Erdős-Pósa for pumpkins and approximation

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König’s theorem

Theorem (König, 1931)

For every bipartite graph, \(|\text{maximum matching}| = |\text{minimum vertex cover}|\).
König’s theorem

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**Theorem (König, 1931)**

*For every bipartite graph, $|\text{maximum matching}| = |\text{minimum vertex cover}|$.*

- min-max theorem for packing and covering edges;
- knowing one parameter gives the other one.
Packing and covering

Can we generalize it to

- packing/covering of larger classes; and
- in more general graphs?
Packing and covering

Can we generalize it to
- packing/covering of larger classes; and
- in more general graphs?

What about packing cycles in graphs?
The case of cycles

Packing number **pack** maximum number of vertex-disjoint cycles in $G$; Covering number **cover** minimum size $X \subseteq V$ s.t. $G \setminus X$ forest (FVS).
The case of cycles

Packing number \textbf{pack} maximum number of vertex-disjoint cycles in \( G \);
Covering number \textbf{cover} minimum size \( X \subseteq V \) s.t. \( G \setminus X \) forest (FVS).

pack = ?
The case of cycles

Packing number **pack** maximum number of vertex-disjoint cycles in $G$;
Covering number **cover** minimum size $X \subseteq V$ s.t. $G \setminus X$ forest (FVS).

```
pack \geq 2
```
The case of cycles

Packing number \textit{pack} maximum number of vertex-disjoint cycles in \( G \);
Covering number \textit{cover} minimum size \( X \subseteq V \) s.t. \( G \setminus X \) forest (FVS).

\[ \text{pack} \geq 4 \]
The case of cycles

Packing number \textbf{pack} maximum number of vertex-disjoint cycles in $G$;
Covering number \textbf{cover} minimum size $X \subseteq V$ s.t. $G \setminus X$ forest (FVS).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{grid}
\end{figure}

pack $\geq 4$
The case of cycles

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\text{pack} = 4
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cover = ?
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$$\text{pack} = 4 \quad \text{cover} = ?$$
The case of cycles

Packing number **pack** maximum number of vertex-disjoint cycles in $G$;
Covering number **cover** minimum size $X \subseteq V$ s.t. $G \setminus X$ forest (FVS).

![Diagram showing pack = 4 and cover ≤ 6](image-url)
Are **pack** and **cover** related?

Theorem (Erdős & Pósa, 1962)

Every graph has $k$ vertex-disjoint cycles or a set of $O(k \log k)$ vertices hitting every cycle. In other words: $f(k) = O(k \log k)$. 
Are \textbf{pack} and \textbf{cover} related?

\[
\text{pack}(G) \leq \text{cover}(G)
\]
Erdős-Pósa Theorem

Are pack and cover related?

\[
\text{pack}(G) \leq \text{cover}(G) \leq f(\text{pack}(G))
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In other words: \( f(k) = O(k \log k) \).
Packing and covering models

Given $H$ and $G$,

**Packing** How many vertex-disjoint sgr. of $G$ can be contracted to $H$?

**Covering** How many vertices to remove in $G$ to get an $H$-minor-free graph?
Packing and covering models

Given $H$ and $G$,

**Packing** How many vertex-disjoint sgr. of $G$ can be contracted to $H$? (maximization problem)

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Packing and covering models

Given $H$ and $G$,

**Packing** How many vertex-disjoint sgr. of $G$ can be contracted to $H$? (maximization problem)  
$\Rightarrow$ packing number $\text{pack}_H(G)$

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Packing and covering models

Given $H$ and $G$,

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→ covering number $\text{cover}_H(G)$

Previously: $H = \bullet\bullet$ and $H = \bullet\circ\bullet$. 

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The Erdős-Pósa property

$H$ has the ÉP-property if:

$$\forall G, \text{ cover}_H(G) \leq f(\text{pack}_H(G)).$$
$H$ has the ŠP-property if:

$$\forall G, \; \text{cover}_H(G) \leq f(\text{pack}_H(G)).$$

$f$: gap.
$H$ has the ŠP-property if:

$$\forall G, \ \text{cover}_H(G) \leq f(\text{pack}_H(G)).$$

$f$: gap.

Two questions: What classes and with which gap?
The Erdős-Pósa property

$H$ has the ĖP-property if:

$$\forall G, \text{cover}_H(G) \leq f(\text{pack}_H(G)).$$

$f$: gap.

Two questions: What classes and with which gap?

Previously:

- $\bullet\bullet\bullet$ has the ĖP-property with gap $k$;
- $\bullet\circ\bullet$ has the ĖP-property with gap $O(k \log k)$. 
The Erdős-Pósa property of planar models

Theorem (Robertson & Seymour, 1986)

\[ H \text{ has the } \tilde{\mathcal{E}} \mathcal{P}\text{-property } \iff H \text{ planar.} \]
The Erdős-Pósa property of planar models

Theorem (Robertson & Seymour, 1986)

$H$ has the $\tilde{E}$P-property $\iff H$ planar.

Theorem (Chekury & Chuzhoy, 2013)

$\forall H$ planar, $f_H(k) = O(k \text{ polylog } k)$. 
Hitting and harvesting pumpkins

\( r \)-pumpkin \( \theta_r \): graph with 2 vertices and \( r \) edges.

**Theorem (Fiorini, Joret and Sau)**

\[ f_{\theta_r} = O(k \log k). \]

**Theorem (Joret, Paul, Sau, Saurabh and Thomassé, 2011)**

There is a \( O(\log(n)) \)-approximation for \( \text{pack}_{\theta_r} \) and \( \text{cover}_{\theta_r} \).
Main result

Theorem (Chatzdimitriou, R., Sau, Thilikos)

There is an $O(\log(\text{OPT}))$-approximation for $\text{pack}_{\theta_r}$ and $\text{cover}_{\theta_r}$.
Main result

Theorem (Chatzdimitriou, R., Sau, Thilikos)

There is an $O(\log(OPT))$-approximation for $\text{pack}_{\theta_r}$ and $\text{cover}_{\theta_r}$.

Ingredients:
- a protrusion-based reduction;
- a algorithm to extract a big packing or a protrusion to reduce.
### The approximation

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
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<tbody>
<tr>
<td>$\exists$ packing $\geq k$</td>
<td>Yes</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Yes</td>
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<td>No</td>
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<tr>
<td>$\exists$ cover $\leq k \log k$</td>
<td>No</td>
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<td>No</td>
<td>Yes</td>
<td>...</td>
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<td>...</td>
<td>...</td>
<td>Yes</td>
</tr>
<tr>
<td>Possible $A(G, k)$</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P/C</td>
<td>P/C</td>
<td>P/C</td>
<td>P/C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

Assume $A(G, k)$ outputs either a **packing** $\geq k$ or a **cover** $\leq k \log k$. 
Assume $A(G, k)$ outputs either a packing $\geq k$ or a cover $\leq k \log k$.

1. run $A$ on $(G, k)$ for every $k \in \{0, \ldots, n\}$;
The approximation

<table>
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<tr>
<th>( k )</th>
<th>0</th>
<th>( \ldots )</th>
<th>( \ldots )</th>
<th>( \ldots )</th>
<th>( \ldots )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
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<td>( \exists ) packing ( \geq k )</td>
<td>Yes</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>Yes</td>
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<td>Yes</td>
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<td>( \ldots )</td>
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<td>Possible ( A(G, k) )</td>
<td>P</td>
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<td>P</td>
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Assume \( A(G, k) \) outputs either a packing \( \geq k \) or a cover \( \leq k \log k \).

1. run \( A \) on \( (G, k) \) for every \( k \in \{0, \ldots, n\} \);
2. returns the largest \( k \) s.t. \( A(G, k) \) is a packing.
The reduction

Boundary

Protrusion

Reduction

“Gadget”

Picture by Felix Reidl
How can a cycle invade a given protrusion?

cycle-free protrusion

rest of the graph
How can a cycle invade a given protrusion?

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How can a cycle invade a given protrusion?

- defines an equivalence relation;
- safe move: reduce to an equivalent protrusion;
- how to compute the equivalence class?
Decomposing protrusions

protrusion
Decomposing protrusions

protrusion → decomposed protrusion
Decomposing protrusions

- partition of the vertex set;
- edges between adjacent nodes only;
- boundary of small size;
- node $\rightarrow$ protrusion $\rightarrow$ eq class.
Decomposing protrusions

- partition of the vertex set;
- edges between adjacent nodes only;
- boundary of small size;
- node $\rightarrow$ protrusion $\rightarrow$ eq class.
A long path in the decomposition tree
A long path in the decomposition tree

Long path $\rightarrow$ repetition.
A long path in the decomposition tree

Long path $\rightarrow$ repetition.
Large degree in the decomposition tree

Discard a redundant child.
Large degree in the decomposition tree

Large degree $\rightarrow$ repetitions.
Large degree in the decomposition tree

Large degree $\rightarrow$ repetitions.
Discard a redundant child.
When the reduction ends...

- diameter $< f(r)$;
- degree $< h(r)$;
When the reduction ends...

- diameter $< f(r)$;
- degree $< h(r)$;

→ reduced protrusions have constant size.
When the reduction ends...

diameter $< f(r)$;
degree $< h(r)$;

$\rightarrow$ reduced protrusions have constant size.
each reduction step takes $O(n)$ time (reduce from leaves to root).
The reduction: recap

Given $G$ and an a large protrusion of small boundary,
The reduction: recap

Given $G$ and an a large protrusion of small boundary,
- we can reduce it to a smaller graph $G'$;
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- $G' \sim G$ for the problems of covering and packing;
Given $G$ and an a large protrusion of small boundary,

- we can reduce it to a smaller graph $G'$;
- $G' \sim G$ for the problems of covering and packing;
- the reduction takes $O(n)$ time.
Finding protrusions

How to find these protrusions?

**Theorem (Chatzdimitriou, R., Sau, Thilikos)**

*Given* $G$ *large enough, we can compute*
- a $\theta_r$-model of $G$ of *small size*, or;
- a *large protrusion* of *small boundary*, or;
- an subgraph contractible to some $H$ with *large* $\delta$, *in* $O(m)$ *steps.*
we assume that girth $> 4d + 2$ (otherwise we can find a small cycle);
Sketch of the proof (for cycles)

- we assume that girth $> 4d + 2$ (otherwise we can find a small cycle);
- find a maximal $2d$-scattered set (centers);
we assume that girth \( > 4d + 2 \) (otherwise we can find a small cycle);
find a maximal \( 2d \)-scattered set (centers);
grow trees from centers;
Sketch of the proof (for cycles)

- we assume that girth $> 4d + 2$ (otherwise we can find a small cycle);
- find a maximal $2d$-scattered set (centers);
- grow trees from centers;
- $\not\exists$ long path (otherwise: large protrusion of small boundary);
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- grow trees from centers;
- $\not\exists$ long path (otherwise: large protrusion of small boundary);
- $\rightarrow$ expansion;
Sketch of the proof (for cycles)

- we assume that girth $> 4d + 2$ (otherwise we can find a small cycle);
- find a maximal $2d$-scattered set (centers);
- grow trees from centers;
- $\exists$ long path (otherwise: large protrusion of small boundary);
  $\rightarrow$ expansion;
  $\rightarrow$ every tree has exponentially many leaves;
we assume that girth $> 4d + 2$ (otherwise we can find a small cycle);
find a maximal $2d$-scattered set (centers);
grow trees from centers;
$\not\exists$ long path (otherwise: large protrusion of small boundary);
$\rightarrow$ expansion;
$\rightarrow$ every tree has exponentially many leaves;
$\rightarrow$ many edges between trees;
an subgraph contractible to some $H$ with large $\delta$. 
Finding a packing or a covering

Input: \((G, k)\).
Output: a packing \(\geq k\) or a cover \(\leq k \log k\).
Finding a packing or a covering

Input: $(G, k)$.
Output: a packing $\geq k$ or a cover $\leq k \log k$.

The previous algorithm gives:
- a $\theta_r$-model of $G$ of small size: add to a partial packing $\mathcal{P}$;
Finding a packing or a covering

Input: \((G, k)\).
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The previous algorithm gives:
- a \(\theta_r\)-model of \(G\) of small size: add to a partial packing \(P\);
- a large protrusion of small boundary: reduce;
- an subgraph contractible to some \(H\) with large \(\delta\): \(\exists\) a large packing.

Then:
- if \(P \geq k\) then we output \(P\);
- if \(G\) is \(\theta_r\)-free, then \(P\) is a small cover.
Finding a packing or a covering

**Input:** \((G, k)\).
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The previous algorithm gives:
- a \(\theta_r\)-model of \(G\) of **small size**: add to a partial packing \(\mathcal{P}\);
- a **large protrusion** of **small boundary**: reduce;
- an subgraph contractible to some \(H\) with **large** \(\delta\): \(\exists\) a large packing.
Finding a packing or a covering

Input: \((G, k)\).
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The previous algorithm gives:
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- a large protrusion of small boundary: reduce;
- an subgraph contractible to some \(H\) with large \(\delta\): \(\exists\) a large packing.

Then:
- if \(\mathcal{P} \geq k\) then we output \(\mathcal{P}\);
- if \(G\) is \(\theta_r\)-free, then \(\mathcal{P}\) is a small cover.

A constructive algorithm?
Constructing the packing

Easy: \( \delta(G) \geq r \Rightarrow G \geq \theta_r \) (can be found in linear time).
Constructing the packing

Easy: $\delta(G) \geq r \Rightarrow G \supseteq \theta_r$ (can be found in linear time).
The same for packings?

**Theorem (Stiebitz, 1996)**

In every $G$ s.t. $\delta(G) \geq k(r + 1) - 1$, there is a partition $(V_1, \ldots, V_k)$ s.t. 

$$\forall i, \delta(G[V_i]) \geq r.$$
Easy: $\delta(G) \geq r \Rightarrow G \geq \theta_r$ (can be found in linear time).

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*It can be found in polynomial time* [Bazgan, Tuza and Vanderpooten, 2007].
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It can be found in polynomial time [Bazgan, Tuza and Vanderpooten, 2007].

Can we find it in linear time?
Approximation algorithm: recap

Given \((G, k)\) and using both

- the reduction algorithm;
- the algorithm to find protrusions,
Approximation algorithm: recap

Given \((G, k)\) and using both

- the reduction algorithm;
- the algorithm to find protrusions,

we are able to

- answer \(\text{pack}_{\theta_r} \geq k\) or \(\text{cover}_{\theta_r} \leq k \log k\) in \(O(n^2)\)-time;
- extract the corresponding object in polynomial time.
The edge variant

Given $G$ and $H$

- **Packing** How many edge-disjoint sgr. of $G$ can be contracted to $H$?

- **Covering** How many edges to remove in $G$ to get an $H$-minor-free graph?
The edge variant

Given $G$ and $H$

- **Packing**: How many edge-disjoint sgr. of $G$ can be contracted to $H$? → packing number
- **Covering**: How many edges to remove in $G$ to get an $H$-minor-free graph? → covering number
The Erdős-Pósa theorem holds for the edge variant.
The Erdős-Pósa theorem holds for the edge variant.

**Theorem (R., Sau, Thilikos)**

\[ \theta_r \text{ has the edge-EP-property with a gap } \text{poly}(r, k). \]
A few results on the edge variant

The Erdős-Pósa theorem holds for the edge variant.

**Theorem (R., Sau, Thilikos)**

\( \theta_r \) has the edge-ÉP-property with a gap \( \text{poly}(r, k) \).

Not known whether all planar graphs have the edge-ÉP-property.
The Erdős-Pósa theorem holds for the edge variant.

**Theorem (R., Sau, Thilikos)**

\[ \theta_r \text{ has the edge-\text{\textAAEP}-property with a gap } \text{poly}(r, k). \]

Not known whether all planar graphs have the edge-\text{\textAAEP}-property.

Our algorithm can deal with this variant.
Remember, we have an algorithm which gives:

- a $\theta_r$-model of $G$ of small size;
- a large protrusion of small boundary;
- an subgraph contractible to some $H$ with large $\delta$. 
Constructive version for the edge variant

Remember, we have an algorithm which gives:

- a $\theta_r$-model of $G$ of small size;
- a large protrusion of small boundary;
- an subgraph contractible to some $H$ with large $\delta$.

In the last case we can extract an edge-disjoint packing in constant time.
Constructive version for the edge variant

Remember, we have an algorithm which gives:

- a $\theta_r$-model of $G$ of small size;
- a large protrusion of small boundary;
- an subgraph contractible to some $H$ with large $\delta$.

In the last case we can extract an edge-disjoint packing in constant time.
→ same complexity for both existential and constructive version.
Future work and open problems

extension to other graphs?
Future work and open problems

- extension to other graphs?
- extension to topological minor models?
Future work and open problems

- extension to other graphs?
- extension to topological minor models?
- better running time for the constructive version?
  i.e. prove that in $G$ s.t. $\delta(G) = O(kr)$ we can find $k \cdot \theta_r$ in $O(n)$-time?
Future work and open problems

- extension to other graphs?
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- better running time for the constructive version?
  i.e. prove that in $G$ s.t. $\delta(G) = O(kr)$ we can find $k \cdot \theta_r$ in $O(n)$-time?
- $H$ has the edge-EP-property $\iff$ $H$ planar.
  known: direction $\Rightarrow$.
Future work and open problems

- extension to other graphs?
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  i.e. prove that in $G$ s.t. $\delta(G) = O(kr)$ we can find $k \cdot \theta_r$ in $O(n)$-time?
- $H$ has the edge-\(\tilde{E}\P\)-property $\iff$ $H$ planar.
  known: direction $\Rightarrow$.
- does every $H$ have the \(\tilde{E}\P\)-property with gap $O(k \log k)$?
  known: $O(k \text{ polylog } k)$
Future work and open problems

- extension to other graphs?
- extension to topological minor models?
- better running time for the constructive version? i.e. prove that in $G$ s.t. $\delta(G) = O(kr)$ we can find $k \cdot \theta_r$ in $O(n)$-time?
- $H$ has the edge-EP-property $\iff H$ planar. known: direction $\Rightarrow$.
- does every $H$ have the EP-property with gap $O(k \log k)$? known: $O(k \text{ polylog } k)$
- does every $H$ have the edge-EP-property with gap $O(k \log k)$? known: $\text{poly}(k, r)$ for $\theta_r$ and $O(k \log k)$ for cycles.
Future work and open problems

- extension to other graphs?
- extension to topological minor models?
- better running time for the constructive version?
  i.e. prove that in $G$ s.t. $\delta(G) = O(kr)$ we can find $k \cdot \theta_r$ in $O(n)$-time?
- $H$ has the edge-$\mathcal{EP}$-property $\iff H$ planar.
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- does every $H$ have the $\mathcal{EP}$-property with gap $O(k \log k)$?
  known: $O(k \text{ polylog } k)$
- does every $H$ have the edge-$\mathcal{EP}$-property with gap $O(k \log k)$?
  known: $\text{poly}(k, r)$ for $\theta_r$ and $O(k \log k)$ for cycles.

Thank you!

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