w^p-1}{p}\frac{b-q}{B})^{\frac{1}{b}}$ the inequality (1.9) holds true and so (1.3) does.
Let $z = [(1 - w^{-q})(\frac{a+p}{qA})]^{\frac{1}{a}} \wedge 1$.
The second case is $t \leq zm, t/s > w$. It is treated similary. Since $tu \leq t/z \leq m$ for $u \leq 1$ by (1.1) the left side of (1.10) is estimated from above by $A\int_0^1 u^{p+a-1}z^a F(\frac{t}{z})du = \frac{A}{p+a}z^a F(\frac{t}{z})$. The right side of (1.10) is estimated from below by $\int_{1/w}^1 u^{q-1}F(\frac{t}{\sigma w})du \geq \frac{1-w^{-q}}{q}F(t/z)$, because $\sigma w < z$. Thus the inequality (1.10) is fulfilled and so (1.3) does.
The third case is $t/s \leq w$. Then the left side of the inequality (1.9) is estimated from above by $\int_1^{t/s} u^{q-1}T(\frac{s}{\sigma})du = \frac{(\frac{t}{s})^q-1}{q}T(\frac{s}{\sigma})$ and the right side of

the inequality (1.9) is estimated from below by $\int_1^{t/s} u^{p-1}T(t)du = \frac{(\frac{t}{s})^p-1}{p}T(t)$. Thus (1.9) is fulfilled if $T(\frac{s}{\sigma})/T(t) \le \frac{q}{p}\frac{(\frac{t}{s})^p-1}{(\frac{t}{s})^q-1}$.
In a similiar way we show that (1.10) holds true if $F(t)/F(\frac{s}{\sigma}) \le \frac{p}{q}\frac{1-(\frac{s}{t})^q}{1-(\frac{s}{t})^p}$. Hence at least one of the inequalities (1.9), (1.10) is verified if

$$\frac{T(\frac{s}{\sigma})}{T(t)}\frac{F(t)}{F(\frac{s}{\sigma})} \le \frac{q}{p}\frac{(\frac{t}{s})^p-1}{(\frac{t}{s})^q-1}\frac{p}{q}\frac{1-(\frac{s}{t})^q}{1-(\frac{s}{t})^p} = (\frac{t}{s})^{p-q}. \tag{1.11}$$

Since $\sigma < w^{-1} < s/t$ we get $t < s/\sigma$. If $t \ge m$ then by (1.2) the left side of (1.11) is bounded from above by $\frac{T(\frac{s}{\sigma})}{T(t)} \le B(\sigma\frac{t}{s})^b$ which implies the inequality (1.11) because $\sigma^b \le B^{-1}w^{p-q-b}$. Similary if $s/\sigma < m$ then the left side is bounded from above by $\frac{F(t)}{F(\frac{s}{\sigma})} \le A(\sigma\frac{t}{s})^a$ which yields (1.11) because $\sigma^a \le A^{-1}w^{p-q-a}$. If $t \le m \le s/\sigma$ then the left side of (1.11) is estimated from above by

$$\frac{T(\frac{s}{\sigma})}{T(m)}\frac{F(t)}{F(m)} \le B(\frac{\sigma m}{s})^b A(\frac{t}{m})^a \le AB(\sigma\frac{t}{s})^{a\wedge b}.$$

Since $\sigma \le w^{-1}(\frac{w^{p-q}}{AB})^{\frac{1}{a\wedge b}}$ the inequality (1.11) holds in the third case and so (1.3) does.
It remains to consider the case $t \ge sw$, $sw < m$, $t \ge mz$. By (1.1)
$(E(s\vee\xi\wedge t)^p)^{\frac{1}{p}} \ge E(\xi\wedge(m\wedge t))^{\frac{1}{p}} = ((t\wedge m)^p - \int_0^{t\wedge m} pu^{p-1}F(u)du)^{\frac{1}{p}} \ge ((t\wedge m)^p - A(t\wedge m)^{-a}F(t\wedge m)\int_0^{t\wedge m} pu^{p+a-1}du)^{\frac{1}{p}} \ge (t\wedge m)(1-\frac{Ap}{2(p+a)})^{\frac{1}{p}}$
and by (1.2) we get $(E(s\vee\sigma\xi\wedge t)^q)^{\frac{1}{q}} \le (E((s\vee\sigma m)\vee\sigma\xi)^q)^{\frac{1}{q}} =$
$((s\vee\sigma m)^q + \int_{s\vee\sigma m}^{\infty} qu^{q-1}T(u)du)^{\frac{1}{q}} \le$
$((s\vee\sigma m)^q + B(s\vee\sigma m)^bT(s\vee\sigma m)\int_{s\vee\sigma m}^{\infty} qu^{q-b-1}du)^{\frac{1}{q}} \le (s\vee\sigma m)(1+\frac{Bq}{2(b-q)})^{\frac{1}{q}}$.
And (1.3) follows in this case because $(s\vee\sigma m)w \le (t\wedge m)$.

The last estimations applied to $s = 0, t = \infty, \sigma = 1$ yield the following inequality valid for all $0 < q < b$ and $\xi \in V\Lambda(a, A; b, B)$

$$m(1-\frac{Aq}{2(q+a)})^{\frac{1}{q}} \le ||\xi||_q \le m(1+\frac{Bq}{2(b-q)})^{\frac{1}{q}} \tag{1.12}$$

Theorem 2. Let $p < 0 < q$. If $\xi \in HV\Lambda(p, q, \sigma)$ then
$\xi \in V\Lambda(-p, P; q, Q)$, where
$P = \sigma^p(2^{\frac{q-p}{q}} - 2)(-\frac{q}{p}\vee 1)$ and $Q = \sigma^{-q}(2^{\frac{p-q}{p}} - 2)(-\frac{p}{q}\vee 1)$.

Proof. As shown in the proof of Theorem 1 if $\xi \in HV\Lambda(p,q,\sigma)$ then (1.7) and (1.8) hold true for all $0 < s < t$. The left side of the inequality (1.7) is estimated from below by $(1+((\frac{t}{s})^q-1)T(\frac{t}{\sigma}))^{\frac{1}{q}}$ and the right side of the inequality (1.7) is estimated from above by $(1+((\frac{t}{s})^p-1)T(s))^{\frac{1}{p}}$.Thus the inequality (1.7) implies the following one
$1+((\frac{t}{s})^q-1)T(\frac{t}{\sigma}) \le (1+((\frac{t}{s})^p-1)T(s))^{\frac{q}{p}}$. For $r<0$ we have the inequality

$$(1+x)^r \le (2-2^{1-r})x \text{ for } x \in [-\tfrac{1}{2},0]$$

Since $-\frac{1}{2} < ((\frac{t}{s})^p-1)T(s) \le 0$ for $s > m = \min M$ the inequality (1.9) gives $((\frac{t}{s})^q-1)T(\frac{t}{\sigma}) \le ((2^{\frac{p-q}{p}}-2)(1-(\frac{t}{s})^p)T(s)$ for $s>m, m \in M$. Applying the inequality

$$1-y^p \le (-\tfrac{p}{q} \vee 1)(1-y^{-q}) \text{ for } y \ge 1$$

we obtain $T(\frac{t}{\sigma}) \le (2^{\frac{p-q}{p}}-2)(-\frac{p}{q} \vee 1)(\frac{t}{s})^{-q}T(s)$ for $s > m$ and hence for $s = m$ as well. This inequality can be written as $T(t) \le Q(\frac{s}{t})^q T(s)$ for $Q = (2^{\frac{p-q}{p}}-2)(1 \vee -\frac{p}{q})\sigma^q$ for all $s<t$ such that $s \ge m$ for some $m \in M$ and $\sigma t > s$. Since $B(\frac{s}{t})^q > 1$ for $\sigma t \le s$ the above inequality holds for all $s<t$ and $s \ge m$ for some $m \in M$. Thus the condition (1.2) is fullfilled. To prove (1.1) we could proceed in a similar way however it is simpler to apply what has been proven to ξ^{-1} and use (1.4) as well to note that ξ fullfils (1.1) if and only if ξ^{-1} fulfils (1.2) with $B=A$, $b=a$. As a result ξ fullfils (1.2) with $a=-p$ and $P = \sigma^{-1}(2^{\frac{q-p[A}{q}}-2)(-\frac{q}{p} \vee 1)$.

The poperty given in the next Proposition steams aplications of hypercontractivity. It is quite well know for $p>0$. Since we need the case $p<0<q$ and its proof uses different arguments we provide its proof.

Proposition 1. Let $p<0<q$, $h: R_+^n \to R_+$ be a Borel map and let $(\eta_1,\xi_1),..,(\eta_n,\xi_n)$ be a sequence of independent random vectors in R^2. If for each $1 \le i \le n$ and $x_1,..,x_{i-1},x_{i+1},..,x_n \in R$ there holds
$(Eh^q(x_1,.,x_{i-1},\eta_i x_{i+1},.,x_n))^{\frac{1}{q}} \le (Eh^p(x_1,.,x_{i-1},\xi_i,x_{i+1},.,x_n))^{\frac{1}{p}}$ then

$$(Eh^q(\eta_1,\eta_2,\dots,\eta_n))^{\frac{1}{q}} \le (Eh^p(\xi_1,\xi_2,\dots,\xi_n))^{\frac{1}{p}}.$$

Proof. The proof depends on the following statement :
If ξ,ζ are two independent vectors in R^i and in R^j, $f: R^i \times R^j \to R^+$ is a Borel function and $r<0$ then for $f=f(\xi,\zeta)$ it is $E_2(E_1 f)^r \le \left(E_1(E_2 f^r)^{\frac{1}{r}}\right)^r$, where $E_1 = E(\cdot|\zeta)$ is the integration on the first variable ξ and $E_2 = E(\cdot|\xi)$ on the variable ζ.

Indeed, by homogenity we can assume that $E_1(E_2 f^r)^{\frac{1}{r}} = 1$. So we have to show that $E_2(E_1 f)^r \leq 1$. Since the function x^r is a convex on R^+ we get by the Jensen Ineqauality $E_2(E_1 f)^r = E_2\big(E_1((E_2 f^r)^{\frac{1}{r}} f/(E_2 f^r)^{\frac{1}{r}}\big)^r \leq E_2(E_1((E_2 f^r)^{\frac{1}{r}}(f^r/(E_2 f^r))\big) = 1$.
We prove by induction that for $i = 1, .., n$ and each $x_{i+1}, .., x_n \in R_+$ it is

$$(Eh^q(\eta_1, .., \eta_i, x_{i+1}, .., x_n)))^{\frac{1}{q}} \leq (Eh^p(\xi_1, .., \xi_i, x_{i+1}, .., x_n))^{\frac{1}{p}}.$$

For $i = 1$ it is obvious. Assume it holds true for some $i < n$. For fixed $x_{i+2}, .., x_n$ let $f(x_1, ..., x_{i+1}) = h^p(x_1, .., x_{i+1}, x_{i+2}, .., x_n)$. Let $\xi = (\xi_1, .., \xi_i)$, $\eta = (\eta_1, .., \eta_i)$ and $r = \frac{q}{p}$. To complete the induction we have to prove that $Ef^r(\eta, \eta_{i+1}) \leq \big(Ef(\xi, \xi_{i+1})\big)^r$. We have $Ef^r(\eta, \eta_{i+1}) = E_2(E_1 f(\eta, \eta_{i+1}))^r \leq (E_1(E_2(f^r(\xi, \eta_{i+1}))^{\frac{1}{r}}\big)^r \leq \big(Ef(\xi, \xi_{i+1})\big)^r$, where the first inequality follows from the induction assumption, the second one by the statement given at the begining specified to $\zeta = \xi_{i+1}$ and the last one since by the proposition assumptions $E_2 f^r(\xi, \eta_{i+1}) \leq (E_2 f(\xi, \xi_{i+1}))^r$.

Corollary 1 Let $g : R_+^n \to R_+$ be a Borel function such that in each variable separately it is of the form $s \vee x \wedge t$ for some $0 \leq s \leq t \leq \infty$. Moreover assume that g is 1-homogeneous function, i.e. $rg(x_1, \ldots, x_n) = g(rx_1, \ldots, rx_n)$ for all $r > 0, x_1, \ldots, x_n \geq 0$.
Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent positive random variables.
If $\xi_i \in HV\Lambda(p, q, \sigma),\ \ i = 1, ., n$ then $g(\xi_1, .., \xi_n) \in HV\Lambda(p, q, \sigma)$.
If $\xi_i \in V\Lambda(a, A; b, B),\ \ i = 1, .., n$ then $g(\xi_1, .., \xi_n) \in V\Lambda(p, P; q, Q)$, for each $-a < p < 0 < q < b$ and P, Q are as in Theorem 1 and σ as in Theorem 2.

Proof. We apply Proposition 1 to the sequence $\eta_i = \sigma\xi_i$, $i = 1, .., n$ and the function $h = s \vee g \wedge t$. It gives the first statement since then
$(E(s \vee \sigma g(\xi_1, ..., \xi_n) \wedge t)^q)^{\frac{1}{q}} = (E(s \vee g(\sigma\xi_1, ..., \sigma\xi_n) \wedge t)^q)^{\frac{1}{q}} \leq$
$(E(s \vee g(\xi_1, ..., \xi_n) \wedge t)^p)^{\frac{1}{p}}$. The second statement is a direct consequence of the first one and Theorems 1,2.

Reliability

For $u, t \in R_+$ let $h(u, t) = 1$ if $u \leq t$ and 0 otherwise, i.e. $h(u, t) = I_{[0,t]}(t)$, where I_D is the indicator of function of the set D. Let $N_n =: \{1, \ldots, n\}$.
For $S \subset N_n$ and $x = (x_1, \ldots, x_n) \in R_+^n$ let $h_S(x, t) =: \prod_{i \in S} h(x_i, t)$, ($h_\emptyset(x, t) \equiv 1$ for the empty set $\emptyset$).
Let $k_S,\ \ S \subset N_n$ be a system of integer numbers such that
$\mu(A) =: \sum_{S \subset A} k_S$ is a nondecreasing function of $A \subset N_n$,

i.e. $\mu(A) \leq \mu(B)$ for each $A \subset B \subset I_n$. The last condition is equivalent to $\sum_{i \in S \subset A} k_S \geq 0$ for each $A \subset N_n$ and $i \in A$.
Given such a system $k_S, S \subset N_n$ we define the function $f : R_+^n \times R_+ \to R$ by

$$f(x,t) = \sum_{S \subset N_n} k_S h_S(x,t) = \mu(\{i \in N_n : x_i \leq t\}).$$

The function $f(x,t)$ is right continuous and nondecreasing in t for each x.

Definition 3. We call $g(x) =: \inf\{t \geq 0 : f(x,t) \geq 0\}$ *reliability function* of the system $\{k_S, S \subset N_n\}$.

Since $f(x,t)$ is a right continuous and nondecreasing function we obtain

(2.1) $\qquad g(x) \leq t$ if and only if $f(x,t) \geq 0$

The reliability function fullfils both assumptions of Corollary 1, Indeed since $f(rx, rt) = f(x,t)$ for all $x \in R_+^n, t, r \in R_+$ the function g is 1-homogenous.
For fixed $i \in N_n$ and fixed all coordinates other then the i-th one $f(x,t)$ as the funciton of x_i, t can be written as
$f_1(t) I_{\{x_i \leq t\}} + f_2(t) I_{\{x_i > t\}}$, where $f_1(t) = \mu(\{j \in I_n : j \neq i, x_j \leq t\} \cup \{i\})$ and $f_2(t) = \mu(\{j \in I_n : j \neq i, x_j \leq t\} \setminus \{i\})$.
The functions $f_1(t)$, $f_2(t)$ are right contninuous and nondecreasing.
Let $u = \inf\{t : f_1(t) \geq 0\}$ and $v = \inf\{t : f_2(t) \geq 0\}$. By the monotonicity of μ we get $f_1(t) \geq f_2(t)$ and hence $u \leq v$. If $x_1 < u$ then for $t < u$, $f_1(t), f_2(t) < 0$ and therefore $f(t,x) < 0$ and for $t = u$ $f(x,u) = f_1(u) \geq 0$. Thus $g(x) = u$ for $x_i < u$. If $u \leq x_i < v$ then for any $t < x_i$ it is $f(x,t) = f_2(t) < 0$. For $t = x_i$ we get $f(t,x) = f_1(x_i) \geq f_1(u) \geq 0$. Hence $g(x) = x_i$ for $u \leq x_i < v$. In the case $x_i \geq v$ we get $f_1(t), f_2(t) \geq 0$ for $t \geq v$ and thus $f(x,t) \geq 0$. For $t < v \leq x_1$ we get $f(t,x) = f_2(t) < 0$. These prove that $g(x) = v$ for $t \geq v$. Altogether we proved that $g(x)$ as a function of the variable x_i coincides with the function $u \wedge x_i \vee v$ for some $0 \leq u \leq v \leq \infty$. Thus the reliability function satisfies the first assumpion from Corollary 1 as well.
As a consequence of the above considerations and Corollary 1, ii. we obtain

Corollary 2. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a sequence of independent positive random variables such that $\xi_i \in V\Lambda(a, A; B, b)$ for each $i = 1, .., n$ and let $g : R_+^n \to R_+$ be a reliability function of a system as above. Then
$g(\xi) \in V\Lambda(-p, P; q, Q)$ if $-a < p < 0 < q < b$ and P, Q are as in Corollary 1.

In the sequel $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence of independent positive random variables such that their distribution functions $F_i(t) = P(\xi_i \le t)$ are continuous. The reliability function $g : R_+^n \to R_+$ of a system $k_S, S \subset N_n$ is as in Definition 3. We assume additionally that $k_\emptyset < 0$. For $S \subset I_n$ we put $F_S(t) = \prod_{i \in S} F_i(t)$, $(F_\emptyset(t) \equiv 1)$ and $E(t) = Ef(\xi, t) = \sum_{S \subset N_n} k_S F_S(t)$. Moreover let

$$R = \inf\{t > 0 : E(t) > -1/2\}.$$

$E(t)$ is a nondecreasing and continuous function. Hence $E(R) = -1/2$.

Assume that for some $c > 0$ we have estimates

$$(2.2) \qquad 1 - c \ge P(f(\xi, R) + 1/2 \ge 0) \ge c$$

Since $f(\xi, t)$ is integer valued $f(\xi, R) + 1/2 \ge 0$ if and only if $f(\xi, R) \ge 0$ and hence by (2.1) if and only if $g(\xi) \le R$. Thus (2.2) implies $P(g(\xi) \le R) \ge c$. Similary $f(\xi, R) + 1/2 < 0$ if and only if $g(\xi) > R$. Hence $P(g(\xi) > R) \ge c$. By Corollary 2 $g(\xi)) \in V\Lambda(-p, P; q, Q)$. Therefore for each $m \in M(g(\xi))$ these inequalities imply

$$(2.3). \qquad m \le R(\frac{2c}{P})^{\frac{1}{p}}, \;\; m \ge R(\frac{2c}{Q})^{\frac{1}{q}}$$

For each random variable ζ easy consequences of the Hölder Inequality are:

$$P(\zeta - E\zeta > 0) \ge \frac{\left(E|\zeta - E\zeta|\right)^2}{4E(\zeta - E\zeta)^2} \ge \frac{\left(E(\zeta - E\zeta)^2\right)^2}{4E(\zeta - E\zeta)^4}.$$

Taking $\zeta = f(\xi, R),\; -f(\xi, R)$ in view of $|f(\xi, R) + 1/2| \ge 1/2$ we get

$$(2.4) \qquad c =: \max\{\frac{1}{16E(f(\xi,R) - E(R))^2}, \frac{\left(E(f(\xi,R) - E(R))^2\right)^2}{4E(f(\xi,R) - E(R))^4}\} \quad \text{fullfils (2.2).}$$

Theorem 3. Under the above assumption for any $-a < p < 0 < q < b$

$$R(\frac{2c}{Q})^{\frac{1}{q}} \le M(g(\xi)) \le R(\frac{2c}{P})^{\frac{1}{p}}.$$

Additionally if $b > 1$ then for any $-a < p < 0$ and $1 < q < b$ it is

$$R(\frac{2c}{Q})^{\frac{1}{q}}\big((1 - \frac{P}{2(1-p)}) \vee \frac{1}{2}\big) \le Eg(\xi) \le R(\frac{2c}{P})^{\frac{1}{p}}(1 + \frac{Q}{2(q-1)}).$$

were P, Q are as in Corollary 1 and c as in (2.2) or (2.4).

Proof. The proof is a quick consquence of (2.3) and (1.12) applied to $q = 1$ beside a trivial inequality $M(g(\xi))/2 \leq Eg(\xi)$.

Example

Let $\{k_S, S \subset N_n\}$ be the system such that $k_S = 1$ if S consists of a single element in N_n, $k_\emptyset = -k$, where $k \in N_n$ is a fixed integer and $k_S = 0$ for all other $S \subset N_n$. Then the raliability function $g_k(x)$ of this system coincides with the $k - th$ smallest value among the numbers $x_1, \ldots, x_n$, i.e. $g_k(x) = \min_{S \subset N_n, |S|=k} \max_{i \in S} x_i$. If $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence of positive random variables then $g_k(\xi)$ we call the *k-th oder statisitcs* of the sequence ξ. To estimate the constant c in (2.4) let us observe that in this case $f(\xi, R) = \sum_{i=1}^n \eta_i$, where $\eta_i = I_[0, R](\xi_i) - F_i(R)$. Then $E\eta_i = 0$ and $E\eta_i^2 = \rho_i$, where $\rho_i = F_i(R)(1 - F_i(R)$. Hence $\rho =: E(f(\xi) - E(R))^2 = \sum_{i=1}^n E\eta_i^2 = \sum_{i=1}^n \rho_i^2$, and $E(f(\xi) - E(R))^4 = \sum_{i=1}^n \rho_i(1 - 6\rho_i) + 3(\sum_{i=1}^n \rho_i^2)^2 \leq \rho + 3\rho^2$.
By (2.4) we get $c^{-1} \leq 4 \min\{4\rho, 3 + \rho^{-1}\} \leq 16$.

Computing A,B.

1. If ξ uniformly distributed in an interval $[0, r]$ then for each $b \geq 1$

$$\xi \in V\Lambda(1, 1; b, (\frac{2b}{b+1})^{b+1} \frac{1}{b}).$$

2. If ξ has the exponential distribution , $W(\lambda)$, then for each $b \geq 1$

$$\xi \in V\Lambda(1, 2\ln 2; b, 2(\frac{b}{e \ln 2})^b).$$

3. Similary for any distribution with logarithmicaly concave tails there is a way to give optimal bounds for the constants A, B.

Computing R.
In the case of (ξ_i) uniformly distributed R is a zero of some polynomial of degree $\max\{|S| : k_S \neq 0\}$.

When each ξ_i has the distribution $W(r_i)$ then $x = e^{-R}$ satisfies $\sum_{S \subset I_n} k_S x^{\kappa_S} = 0$ where $\kappa_S = \sum_{i \in S} \lambda_i$.