1 Praca domowa III

Termin oddawania: 29.04.2016. U is a bounded, open subset of $\mathbb{R}^d$ with a smooth boundary.

Zadanie 1. Let $u_k \in H^1(U)$ be such that $\int_U |\nabla u_k|^2 dx \le C$ for some constant $C > 0$ and all $k = 1, 2, \ldots$. Prove that if $\mathrm{Tr}(u_k) = g \in L^2(\partial U)$ then the sequence $\|u_k\|_{H^1(U)}$ is uniformly bounded.

Definition 1.1. *Given $\alpha > 0$, by $C^{0,\alpha}(X)$ we denote the (Banach) space of all Hölder continuous functions f defined on X, with the norm given by*

$$\|f\|_{0,\alpha} := \|f\|_{C(X)} + [f]_{0,\alpha}, \tag{1.1}$$

where

$$[f]_{0,\alpha} := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{1.2}$$

Zadanie 2. Let A be a bounded set in $C^{0,\alpha}([0,1])$. Prove that A is a compact set in $C^{0,\beta}([0,1])$ for all $0 < \beta < \alpha$.

Zadanie 3. Find all functions $u \in L^1(\mathbb{R})$, satisfying

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x - y) e^{-\frac{|y|^2}{2}} dy = \frac{1}{\sqrt{2}} e^{-\frac{|x|^2}{4}}. \tag{1.3}$$

Zadanie 4. Take $u \in C^\infty(\mathbb{R}^d)$ such that $\mathrm{supp}\hat{u} \subset B(0, \lambda R)$. Prove that for all $1 \le p \le q \le \infty$ we have

$$\|u\|_{L^q(\mathbb{R}^d)} \le C_R \lambda^{\frac{n}{p} - \frac{n}{q}} \|u\|_{L^p(\mathbb{R}^d)},$$

where $C_R > 0$ is a constant depending only on R. Here $B(0, \lambda R)$ denotes as usual a ball centered in 0 with radius λR.

Proposition 1.1 (Young's inequality (AKA the Hint))**.** *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ with $1 \le p, q, r \le \infty$. Then*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$$