Two approximation algorithms for ATSP with strengthened triangle inequality

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Asymmetric Traveling Salesman Problem (ATSP)

Problem Statement

INPUT:
A complete graph $G = (V, E)$ with a weight function $w : V^2 \rightarrow \mathbb{R}_{\geq 0}$.

OUTPUT:
A minimum weight Hamiltonian cycle in G.

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Some bad news...

In the general version ATSP does not admit $f(n)$-approximation, for any polynomially computable function $f(n)$, unless $\mathrm{P} = \mathrm{NP}$.

ATSP with Triangle Inequality

An extra assumption which makes some approximation possible is the triangle inequality:

The triangle inequality

$$w(x, y) \leq w(x, z) + w(z, y) \quad \text{for all distinct } x, y, z.$$

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Approximation of ATSP with triangle inequality

- Symmetric variant has a $\frac{3}{2}$-approximation (Christofides),
- $\log n$-approximation (Frieze, Galbiati, Maffioli 1982),
- $0.999 \log n$-approximation (Bläser 2003),
- $0.842 \log n$-approximation (Kaplan et al. 2005),
- $\frac{2}{3} \log n$-approximation (Feige and Singh 2008),
- $O(1)$-approximation can still exist.

ATSP with Stregthened Triangle Inequality

Let γ be a constant, $\gamma \in [\frac{1}{2}, 1)$.

Strengthened triangle inequality

$$w(x, y) \leq \gamma(w(x, z) + w(z, y)) \quad \text{for all distinct } x, y, z.$$

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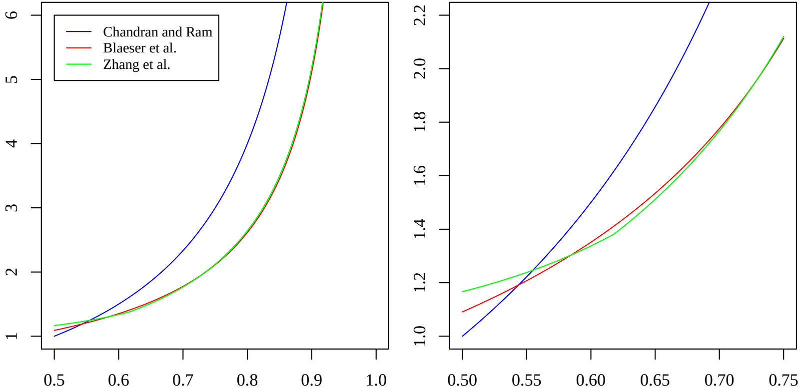
Approximation of ATSP with strenghtened triangle inequality

- $\frac{\gamma}{1-\gamma}$-approximation (Chandran and Ram, STACS 2002),
- $1/(1 - \frac{1}{2}(\gamma + \gamma^3))$-approximation (Bläser, ICALP 2003),
- $\frac{1+\gamma}{2-\gamma-\gamma^3}$-approximation (Bläser et al., J. Discr. Alg. 2006),
- $\frac{\gamma^3}{1-\gamma^2} + \max\{1, \frac{\gamma+\gamma^2+1}{2}\}$-approximation (Zhang, Li & Li, J. Alg. 2009),

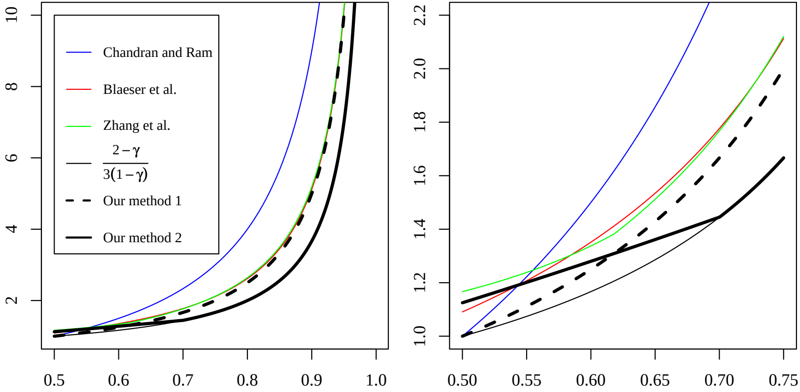
Previous Results



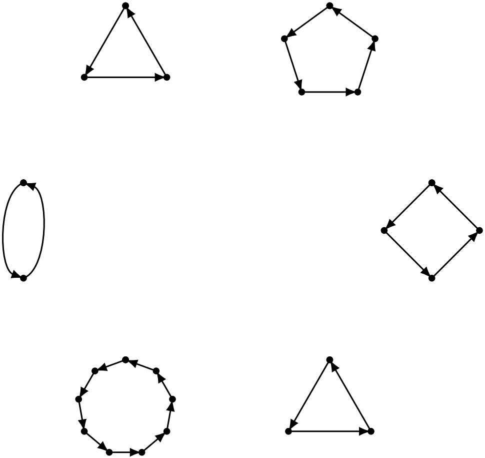
Our Results

- A simple $\frac{1}{2(1-\gamma)}$-approximation,
- An algorithm with approximtaion ratio of
  - $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$ when $\gamma \in (\gamma_0, 1]$ where $\gamma_0 \approx 0.7003$,
  - $\frac{1}{2}(1+\gamma)^2 + \epsilon$ for any $\epsilon > 0$ when $\gamma \in [\frac{1}{2}, \gamma_0]$.

Comparision of Results

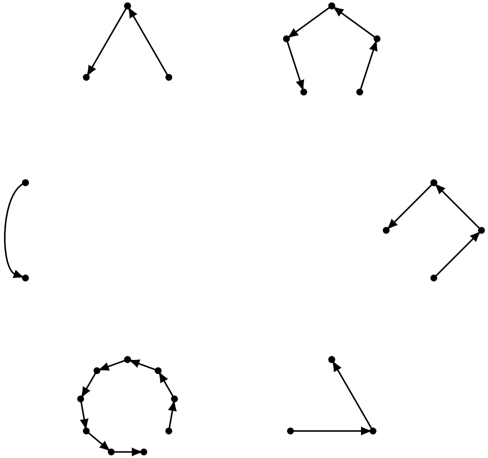


A simple algorithm



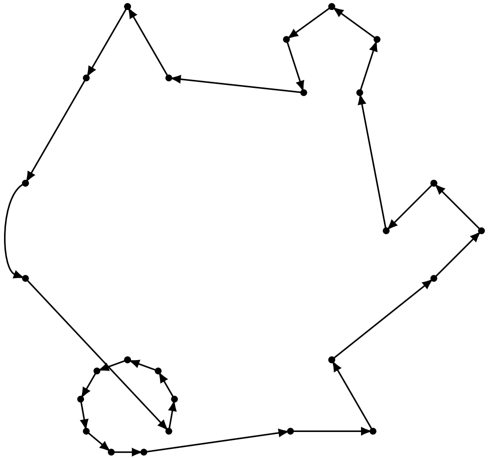
1. Find a minimum weight cycle cover
(doable in polynomial time)

A simple algorithm



2. Remove a random edge from every cycle

A simple algorithm



3. Patch the resulting paths into a Hamiltonian cycle (arbitrarily)

A simple algorithm: Analysis

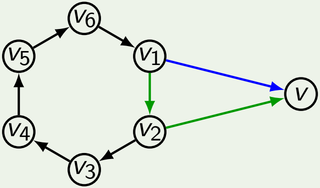
Consider a cycle $C = v_1 v_2 \dots v_l v_\ell$ from the cycle cover.
C is replaced by a (directed) path P and an edge uv, where u is the last vertex on P.

Bounding $E[w(P) + w(uv)]$

$$E[w(P) + w(uv)] = w(C) - \frac{w(C)}{\ell} + \frac{\sum_{i=1}^{\ell} w(v_i v)}{\ell} \tag{1}$$

From the strengthened triangle inequality, for $i = 1, \dots, \ell$:

$$w(v_i, v) \leq \gamma \left(w(v_i, v_{i+1}) + w(v_{i+1}, v)\right)$$



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$$w(v_i, v) \leq \gamma \left(w(v_i, v_{i+1}) + w(v_{i+1}, v)\right)$$

By summing the above inequality over all $i = 1, \ldots, \ell$, we get

$$\sum_{i=1}^{\ell} w(v_i, v) \leq \gamma \left(w(C) + \sum_{i=1}^{\ell} w(v_i, v) \right)$$

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$$\sum_{i=1}^{\ell} w(v_i, v) \leq \gamma \left(w(C) + \sum_{i=1}^{\ell} w(v_i, v) \right)$$

Hence,

$$\sum_{i=1}^{\ell} w(v_i, v) \leq \frac{\gamma}{1-\gamma} w(C). \quad (2)$$

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$$\sum_{i=1}^{\ell} w(v_i, v) \leq \frac{\gamma}{1-\gamma} w(C). \tag{2}$$

Finally, from (1) and (2):

$$E[w(P) + w(uv)] \leq \frac{\ell - 1 - (\ell - 2)\gamma}{\ell(1-\gamma)} w(C).$$

A simple algorithm: Analysis

Consider a cycle $C = v_1 v_2 \ldots v_\ell v_1$ from the cycle cover.
C is replaced by a (directed) path P and an edge uv, where u is the last vertex on P.

Lemma 1

$$E[w(P) + w(uv)] \leq \frac{\ell - 1 - (\ell - 2)\gamma}{\ell(1-\gamma)} w(C).$$

Since $\ell \geq 2$, and $\frac{\ell - 1 - (\ell - 2)\gamma}{\ell(1-\gamma)} = 1 + \frac{2\gamma - 1}{\ell(1-\gamma)}$ is decreasing in ℓ,

Corollary 1

$$E[w(P) + w(uv)] \leq \frac{1}{2(1-\gamma)} w(C).$$

Corollary 2

The expected weight of the resulting Hamiltonian cycle is at most $\frac{1}{2(1-\gamma)} w(\mathcal{C})$, where $\mathcal{C}$ is the initial cycle cover.

A simple algorithm: Analysis

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Corollary 3

The expected weight of the resulting Hamiltonian cycle is at most $\frac{1}{2(1-\gamma)} \mathrm{OPT}$.

What have we got?

Theorem 1

There is a randomized algorithm for the ATSP problem with strengthened triangle inequality with expected approximation ratio of $\frac{1}{2(1-\gamma)}$.

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There is a randomized algorithm for the ATSP problem with strengthened triangle inequality with expected approximation ratio of $\frac{1}{2(1-\gamma)}$.

After derandomizing it with the standard method of conditional expectation we get:

Theorem 2

There is a deterministic algorithm for the ATSP problem with strengthened triangle inequality with approximation ratio of $\frac{1}{2(1-\gamma)}$.

Optimality

This is optimal if we use the cycle cover relaxation.

Another relaxation

In the better algorithm we use a different TSP relaxation:

Theorem 3 (Kaplan, Lewenstein, Shafrir and Sviridenko 2003)

Let G be a directed weighted graph. One can find in polynomial time a pair of cycle covers $\mathcal{C}_1$, $\mathcal{C}_2$ such that

1. *$\mathcal{C}_1$ and $\mathcal{C}_2$ share no 2-cycles,*
2. *$w(\mathcal{C}_1) + w(\mathcal{C}_2) \leq 2\mathrm{OPT}$, where OPT is the weight of the minimum weight Hamiltonian cycle in G.*

Our algorithm begins with finding such a pair.
In what follows, G will refer to the 2-regular directed graph corresponding to $\mathcal{C}_1 \cup \mathcal{C}_2$.

Our Approach

General Idea

- Replace every connected component of G by a (light) path and patch the paths to a Hamiltonian cycle.

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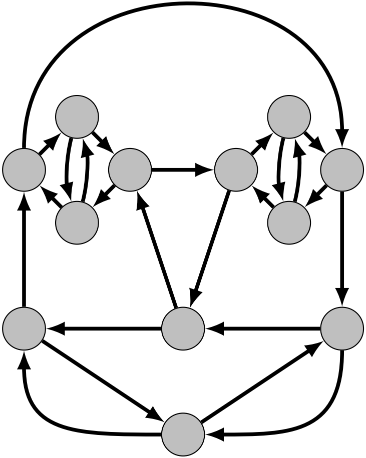
- Replace every connected component of G by a (light) path and patch the paths to a Hamiltonian cycle.
- We process the *small* components and *large* components in a different manner.
- Here, *large* means with at least $K = f(\gamma)$ vertices, for a certain function f. (Note that $K = O(1)$ for any fixed γ).

Large Components

For large components we proceed in two stages:

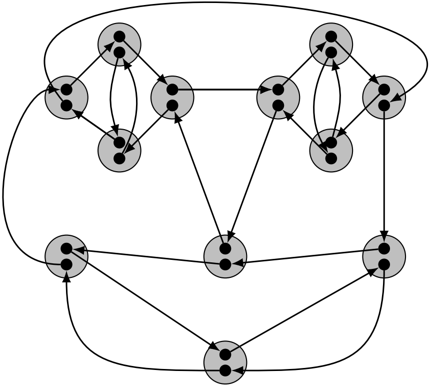
1. Replace each connected component Q by a (light) simple cycle incident to all vertices of Q,
2. Break the cycles and patch them into a path as in the simple algorithm.

Transforming a large components to a cycle



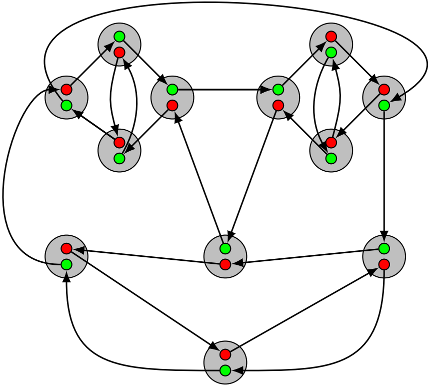
Consider a connected component.

Transforming a large components to a cycle



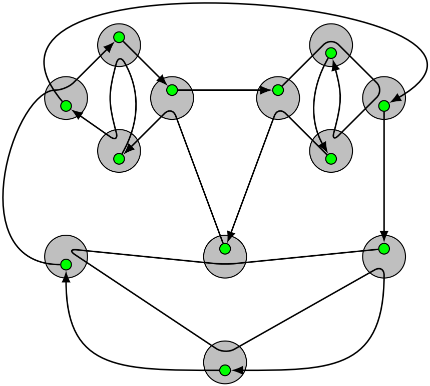
Find an Eulerian cycle in the component. Eulerian cycle = closed walk with each vertex appearing twice.

Transforming a large components to a cycle



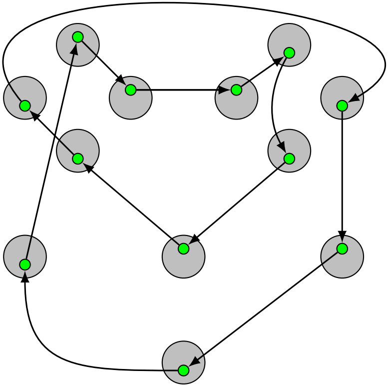
For each vertex choose one of the two occurances (green) uniformly at random.

Transforming a large components to a cycle



Go along the Eulerian cycle and stop only in the green ocurrences of vertices.

Transforming a large components to a cycle



Thus we've got a simple cycle C.

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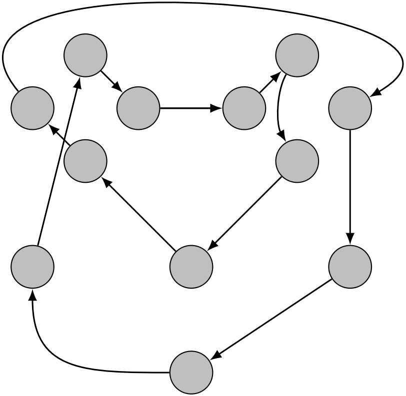

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Large cycles: Analysis

Let $\mathcal{E} = v_1, v_2, \ldots, v_{2|V(Q)|}, v_1$ be the Eulerian cycle we use. Let $C = v_{a_1} v_{a_2} \ldots v_{a_{|Q|}} v_{a_1}$ be the resulting simple cycle.

Shortcutting Lemma

Let $v_p v_q$ be an edge of C. Then
$w(v_p v_q) \leq \gamma w(v_p v_{p+1}) + \gamma^2 \sum_{i=p}^{q-2} w(x_i x_{i+1}) + \gamma w(x_{q-1} x_q)$.

So, each edge $v_p v_q$ of the Eulerian cycle contributes to the weight of the resulting simple cycle at most $\gamma^c w(v_p v_q)$ for some c.

- $\leq 1 \cdot w(v_p v_q)$ if both p and q are green (probablity $\frac{1}{4}$),
- $\leq \gamma \cdot w(v_p v_q)$ if p is green and q is red (probablity $\frac{1}{4}$),
- $\leq \gamma \cdot w(v_p v_q)$ if p is red and q is green (probablity $\frac{1}{4}$),
- $\leq \gamma^2 \cdot w(v_p v_q)$ if both p and q are red (probablity $\frac{1}{4}$).

$\Longrightarrow$ expected contribution of $v_p v_q$ is $\leq \frac{1}{4}(1+\gamma)^2 w(v_p v_q)$.

Large cycles: Analysis

Lemma 4 (again, we can derandomize...)

Let C be the resulting simple cycle. Then $w(C) \leq \frac{1}{4}(1+\gamma)^2 w(Q)$.

Then cycles are broken and patched to a path like in the simple algorithm.

Corollary 5

Assume we call a cycle long when it has length at least K. Long cycles $C_1, \ldots, C_s$ contribute to the solution by at most $\frac{1}{4}(1+\gamma)^2 \cdot \left(1 + \frac{2\gamma-1}{K(1-\gamma)}\right)(w(C_1) + \ldots + w(C_s))$.

Lemma 6 (Proof skipped)

Short cycles $C_{s+1}, \ldots, C_r$ contribute to the solution by at most $\left(\frac{2-\gamma}{6(1-\gamma)} + O\left(\frac{1}{n}\right)\right)(w(C_{s+1}) + \ldots + w(C_r))$.

Approximation ratio of the better algorithm

Assume we call a cycle long when it has length at least K.
The weight of the returned Hamiltonian cycle is at most

$$\max\left\{\tfrac{1}{4}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right),\frac{2-\gamma}{6(1-\gamma)}+O\left(\tfrac{1}{n}\right)\right\}w(\mathcal{C}_1\cup\mathcal{C}_2)\leq$$

$$\max\left\{\tfrac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right),\frac{2-\gamma}{3(1-\gamma)}+O\left(\tfrac{1}{n}\right)\right\}\cdot 2\mathrm{OPT}\leq$$

$$\max\left\{\tfrac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right),\frac{2-\gamma}{3(1-\gamma)}+O\left(\tfrac{1}{n}\right)\right\}\mathrm{OPT}.$$

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- $\frac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right)$ is small when K is large.
- There is a number $\gamma_0 \approx 0.7003$ s.t. when $\gamma \in (\gamma_0, 1)$, we have $\frac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right) \leq \frac{2-\gamma}{3(1-\gamma)}$ for some $K = f(\gamma)$.

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- So, when $\gamma \in (\gamma_0, 1)$, approximation ratio is $\frac{2-\gamma}{3(1-\gamma)} + O\left(\frac{1}{n}\right)$.

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- So, when $\gamma\in(\gamma_0,1)$, approximation ratio is $\frac{2-\gamma}{3(1-\gamma)}+O\left(\frac{1}{n}\right)$.
- For $\gamma\in[\frac{1}{2},\gamma_0]$, $\frac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right)\geq\frac{2-\gamma}{3(1-\gamma)}$.

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- So, when $\gamma \in (\gamma_0, 1)$, approximation ratio is $\frac{2-\gamma}{3(1-\gamma)} + O\left(\frac{1}{n}\right)$.
- For $\gamma \in [\frac{1}{2}, \gamma_0]$, $\frac{1}{2}(1+\gamma)^2\cdot\left(1+\frac{2\gamma-1}{K(1-\gamma)}\right) \geq \frac{2-\gamma}{3(1-\gamma)}$.
- So, by taking K large enough, then we get a ratio of $\frac{1}{4}(1+\gamma)^2 + \epsilon$ for any $\epsilon > 0$.

Conclusion

- We showed a simple algorithm with approximtaion ratio $\frac{1}{2(1-\gamma)}$. It is optimal w.r.t. the cycle cover relaxation.
- We showed a algorithm with approximtaion ratio of
  - $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$ when $\gamma \in (\gamma_0, 1]$ where $\gamma_0 \approx 0.7003$,
  - $\frac{1}{2}(1+\gamma)^2 + \epsilon$ for any $\epsilon > 0$ when $\gamma \in [\frac{1}{2}, \gamma_0]$.

  It is optimal w.r.t. the double cycle cover relaxation for $\gamma \in (\gamma_0, 1)$.
- Open: get a ratio $\frac{2-\gamma}{3(1-\gamma)}$ for all γ.
- ... or an even better ratio!

The end

Thank you for your attention!