The Kleene-Schützenberger theorem

- Rational power series (or languages, streams): power series characterizable by rational expressions (over arbitrary semirings $S$).
- Recognizable power series (or languages, streams): power series that can be recognized by a weighted automaton.
- Proven by Kleene for $\mathbb{B}$ (Kleene’s theorem), by Schützenberger for $\mathbb{Z}$ and by Eilenberg for arbitrary semirings $S$.
- Coalgebraic proof by Rutten for $\mathbb{B}$ in both directions, and for arbitrary semirings in the rational $\rightarrow$ recognizable direction.
Formal power series

Given a semiring $S$ and a finite alphabet $A$, let $S\langle\langle A\rangle\rangle$ denote the function space:

$$\{\sigma \mid \sigma \in A^* \to S\}$$

We assign a semiring structure to $S\langle\langle A\rangle\rangle$ (we use 1 to denote the empty word):

- $0(w) = 0$
- $1(w) = \text{if } w = 1 \text{ then } 1 \text{ else } 0$
- $(\sigma + \tau)(w) = \sigma(w) + \tau(w)$
- $(\sigma \tau)(w) = \sum_{uv=w} \sigma(u)\tau(v)$

Also: alphabet injections $A \to S\langle\langle A\rangle\rangle$:

$$a(w) = \text{if } w = a \text{ then } 1 \text{ else } 0$$
Formal power series (2)

We can also assign *output* and *derivative* operators $O$ and $\Delta$ on $S\langle\langle A\rangle\rangle$

\[
O(\sigma) = \sigma(1) \\
\Delta(\sigma)(a)(w) = \sigma(aw)
\]

and will simply write $\sigma_a$ for $\Delta(\sigma)(a)$.

The semiring structure on $S\langle\langle A\rangle\rangle$ now can be characterized using the following *behavioural differential equations*:

\[
\begin{align*}
O(0) &= 0 & 0_a &= 0 \\
O(1) &= 1 & 1_a &= 0 \\
O(b) &= 0 & b_a &= \text{if } b = a \text{ then } 1 \text{ else } 0 \\
O(\sigma + \tau) &= O(\sigma) + O(\tau) & (\sigma + \tau)_a &= \sigma_a + \tau_a \\
O(\sigma \tau) &= O(\sigma)O(\tau) & (\sigma \tau)_a &= \sigma_a \tau + O(\sigma)\tau_a
\end{align*}
\]
We call a power series $\sigma \in S\langle A \rangle$ a *polynomial* iff for only finitely many $w \in A^*$, $\sigma(w) \neq 0$.

The set of polynomials in $S\langle A \rangle$ is denoted by $S\langle A \rangle$.

We call a power series $\sigma \in S\langle A \rangle$ *proper* iff $O(\sigma) = 0$. 

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Polynomials and proper series
Recognizable series

Some equivalent characterizations:

- A power series is $S$-recognizable iff it occurs as the solution to a linear system of behavioural differential equations.

- A power series $\sigma_0$ is $S$-recognizable iff there is a finite set $\Sigma = \{\sigma_0, \ldots, \sigma_k\}$ s.t. for each $\sigma \in \Sigma$ and each $a \in A$, $\sigma_a$ is a linear combinations of elements from $\Sigma$.

- A power series $\sigma$ is $S$-recognizable iff there is a $k \in \mathbb{N}$, and there are $c_{ij}, b_i \in S$, such that $\sigma$ occurs as a component of the unique solution in $S\llangle A \rrangle$ to the system of equations

\[
x_i = b_i + \sum_{a \in A} a \sum_{j \leq n} c_{ij} x_j
\]

- A power series $\sigma$ is $S$-recognizable iff it is contained in a stable finitely generated submodule of $S\llangle A \rrangle$. 
Recognizable series (2)

▷ A power series $\sigma$ is $S$-recognizable iff it occurs in the final coalgebra mapping of the determinization of a $S \times (S_X^X)^A$-coalgebra, as follows:

\[
\begin{array}{c}
X \subset \eta \quad S_\omega^X \quad [\ldots]\quad S\langle\langle A\rangle\rangle \\
(S \times (S_\omega^X)^A) \quad \ldots \quad S \times S\langle\langle A\rangle\rangle^A
\end{array}
\]

▷ A power series $\sigma$ is $S$-recognizable iff $\sigma$ is accepted by a finite $S$-weighted automaton.
The star operator

The star operator can be defined in several ways:

- If we assume a topological structure on $S$ (i.e. $S$ is a topological semiring), we can define $\sigma^*$ as the limit

  $$\sigma^* = \lim_{n \to \infty} \sum_{i=0}^{n} \sigma^i$$

  (wherever this limit exists).

- Simple coinductive definition: $\sigma^*$ is defined iff $\sigma$ is proper, and in this case $\sigma^*$ is defined as:

  $$O(\sigma^*) = 1 \quad (\sigma^*)_a = \sigma_a(\sigma^*)$$

For any semiring, we can obtain a topological semiring by assuming the discrete topology on $S$. The coinductive definition of the star is always compatible with this definition.
Given a set $X \subseteq S\langle\langle X\rangle\rangle$, the class of $S$-rational power series in $X$ $\text{Rat}_S[X]$ can be defined as the smallest subset of $S\langle\langle A\rangle\rangle$ such that

1. $X \subseteq \text{Rat}_S[X]$
2. $S\langle X\rangle \subseteq \text{Rat}_S[X]$
3. $\text{Rat}_S[X]$ is closed under the operators $+$ and $\cdot$.
4. If $\sigma \in \text{Rat}_S[X]$ and $\sigma$ is proper, then $\sigma^*$ in $\text{Rat}_S[X]$

We call a power series simply $S$-rational if it is $S$-rational in the empty set.

Any element of $\text{Rat}_S[X]$ can be described using a rational (regular) expression with variables in $X$. 

Rational power series
Rational to recognizable

Induction on size of regular expressions. Base cases trivial. If \( \sigma_0 \) and \( \tau_0 \) are recognizable, there are \( \Sigma \) and \( T \) with \( \sigma_0 \in \Sigma \), \( \tau_0 \in T \), s.t. for each \( \sigma \in \Sigma \) and \( \tau \in T \) and \( a \in A \), \( \sigma_a \) and \( \tau_a \) can be written as a linear combination of elements of \( \Sigma \) and \( T \), respectively.

▸ \( (\sigma + \tau)_a = \sigma_a + \tau_a \) so \( \Sigma \cup T \cup \{\sigma + \tau\} \) again has the required property (i.e. ‘is a stable finitely generated \( S \)-submodule of \( S\langle\langle A\rangle\rangle \)).

▸ For \( (\sigma \tau)_a = \sigma_a \tau + o(\sigma)\tau_a \) so \( \{\sigma \tau \mid \sigma \in \Sigma, \tau \in T\} \cup T \) has the required property.

▸ If \( \sigma \) is proper, \( (\sigma^*)_a = \sigma_a \sigma^* \), and \( (v\sigma^*)_a = v_a \sigma^* + o(v)\sigma_a \sigma^* \), so \( \{v\sigma^* \mid v \in \Sigma\} \cup \{\sigma^*\} \) has the required property.
Lemma

Given any $\sigma, \tau \in S\langle A \rangle$ with $\tau$ proper, the unique solution to the equation

$$x = \sigma + \tau x$$

is given by:

$$x = \tau^* \sigma$$
Lemma

Given a \( k \in \mathbb{N} \) and a family of \( r_{ij} \ (i,j \leq k) \) that are proper and \( S \)-rational in \( X \) for all \( i,j \), as well as a family of \( p_i \ (i \leq k) \) that are \( S \)-rational in \( X \) for all \( i \), the system of equations with components

\[
x_i = p_i + \sum_{j=0}^{k} r_{ij}x_j
\]

for all \( i \leq k \) has a unique solution, and each \( x_i \) is \( S \)-rational in \( X \).

Proof: Natural induction on \( k \).

Base case, if \( k = 0 \), there is a single equation

\[
x_0 = p_0 + r_{00}x_0
\]

and Arden’s rule now gives a unique solution

\[
x_0 = (r_{00})^* p_0
\]

which is \( K \)-rational in \( X \) again.
Inductive case: if $k = n + 1$, write the last equation in the system as

$$x_k = p_k + \sum_{j=0}^{n} r_{kj}x_j + r_{kk}x_k,$$

apply Arden’s rule:

$$x_k = (r_{kk})^* \left( p_k + \sum_{j=0}^{n} r_{xj}x_j \right)$$
General unique solution lemma (left version) (3)

Substituting this equation for $x_k$ into the equations $x_i$ for $i \leq n$ gives

$$x_i = p_i + r_{ik}(r_{kk})^*p_k + \sum_{j=0}^{n}(r_{ij} + r_{ik}(r_{kk})^*r_{kj})x_j$$

Now set

$$q_i := p_i + r_{ik}(r_{kk})^*p_k \quad \text{and} \quad s_{ij} := r_{ij} + r_{ik}(r_{kk})^*r_{kj}$$

and we get a system in $n$ variables:

$$x_i = q_i + \sum_{j=0}^{n} s_{ij}x_j$$

By IH, this system has a unique solution (with each component rational in $X$), and it follows that the original system has a unique solution, too (again, with each component rational in $X$).
If $\sigma$ is $S$-recognizable, it occurs as a solution to a system of $n + 1$ equations

$$x_i = b_i + \sum_{a \in A} a \sum_{j \leq n} c_{ij} x_j$$

or equivalently

$$x_i = b_i + \sum_{j \leq n} \left( \sum_{a \in A} a c_{ij} \right) x_j$$

Because each $b_i$ is rational, and all $\sum_{a \in A} a c_{ij}$ are rational and proper, it follows from the preceding lemma that the system has a unique solution and all $x_i$ are rational.
Two equivalent characterizations:

- A power series $\tau$ is $S$-algebraic iff there is a finite set $\Sigma$ with $\tau \in \Sigma$, s.t. for each $\sigma \in \Sigma$ and each $a \in A$, $\sigma a$ can be written as a polynomial over $\Sigma$.

- A power series $\tau$ is $S$-algebraic iff there is a finite set $\Sigma$ with $\tau \in \Sigma$, s.t. for each $\sigma \in \Sigma$ and each $a \in A$, $\sigma a$ is $S$-rational in $\Sigma$.

Algebraic power series generalize context-free languages, in the sense that a language is context-free iff it is $\mathbb{B}$-algebraic.
Proper systems and their solutions

The traditional way of obtaining (constructively) algebraic series is as solutions to proper systems of equations, generalizing CF grammars. The systems of equations consist of a finite $X$ and a mapping:

$$ p : X \rightarrow S\langle X + A \rangle $$

A system is called proper iff for all $x \in X$, $p(x)(1) = 0$, and for all $x, y \in X$, $p(x)(y) = 0$.

A solution is a mapping $[-] : X \rightarrow S\langle A \rangle$ such that for all $x$

$$ [x] = [p(x)]^{#} $$

where $[-]^{#}$ is the inductive extension of $[-]$.

A solution $[-]$ is strong iff for all $x \in X$, $O[x] = 0$. 
Proper systems can be represented as

\[ x_i = \sum_{j=0}^{k} x_j q_{ij} + \sum_{a \in A} a r_{ia} \]

with \( q_{ij} \) rational in \( X \) and proper, and \( r_{ia} \) rational in \( X \). Assuming that we have a strong solution, we take the derivative to obtain:

\[ (x_i)_a = r_{ia} + \sum_{j=0}^{k} (x_j)_a q_{ij} \]

Now apply the (right version of the) unique solution lemma to conclude that all \( (x_i)_a \) are rational in \( X \).
Conclusions and future work

- A uniform way of presenting two different results via a sufficiently generally formulated lemma: the Kleene-Schützenberger theorem and the construction of the Greibach Normal form from proper systems.
- The construction of the GNF does, unlike traditional presentations, not require a detour via the Chomsky Normal form.
- The construction of the GNF transforms a proper system in $n$ nonterminals into a GNF-system in $2n + |A|$ nonterminals, less than the $n^2 + n$ nonterminals yielded by Rosenkrantz’ procedure.
- Future work: investigate the connections with other limit notions/topologies, unique solutions vs. least solutions, $\epsilon$-transitions and construction of proper systems from arbitrary systems.