1. Let $\Omega = B(10,1) \in R^n$ and let

$$L = -y\partial_x^2 - \partial_y^2 - \mu x^2 u.$$

Prove that for any $f \in L^2(\Omega)$ and all sufficiently small $\mu > 0$ the problem

$$\begin{aligned} Lu &= f \qquad && \text{in } \Omega \\ u &= 0 \qquad && \text{in } \partial\Omega \end{aligned}$$

Has a weak solution in $H^1(\Omega)$. Provide the weak formulation.

2. Let $\phi \in C_c^\infty(R^n)$. Prove that the Fourier transform and the inverse Fourier transform of ϕ belongs to $L^p(R^n)$ for any $p \in [1, \infty]$.

3. Consider the Dirichlet problem

$$-div(\nabla u + au) = f,$$

where $f \in H^{-1}(\Omega)$, $\Omega \subset R^n$ is a bounded domain with a smooth boundary. The vector field a is divergence free in a weak sense. Show that the above problem has a unique solution in $H_0^1(\Omega)$.

4. Let us consider the functional

$$\int_0^1 (x + u(x))^2 \, dx$$

defined on the space of continuous functions on $[0,1]$ satisfying boundary data $u(0) = u(1) = 0$. Show that this functional is continuous and bounded, but does not admit the minimum. This is the counterexample to the trivial version of Dirichlet's principle.