*Summer School 2000: Geometry of Toric Varieties*

# Lectures 16-17

## Cohomology of line bundles on toric varieties, vanishing theorems.

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In this lecture, we make a brief introduction to sheaf theory, Čech cohomology of sheaves and focus on the particular case of cohomology of line bundles on toric varieties.

## §1. Sheaf Theory, a brief Introduction.

In this section, the references are for example [2] and [3], among many others. *I thank Stéphane Guillermou for a carefull reading of a preliminary version of these notes.*

***Basic Definitions***. Lecture 3 already recalled some basic definitions, let us repeat them here and give some others.

Let $X$ be a topological space. A **pre-sheaf of abelian groups** $\mathcal{A}$ on $X$ consists in the following data:

• for any open subset $U \subset X$, an abelian group $\mathcal{A}(U)$ called the set of sections of $\mathcal{A}$ on $U$;

• for any pair $U \subset V$ of open subsets of $X$, a morphism of groups $r_U^V : \mathcal{A}(V) \to \mathcal{A}(U)$ (called a restriction map) satisfying $r_U^U$ is the identity map and $r_U^V \circ r_V^W = r_U^W$ for any $U \subset V \subset W$.

In the sequel, we will often denote $s_{|U}$ instead of $r_U^V(s)$ for $s \in \mathcal{A}(V)$.

A pre-sheaf $\mathcal{A}$ (of abelian groups) on $X$ is a **sheaf** if it satisfies the two following properties for any family of open subset $(U_i)_{i\in I}$ with $U = \cup_{i\in I} U_i$,

(R1) for any family $s_i \in \mathcal{A}(U_i)$ such that $r_{U_i\cap U_j}^{U_i}(s_i) = r_{U_i\cap U_j}^{U_j}(s_j)$ for any $i, j$ in $I$, there exists $s \in \mathcal{A}(U)$ such that $r_{U_i}^U(s) = s_i$ for any $i$ in $I$;

(R2) for any $s$ and $s'$ in $\mathcal{A}(U)$ such that $r_{U_i}^U(s) = r_{U_i}^U(s')$ for any $i$ in $I$, we have $s = s'$.

***Main Examples***. If $X$ is a topological space, we will denote by $\mathbb{Z}_X$, $\mathbb{Q}_X$, $\mathbb{R}_X$ or $\mathbb{C}_X$ the sheaves of *locally* constant functions with values in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ with the $r_U^V$ being restriction maps in the usual sense. These sheaves will be used to study the topology of $X$.

**Exercise 1.1.** Explain why the pre-sheaf defined by $\mathbb{Z}(U) = \mathbb{Z}$ for any open subset $U$ with the usual restriction maps is not a sheaf.

If $X$ is a smooth manifold, we will denote by $\mathcal{E}_{\mathbb{C}}$ (resp. $\mathcal{E}_{\mathbb{C}}^*$) the sheaf on $X$ of germs of $\mathcal{C}^\infty$ complex valued functions (resp. the sheaf on $X$ of germs of $\mathcal{C}^\infty$ non vanishing complex valued functions), *i.e.* for any open subset $U$ of $X$, $\mathcal{E}_{\mathbb{C}}(U)$ is the additive group of $\mathcal{C}^\infty$ $\mathbb{C}$-valued functions on $U$ (resp. $\mathcal{E}_{\mathbb{C}}^*(U)$ is the multiplicative group of $\mathcal{C}^\infty$ $\mathbb{C}^*$-valued functions on $U$); with the $r_U^V$ being restriction maps in the usual sense.

If $X$ is an algebraic variety and $L$ is a line bundle on $X$, we will denote by $\mathcal{O}_X(L)$ the sheaf of germs of regular sections of $L$, *i.e.* $\mathcal{O}_X(L)(U) = H^0(U, L_{|U})$ for any Zariski open subset $U$, with the $r_U^V$ being restriction maps in the usual sense. If $L$ is the trivial line bundle, the corresponding sheaf is simply denoted by $\mathcal{O}_X$ and is called the structure sheaf of $X$.

***Maps of Sheaves***. A **map of sheaves** (of abelian groups) $f : \mathcal{A} \to \mathcal{B}$ on $X$ is given by a collection of homomorphisms (of groups) $\{f_U : \mathcal{A}(U) \to \mathcal{B}(U)\}$ when $U$ varies among the open subsets of $X$ such that for any pair $U \subset V$, $f_U$ and $f_V$ commute with the restriction maps of $\mathcal{A}$ and $\mathcal{B}$.

The **kernel** of a map $f : \mathcal{A} \to \mathcal{B}$ is the sheaf $\mathrm{Ker}(f)$ given by $\mathrm{Ker}(f)(U) = \mathrm{Ker}(f_U)$ with the restriction maps being those of $\mathcal{A}$ (exercise: check it is a sheaf).

Unfortunately, it is more difficult to define the cokernel of a sheaf, as shown in the following exercise.

**Exercise 1.2.** Let $X$ be a smooth differentiable manifold and $\exp : \mathcal{E}_{\mathbb{C}} \to \mathcal{E}^*_{\mathbb{C}}$ be the map of sheaves given by $\exp_U(h) = \exp(2i\pi h)$ for any $h$ in $\mathcal{E}_{\mathbb{C}}(U)$.

a. Show that $\mathrm{Ker}(\exp) = \mathbb{Z}_X$.

b. Let $X = \mathbb{C} - \{0\}$. Show that the pre-sheaf $\mathcal{C}$ on $X$ given by

$$\mathcal{C}(U) = \mathrm{Coker}(\exp_U : \mathcal{E}_{\mathbb{C}}(U) \to \mathcal{E}^*_{\mathbb{C}}(U))$$

is not a sheaf.

The **cokernel** of a map $f : \mathcal{A} \to \mathcal{B}$ will therefore be defined in the following manner: a section $s$ of $\mathrm{Coker}(f)$ on $U$ is given by an open cover $(U_\alpha)$ of $U$ together with sections $s_\alpha \in \mathcal{B}(U_\alpha)$ such that for all $\alpha, \beta$

$$(s_\alpha)_{|U_\alpha \cap U_\beta} - (s_\beta)_{|U_\alpha \cap U_\beta} \in f_{U_{\alpha\beta}}(\mathcal{A}(U_\alpha \cap U_\beta));$$

two such collections $\{(U_\alpha), (s_\alpha)\}$ and $\{(U'_\alpha), (s'_\alpha)\}$ will be identified if for any $p \in U$ and any $\alpha$ and $\beta$ such that $p \in U_\alpha \cap U'_\beta$, there exists $V \subset U_\alpha \cap U'_\beta$ such that $(s_\alpha)_{|V} - (s'_\beta)_{|V} \in f_V(\mathcal{A}(V))$. (Exercise: check it is a sheaf).

A map of sheaves $f : \mathcal{A} \to \mathcal{B}$ is **injective** if $\mathrm{Ker}(f)$ is the zero sheaf, *i.e.* $\mathrm{Ker}(f)(U) = \{0\}$ for any $U$.

A map of sheaves $f : \mathcal{A} \to \mathcal{B}$ is **surjective** if $\mathrm{Coker}(f)$ is the zero sheaf.

**Exercise 1.3.** Show that a map of sheaves $f : \mathcal{A} \to \mathcal{B}$ is surjective if and only if for any open subset $U$ of $X$, for any $t \in \mathcal{B}(U)$ and any $p \in U$, there exist an open subset $V \subset U$ containing $p$ and $s \in \mathcal{A}(V)$ such that $f_V(s) = t_{|V}$.

Finally, an **exact sequence** of sheaves on $X$ is a sequence:

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

such that $\mathcal{A} = \mathrm{Ker}(g)$ and $\mathcal{C} = \mathrm{Coker}(f)$.

**Exercise 1.4.** Let $X$ be a smooth manifold. Show that

$$0 \to \mathbb{Z}_X \xrightarrow{i} \mathcal{E}_{\mathbb{C}} \xrightarrow{\exp} \mathcal{E}^*_{\mathbb{C}} \to 0$$

is an exact sequence of sheaves, where $i$ is the natural inclusion.

## §2. Čech Cohomology, a brief Introduction.

***Basic Definitions***. Let $X$ be a topological space, $\mathcal{A}$ be a sheaf (of abelian groups) on $X$, and let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open cover of $X$, *i.e.* each $U_\alpha$ is an open subset of $X$ and $X = \cup_{\alpha \in I} U_\alpha$.

*In the sequel, all covers will be supposed locally finite, which means that each compact subset of $X$ meets only a finite number of $U_\alpha$.*

The group of $q$-**cochains** of $\mathcal{U}$ with values in $\mathcal{A}$, which will be denoted by $\mathcal{C}^q(\mathcal{U},\mathcal{A})$ is

$$\mathcal{C}^q(\mathcal{U},\mathcal{A}) = \prod_{(\alpha_0,\dots,\alpha_q)\in I^{q+1}} \mathcal{A}(U_{\alpha_0\dots\alpha_q})$$

where $U_{\alpha_0\dots\alpha_q}$ is equal to $U_{\alpha_0}\cap\dots\cap U_{\alpha_q}$.

Now there is a **coboundary map** $d^q : \mathcal{C}^q(\mathcal{U},\mathcal{A}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{A})$; for $c \in \mathcal{C}^q(\mathcal{U},\mathcal{A})$, $c' = d^q c$ is defined by the following formula:

$$c'_{\alpha_0\dots\alpha_{q+1}} = \sum_{j=0}^{q+1}(-1)^j (c_{\alpha_0\dots\hat{\alpha_j}\dots\alpha_{q+1}})_{|U_{\alpha_0\dots\alpha_{q+1}}}.$$

It is a classical and easy exercise to show that $d^{q+1}\circ d^q = 0$. Therefore, we can define $\check{H}^q(\mathcal{U},\mathcal{A})$ as being the **$q$-th Čech cohomology group of $\mathcal{A}$ with respect to the cover $\mathcal{U}$** by

$$\check{H}^q(\mathcal{U},\mathcal{A}) = \mathrm{Ker}(d^q : \mathcal{C}^q(\mathcal{U},\mathcal{A}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{A}))/\mathrm{Im}(d^{q-1} : \mathcal{C}^{q-1}(\mathcal{U},\mathcal{A}) \to \mathcal{C}^q(\mathcal{U},\mathcal{A})).$$

Usually, the elements in $\mathrm{Ker}(d^q : \mathcal{C}^q(\mathcal{U},\mathcal{A}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{A}))$ are called $q$-**cocycles** and the elements in $\mathrm{Im}(d^{q-1} : \mathcal{C}^{q-1}(\mathcal{U},\mathcal{A}) \to \mathcal{C}^q(\mathcal{U},\mathcal{A}))$ are called $q$-**coboundaries**.

The simplest group to understand is $\check{H}^0(\mathcal{U},\mathcal{A})$:

**Lemma 2.1.** *We have $\check{H}^0(\mathcal{U},\mathcal{A}) = \mathcal{A}(X)$.*

**Proof.** Let $c = (c_\alpha)_{\alpha\in I}$ be a 0-cochain: it means that for any $\alpha,\beta$, we have $c'_{\alpha\beta} := (c_\alpha)_{|U_{\alpha\beta}} - (c_\beta)_{|U_{\alpha\beta}} = 0$ hence by the property (R1) satisfied by $\mathcal{A}$, there is a section $s$ in $\mathcal{A}(X)$ such that $s_{|U_\alpha} = c_\alpha$ for any $\alpha$ and this section is unique since $\mathcal{A}$ satisfies property (R2). □

Finally, let us remark that if $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ and $\mathcal{U}' = (U_\beta)_{\beta\in J}$ are two open covers of $X$, with $\mathcal{U}'$ being finer than $\mathcal{U}$ (*i.e* every open subset $U'_\beta$ of $\mathcal{U}'$ is contained in a open subset $U_{\lambda(\beta)}$ of $\mathcal{U}$ for a certain map $\lambda : J \to I$), then there is a natural map

$$\lambda_q : \mathcal{C}^q(\mathcal{U},\mathcal{A}) \to \mathcal{C}^q(\mathcal{U}',\mathcal{A})$$

defined by $\lambda_q(c)_{\beta_0\dots\beta_q} = c_{\lambda(\beta_0)\dots\lambda(\beta_q)}$. This map induces a map from $\check{H}^q(\mathcal{U},\mathcal{A})$ to $\check{H}^q(\mathcal{U}',\mathcal{A})$ which depends only on $\mathcal{U}$ and $\mathcal{U}'$ (*i.e.* which doesn't depend on $\lambda$).

Now, we denote by $\check{H}^q(X,\mathcal{A})$ the **$q$-th Čech cohomology group** of $\mathcal{A}$ as being the limit of the cohomology groups $\check{H}^q(\mathcal{U},\mathcal{A})$ when the covers $\mathcal{U}$ become finer and finer.

***Interpretation of the First Cohomology Group***. We will now understand $\check{H}^1(\mathcal{U},\mathcal{A})$: let $f : \mathcal{A} \to \mathcal{B}$ be a map of sheaves on $X$. Then $f$ induces a natural map $f_X : \mathcal{A}(X) \to \mathcal{B}(X)$ between groups of global sections.

**Exercise 2.1.** Show that $f$ injective implies that $f_X$ is injective but $f$ surjective doesn't imply that $f_X$ is surjective in general.

Suppose in the sequel that $f : \mathcal{A} \to \mathcal{B}$ is a surjective map of sheaves on $X$ and let $t \in \mathcal{B}(X)$. For any $x \in X$, since $f$ is surjective, there is a small open subset $U_x$ containing $x$ and $s_x \in \mathcal{A}(U_x)$ such that $f_{U_x}(s_x) = t_{|U_x}$. Let $\mathcal{U}'$ the open cover defined by the $U_x$ when $x$ varies in $X$. If $X$ is paracompact, we can extract from $\mathcal{U}'$ a locally finite cover $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ and let $(s_\alpha)_{\alpha\in I}$ be the corresponding collection of sections such that $f_{U_\alpha}(s_\alpha) = t_{|U_\alpha}$ .

Now, for $\alpha$ and $\beta$ in $I$, let $z_{\alpha\beta} = (s_\alpha)_{|U_{\alpha\beta}} - (s_\beta)_{|U_{\alpha\beta}}$. and define $\mathcal{F} = \mathrm{Ker}(f)$. Then the collection of the $z_{\alpha\beta}$ clearly defines a 1-cocycle in $\mathcal{C}^1(\mathcal{U}, \mathcal{F})$. Therefore $t$ defines naturally an element $\xi(t)$ in $\check{H}^1(\mathcal{U}, \mathcal{F})$ and the following easy exercice show that $\xi(t)$ is exactly the obstruction for $t$ being in the image of $f_X$:

**Exercise 2.2.** With the previous notations, show that there exists $s \in \mathcal{A}(X)$ such that $f_X(s) = t$ if and only if $\xi(t) = 0$. In particular, if $\check{H}^1(\mathcal{U}, \mathrm{Ker}(f)) = 0$, then $f_X$ is surjective.

**Exercise 2.3.** Let $\mathcal{U}$ be the open cover of $X = \mathbb{C} - \{0\}$ given by the two open subsets $U_1 = \mathbb{C} - (\mathbb{R}^+ \times \{0\})$ and $U_2 = \mathbb{C} - (\mathbb{R}^- \times \{0\})$. Compute $\check{H}^q(\mathcal{U}, \mathbb{Z}_X)$.

***Relative Cohomology***. For this section, we refer to [4].

Let $X$ be a topological space, $\mathcal{A}$ be a sheaf (of abelian groups) on $X$ and $Z$ be a closed subset of $X$.

We say that $(\mathcal{U},\mathcal{U}')$ is an open cover of $(X, X - Z)$ if $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is an open cover of $X$ and $\mathcal{U}' = (U_\alpha)_{\alpha \in I'}$ is an open cover of $X - Z$ with $I' \subset I$.

The group of $q$-**cochains** of $(\mathcal{U},\mathcal{U}')$ with values in $\mathcal{A}$, which will be denoted by $\mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A})$ is

$$\mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) = \prod_{(\alpha_0,\ldots,\alpha_q) \in I^{q+1}} \Gamma'(U_{\alpha_0\ldots\alpha_q}, \mathcal{A})$$

where $\Gamma'(U_{\alpha_0\ldots\alpha_q}, \mathcal{A}) = \{0\}$ if $(\alpha_0, \ldots, \alpha_q) \in (I')^{q+1}$ and $\Gamma'(U_{\alpha_0\ldots\alpha_q}, \mathcal{A}) = \mathcal{A}(U_{\alpha_0\ldots\alpha_q})$ if not.

Now, there is a coboundary map $d^q : \mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{U}', \mathcal{A})$ defined by the same formula as above, and we define as before

$$\check{H}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) = \frac{\mathrm{Ker}(d^q : \mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{U}', \mathcal{A}))}{\mathrm{Im}(d^{q-1} : \mathcal{C}^{q-1}(\mathcal{U},\mathcal{U}', \mathcal{A}) \to \mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A}))}.$$

Finally we denote by $\check{H}^q(X, X - Z, \mathcal{A})$ the $q$-**th relative Čech cohomology group** of $\mathcal{A}$ as being the limit of the cohomology groups $\check{H}^q(\mathcal{U},\mathcal{U}', \mathcal{A})$ when the covers $(\mathcal{U},\mathcal{U}')$ become finer and finer.

If $s \in \mathcal{A}(U)$, where $U$ is an open subset of $X$, the **support** of $s$, denoted by $\mathrm{supp}(s)$, is

$$\mathrm{supp}(s) = \{x \in U \mid s_x \neq 0\},$$

where $s_x$ is the germ of $s$ at $x$, which means that $x \notin \mathrm{supp}(s)$ if and only if there is an open neighbourhood $V \in U$ of $x$ such that $s_{|V} = 0$. By definition, $\mathrm{supp}(s)$ is closed in $U$.

Now, for any open subset $U$ of $X$, we denote by $\Gamma_Z(U, \mathcal{A})$ the subgroup of $\mathcal{A}(U)$ consisting of sections $s$ such that $\mathrm{supp}(s) \subset U \cap Z$.

**Exercise 2.4.** Let $X$ be a topological space, $\mathcal{A}$ be a sheaf (of abelian groups) on $X$, $Z$ be a closed subset of $X$ and $(\mathcal{U},\mathcal{U}')$ be an open cover of $(X, X - Z)$. Show that $\check{H}^0(\mathcal{U},\mathcal{U}', \mathcal{A}) = \Gamma_Z(X, \mathcal{A})$.

The relative cohomology groups are usually computed with the help of an exact sequence we will know describe. Let us first remark that for any $q$, there is an exact sequence of groups:

$$0 \to \mathcal{C}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) \overset{i_q}{\to} \mathcal{C}^q(\mathcal{U}, \mathcal{A}) \overset{r_q}{\to} \mathcal{C}^q(\mathcal{U}', \mathcal{A}) \to 0.$$

Since $i_q$ and $r_q$ commute with the coboundary maps, it induces an exact sequence (exercise for the reader):

$$\check{H}^q(\mathcal{U},\mathcal{U}', \mathcal{A}) \overset{i_{*q}}{\to} \check{H}^q(\mathcal{U}, \mathcal{A}) \overset{r_{*q}}{\to} \check{H}^q(\mathcal{U}', \mathcal{A}).$$

Let us now construct a map

$$\partial^q : \check{H}^q(\mathcal{U}', \mathcal{A}) \to \check{H}^{q+1}(\mathcal{U}, \mathcal{U}', \mathcal{A}).$$

Let $w \in \mathcal{C}^q(\mathcal{U}', \mathcal{A})$ be a $q$-cocycle representing a cohomology class $\bar{w} \in \check{H}^q(\mathcal{U}', \mathcal{A})$. Since $r_q$ is surjective, there is $v \in \mathcal{C}^q(\mathcal{U}, \mathcal{A})$ such that $r_q(v) = w$. Now $d^q(v) \in \mathrm{Ker}(r_{q+1})$ since $d^q(w) = 0$. Hence, there is a (unique) $u \in \mathcal{C}^{q+1}(\mathcal{U}, \mathcal{U}', \mathcal{A})$ such that $i_{q+1}(u) = d^q(v)$ and $i_{q+2}(d^{q+1}(u)) = d^{q+1} \circ d^q(v) = 0$ hence $d^{q+1}(u) = 0$, and we define $\partial^q(\bar{w}) = \bar{u} \in \check{H}^{q+1}(\mathcal{U}, \mathcal{U}', \mathcal{A})$.

**Exercise 2.5.**
a. Check the details in the previous construction.
b. Show that the following sequence is exact:

$$\check{H}^q(\mathcal{U}, \mathcal{A}) \stackrel{r_{*q}}{\to} \check{H}^q(\mathcal{U}', \mathcal{A}) \stackrel{\partial^q}{\to} \check{H}^{q+1}(\mathcal{U}, \mathcal{U}', \mathcal{A}) \stackrel{i_{*(q+1)}}{\to} \check{H}^{q+1}(\mathcal{U}, \mathcal{A}).$$

By taking the limit on covers, we get the following fundamental exact sequence, which relates the relative cohomology groups with the usual ones:

$$\cdots \to \check{H}^q(X, X - Z, \mathcal{A}) \to \check{H}^q(X, \mathcal{A}) \to \check{H}^q(X - Z, \mathcal{A}) \to \check{H}^{q+1}(X, X - Z, \mathcal{A}) \to \cdots.$$

***Exact Sequences and Cohomology***. Let $0 \to \mathcal{A} \stackrel{f}{\to} \mathcal{B} \stackrel{g}{\to} \mathcal{C} \to 0$ be an exact sequence of sheaves on $X$. Then for each $q \geq 0$, there are natural maps

$$\check{H}^q(X, \mathcal{A}) \stackrel{f_{*q}}{\to} \check{H}^q(X, \mathcal{B}) \stackrel{g_{*q}}{\to} \check{H}^q(X, \mathcal{C})$$

and also a map (constructed as in the previous section)

$$\partial^q : \check{H}^q(X, \mathcal{C}) \to \check{H}^{q+1}(X, \mathcal{A})$$

fitting all together into a **long exact sequence of cohomology groups**:

$$\cdots \to \check{H}^{q-1}(X, \mathcal{C}) \stackrel{\partial^{q-1}}{\to} \check{H}^q(X, \mathcal{A}) \stackrel{f_{*q}}{\to} \check{H}^q(X, \mathcal{B}) \stackrel{g_{*q}}{\to} \check{H}^q(X, \mathcal{C}) \stackrel{\partial^q}{\to} \check{H}^{q+1}(X, \mathcal{A}) \to \cdots.$$

The following example motivates our quest of vanishing theorems: let $L$ be a line bundle on a smooth algebraic variety $X$ and let $D$ be smooth divisor. Let us define by $\mathcal{O}_D(L)$ the following sheaf on $X$: $\mathcal{O}_D(L)(U) = 0$ if $U \cap D = \emptyset$ and $\mathcal{O}_D(L)(U) = H^0(U \cap D, L_{|U \cap D})$ if $U \cap D \neq \emptyset$. Then the restriction maps $H^0(U, L) \to H^0(U \cap D, L_{|U \cap D})$ induce a surjective map of sheaves $\mathcal{O}_X(L) \to \mathcal{O}_D(L)$, the kernel of which being the sheaf of germs of sections of $L$ vanishing along $D$, *i.e.* $\mathcal{O}_X(L - D)$. Now, the exact sequence of sheaves

$$0 \to \mathcal{O}_X(L - D) \to \mathcal{O}_X(L) \to \mathcal{O}_D(L) \to 0$$

gives the long sequence of cohomology

$$\cdots \to \check{H}^0(X, \mathcal{O}_X(L)) \to \check{H}^0(D, \mathcal{O}_D(L)) \to \check{H}^1(X, \mathcal{O}_X(L - D)) \to \cdots.$$

In particular, we get that if $\check{H}^1(X, \mathcal{O}_X(L - D)) = 0$ any global section (on $D$) of the restriction of the line bundle $L$ to $D$ is the restriction of a global section of $L$ on $X$.

***Computational Tools***. The main tools to compute cohomology groups are:
- long exact sequences of cohomology groups we met in the previous section,

- vanishing results, of topological or algebraic nature,
- Leray's theorems on acyclic covers.

We will state them without proofs in the form they will be used later on, the references are again [2], [3] and [4] among many others.

**Serre's Theorem 2.2.** *Let $X$ be an affine algebraic variety over $\mathbb{C}$ and $L$ be a line bundle on $X$. Then*

$$\forall q \geq 1 \quad \check{H}^q(X, \mathcal{O}_X(L)) = 0.$$

**Proposition 2.2.** *Let $X$ be a contractible topological space. Then*

$$\forall q \geq 1 \quad \check{H}^q(X, \mathbb{C}_X) = 0.$$

This last result will be used in case $X$ is an open convex subset of $N_{\mathbb{R}}$.

Let $X$ be a paracompact topological space, $\mathcal{A}$ be a sheaf on $X$ and $\mathcal{U}$ be an open cover of $X$.

**Definition 2.3.** *The cover $\mathcal{U}$ is* **acyclic** *for the sheaf $\mathcal{A}$ if*

$$\forall q \geq 1 \ \forall (\alpha_0, \ldots, \alpha_l) \in I^{l+1} \quad \check{H}^q(U_{\alpha_0 \ldots \alpha_l}, \mathcal{A}) = 0.$$

We now state Leray's Theorem, which will allow us to compute cohomology groups without taking the limit on all covers.

**Leray's Theorem 2.3.** *Let $X$ be a paracompact topological space, $\mathcal{A}$ be a sheaf on $X$.*

a. *If $\mathcal{U}$ is an open cover of $X$, acyclic for the sheaf $\mathcal{A}$, then for any $q \geq 0$,*

$$\check{H}^q(\mathcal{U}, \mathcal{A}) \simeq \check{H}^q(X, \mathcal{A}).$$

b. *If $Z$ is a closed subset of $X$ and $(\mathcal{U}, \mathcal{U}')$ is an open cover of $(X, X - Z)$ with $\mathcal{U}$ being acyclic for the sheaf $\mathcal{A}$, then for any $q \geq 0$,*

$$\check{H}^q(\mathcal{U}, \mathcal{U}', \mathcal{A}) \simeq \check{H}^q(X, X - Z, \mathcal{A}).$$

Another Leray type theorem will allow us to identify the relative cohomology groups with the cohomology of the complex of sections with support: let $X$ be a paracompact topological space, $\mathcal{A}$ be a sheaf on $X$ and $(\mathcal{U}, \mathcal{U}')$ be an open cover of $(X, X - Z)$. Let us define the complex of sections with support in $Z$:

$$\mathcal{C}^q_Z(\mathcal{U}, \mathcal{A}) = \prod_{(\alpha_0, \ldots, \alpha_q) \in I^{q+1}} \Gamma_Z(U_{\alpha_0 \ldots \alpha_q}, \mathcal{A})$$

with the usual restriction maps. Let $\check{\mathcal{H}}^q_Z(\mathcal{U}, \mathcal{A})$ be the $q$-th cohomology group of this complex.

We are now ready to state:

**Leray's Theorem 2.4.** *Let $X$ be paracompact toplogical space, $\mathcal{A}$ be a sheaf on $X$ and $(\mathcal{U}, \mathcal{U}')$ be an open cover of $(X, X - Z)$ with $\mathcal{U} = (U_\alpha)_{\alpha \in I}$. Suppose that*

$$\forall q \geq 1 \ \forall (\alpha_0, \ldots, \alpha_l) \in I^{l+1} \quad \check{H}^q(U_{\alpha_0 \ldots \alpha_l}, U_{\alpha_0 \ldots \alpha_l} - Z, \mathcal{A}) = 0.$$

*Then, for any* $q \geq 0$*:*

$$\check{\mathcal{H}}^q_Z(\mathcal{U},\mathcal{A}) \simeq \check{H}^q(\mathcal{U},\mathcal{U}',\mathcal{A}).$$

**Exercise 2.6.** Compute the relative cohomology groups $\check{H}^q(\mathbb{R}^2, \mathbb{R}^2 - \{0\}, \mathbb{Z})$.

Let us mention here that Čech cohomology compares well with other known cohomology theory, for example the $\check{H}^q(X, \mathcal{O}_X(L))$, when $X$ is a complex algebraic variety and $L$ is a line bundle on $X$ are isomorphic to the Dolbeault cohomology groups associated to the underlying complex analytic structure and the $\check{H}^q(X, \mathbb{C}_X)$, when $X$ is a topological locally contractible and paracompact space, are isomorphic to the singular cohomology groups with complex coefficients.

## §3. The Toric Case.

As usual, we refer to [1] and [5].

Let $X$ be a toric variety, $\Delta$ be the fan defining $X$ and $D_\psi$ be an invariant $T$-Cartier divisor. As usual, $D_\psi = -\sum_{\rho\in\Delta(1)} \psi(v(\rho))D_\rho$ where $\psi$ is a support function on $\Delta$.

On $X$, we have the natural cover $\mathcal{U} = (U_\sigma)_{\sigma\in\Delta}$, each $U_\sigma$ being a toric affine variety as well as an invariant open subset of $X$. By Serre's Theorem, the cover $\mathcal{U}$ is acyclic for the sheaf $\mathcal{O}_X(D_\psi)$, therefore we get, by Leray's Theorem 2.3:

$$\forall q \geq 0 \quad \check{H}^q(X, \mathcal{O}_X(D_\psi)) \simeq \check{H}^q(\mathcal{U}, \mathcal{O}_X(D_\psi)).$$

Moreover, since $D_\psi$ and each $U_\sigma$ are invariant, we already mentioned that $T$ acts algebraically and linearly on each $H^0(U_\sigma, \mathcal{O}_X(D_\psi)_{|U_\sigma})$, therefore on the group of cochains. Hence we get a decomposition into eigenspaces:

$$\mathcal{C}^q(\mathcal{U}, \mathcal{O}_X(D_\psi)) = \bigoplus_{m\in M} \mathcal{C}^q(\mathcal{U}, \mathcal{O}_X(D_\psi))_m$$

where $\mathcal{C}^q(\mathcal{U}, \mathcal{O}_X(D_\psi))_m$ denotes the set of $q$-cochains which are eigenvectors for the character $m$. Since the coboundary map $d^q$ obviously respects this decomposition, we get a decomposition of the cohomology groups:

$$\forall q \geq 0 \quad \check{H}^q(X, \mathcal{O}_X(D_\psi)) = \bigoplus_{m\in M} \check{H}^q(X, \mathcal{O}_X(D_\psi))_m.$$

Now let us fix $m \in M$ and define:

$$Z(\psi, m) = \{n \in |\Delta| \mid \langle m, n\rangle \geq \psi(n)\}.$$

Then, $Z(\psi, m)$ is obviously a closed cone in $|\Delta|$ and we have seen in Lecture 11 that $\chi^m$ is a global section of $\mathcal{O}_X(D_\psi)$ if and only if $Z(\psi, m) = |\Delta|$. Moreover, let us recall that for any cone $\sigma$ in $\Delta$, $\psi$ is defined on $\sigma$ by an element $l_\sigma$. Therefore

$$|\sigma| - Z(\psi, m) = \{n \in |\sigma| \mid \langle m - l_\sigma, n\rangle < 0\}$$

is an open convex subset of $|\sigma|$, where $|\sigma|$ denotes the closed strictly convex cone in $N_{\mathbb{R}}$ spanned by $\sigma$.

We are now ready to prove the following result, which is due to Demazure:

**Theorem 3.1.** *With the previous notations:*

$$\check{H}^q(X,\mathcal{O}_X(D_\psi))_m = \check{H}^q(|\Delta|,|\Delta|-Z(\psi,m),\mathbb{C}_{|\Delta|}).$$

**Proof.** Let us first remark that the open cover $\mathcal{U}=(U_\sigma)_{\sigma\in\Delta}$ is finite and we have by definition:

$$\mathcal{C}^q(\mathcal{U},\mathcal{O}_X(D_\psi)) = \bigoplus_{(\sigma_0,\dots,\sigma_q)\in\Delta^{q+1}} H^0(U_{\sigma_0\cap\cdots\cap\sigma_q},\mathcal{O}_X(D_\psi)_{|U_{\sigma_0\cap\cdots\cap\sigma_q}}),$$

(one should remark that $U_{\sigma_0\cap\cdots\cap\sigma_q}=U_{\sigma_0}\cap\cdots\cap U_{\sigma_q}$).

But since $\chi^m\in H^0(U_{\sigma_0\cap\cdots\cap\sigma_q},\mathcal{O}_X(D_\psi)_{|U_{\sigma_0\cap\cdots\cap\sigma_q}})$ if and only if for any $n\in\sigma_0\cap\cdots\cap\sigma_q$, $\langle m,n\rangle\geq\psi(n)$ (see Lecture 11), we get that $\chi^m\in H^0(U_{\sigma_0\cap\cdots\cap\sigma_q},\mathcal{O}_X(D_\psi)_{|U_{\sigma_0\cap\cdots\cap\sigma_q}})$ if and only if $\sigma_0\cap\cdots\cap\sigma_q\subset Z(\psi,m)$. Therefore,

$$\mathcal{C}^q(\mathcal{U},\mathcal{O}_X(D_\psi))_m = \bigoplus_{(\sigma_0,\dots,\sigma_q)\in\Delta^{q+1}} V(\sigma_0,\dots,\sigma_q)$$

where $V(\sigma_0,\dots,\sigma_q)=\mathbb{C}\chi^m$ if $\sigma_0\cap\cdots\cap\sigma_q$ is contained in $Z(\psi,m)$ and $\{0\}$ if not.

Now, consider the topological space $|\Delta|$ with the closed subspace $Z(\psi,m)$, and for small $\eta>0$, let $Z(\psi,m)_\eta$ be the closed $\eta$-neighbourhood of $Z(\psi,m)$ in $|\Delta|$. Let us moreover consider the following finite open cover of $|\Delta|$: $\mathcal{V}=(|\sigma|_\varepsilon)_{\sigma\in\Delta}\cup(|\sigma|_\varepsilon-Z(\psi,m)_\eta)_{\sigma\in\Delta}$; and the following finite open cover of $|\Delta|-Z(\psi,m)_\eta$: $\mathcal{V}'=(|\sigma|_\varepsilon-Z(\psi,m)_\eta)_{\sigma\in\Delta}$. By construction, $(\mathcal{V},\mathcal{V}')$ is an open cover of $(|\Delta|,|\Delta|-Z(\psi,m)_\eta)$.

Since each $|\sigma|-Z(\psi,m)$ and each $|\sigma|$ are open and convex, each $|\sigma|_\varepsilon-Z(\psi,m)_\eta$ and each $|\sigma|_\varepsilon$ are also open and convex. It implies that each finite intersection of elements of $\mathcal{V}$ is open and convex. Therefore by Proposition 2.2, $\mathcal{V}$ is acyclic for the sheaf $\mathbb{C}_{|\Delta|}$, hence, for any $q\geq0$, we get applying Leray's Theorem 2.3:

$$\check{H}^q(\mathcal{V},\mathcal{V}',\mathbb{C}_{|\Delta|})\simeq\check{H}^q(|\Delta|,|\Delta|-Z(\psi,m)_\eta,\mathbb{C}_{|\Delta|}).$$

Moreover, if $\tau$ is any finite intersection of elements of $\mathcal{V}$, the fundamental exact sequence after Exercise 2.5 and Proposition 2.2 show that for any $q\geq1$

$$\check{H}^q(\tau,\tau-Z(\psi,m)_\eta,\mathbb{C}_{|\Delta|})=0.$$

Therefore, by Leray's Theorem 2.4, we deduce that for any $q\geq0$:

$$\check{\mathcal{H}}^q_{Z(\psi,m)_\eta}(\mathcal{V},\mathbb{C}_{|\Delta|})\simeq\check{H}^q(\mathcal{V},\mathcal{V}',\mathbb{C}_{|\Delta|})\simeq\check{H}^q(|\Delta|,|\Delta|-Z(\psi,m)_\eta,\mathbb{C}_{|\Delta|}).$$

It suffices therefore to compute the complex $\mathcal{C}^\bullet_{Z(\psi,m)_\eta}(\mathcal{V},\mathbb{C}_{|\Delta|})$. But, if $\tau$ is any non empty finite intersection of elements of $\mathcal{V}$, since $\tau$ is convex, we get that $\Gamma_{Z(\psi,m)_\eta}(\tau,\mathbb{C}_{|\Delta|})=\mathbb{C}$ if $\tau\subset Z(\psi,m)_\eta$ and $\{0\}$ if not. Therefore, if one of the $|\sigma|_\varepsilon-Z(\psi,m)_\eta$ (*i.e.* one of the open subsets of the cover $\mathcal{V}'$) appears in the intersection defining $\tau$, then $\Gamma_{Z(\psi,m)_\eta}(\tau,\mathbb{C}_{|\Delta|})=\{0\}$. It follows that the complex $\mathcal{C}^\bullet_Z(\mathcal{V},\mathbb{C}_{|\Delta|})$ is isomorphic to the complex of sections with support in $Z(\psi,m)_\eta$ computed with the open cover $\mathcal{U}'=(|\sigma|_\varepsilon)_{\sigma\in\Delta}$, for which the group of $q$-cochains is:

$$\mathcal{C}^q_{Z(\psi,m)_\eta}(\mathcal{U}',\mathbb{C}_{|\Delta|}) = \bigoplus_{(\sigma_0,\dots,\sigma_q)\in\Delta^{q+1}}\Gamma_{Z(\psi,m)_\eta}(|\sigma_0|_\varepsilon\cap\cdots\cap|\sigma_q|_\varepsilon,\mathbb{C}_{|\Delta|}),$$

*i.e.*

$$\mathcal{C}^q_{Z(\psi,m)_\eta}(\mathcal{U}',\mathbb{C}_{|\Delta|}) = \bigoplus_{(\sigma_0,\dots,\sigma_q)\in\Delta^{q+1}} W(\sigma_0,\dots,\sigma_q)$$

where $W(\sigma_0,\dots,\sigma_q) = \mathbb{C}$ if $\sigma_0 \cap \dots \cap \sigma_q \subset Z(\psi,m)$ and $\{0\}$ if not.

Comparing the two descriptions, and since the boundary map $d^q$ is compatible with these decompositions, we get that for any small $\eta > 0$,

$$\check{H}^q(X, \mathcal{O}_X(D_\psi))_m = \check{H}^q_{Z(\psi,m)_\eta}(|\Delta|, \mathbb{C}_{|\Delta|}).$$

Since for any small $\eta > 0$, the pairs $(|\Delta|, Z(\psi,m))$ and $(|\Delta|, Z(\psi,m)_\eta)$ are homotopically equivalent, the result is proved. □

**Exercise 3.1.** For any $m$ and $n$, compute the cohomology groups of $\mathcal{O}_{\mathbb{P}^n}(m)$ using Theorem 3.1.

## §4. Vanishing Theorems.

It is a general philosophy in complex algebraic or analytic geometry that when a line bundle tends to be positive, it tends to have a lot of sections (*i.e.* a "big" $\check{H}^0$) and few cohomology (*i.e.* "small" $\check{H}^q$ for $q \geq 1$). For example, there is the famous Kodaira's Theorem:

**Kodaira's Theorem 4.1.** *Let $X$ be a smooth complete algebraic variety over $\mathbb{C}$ and $L$ be an ample line bundle on $X$. Then for any $q \geq 1$:*

$$\check{H}^q(X, \mathcal{O}(L + K_X)) = 0.$$

In the toric case, we have a very strong result:

**Theorem 4.2.** *Let $X$ be a toric variety defined by a fan $\Delta$, $D_\psi$ be a $T$-invariant Cartier divisor defined by a support function $\psi$. Suppose that $|\Delta|$ is convex and that $D_\psi$ is globally generated, then for any $q \geq 1$:*

$$\check{H}^q(X, \mathcal{O}(D_\psi)) = 0.$$

**Proof.** Recall that $D_\psi$ is globally generated if and only if $\psi$ is convex. Therefore, for any $m \in M$, $|\Delta| - Z(\psi,m)$ is either empty or convex. Since $|\Delta|$ is also convex, the long exact sequence

$$\check{H}^{q-1}(|\Delta| - Z(\psi,m), \mathbb{C}_{|\Delta|}) \to \check{H}^q(|\Delta|, |\Delta| - Z(\psi,m), \mathbb{C}_{|\Delta|}) \to \check{H}^q(|\Delta|, \mathbb{C}_{|\Delta|}) \to \cdots$$

and Proposition 2.2 show that for any $q \geq 1$:

$$\check{H}^q(|\Delta|, |\Delta| - Z(\psi,m), \mathbb{C}_{|\Delta|}) = 0$$

therefore Theorem 3.1 gives the result. □

Since the trivial line bundle is globally generated on any toric variety, we get:

**Corollary 4.3.** *Let $X$ be a toric variety defined by a fan $\Delta$ with $|\Delta|$ convex. Then for any $q \geq 1$:*

$$\check{H}^q(X, \mathcal{O}_X) = 0.$$

Finally, we have the following consequence:

**Corollary 4.4.** *Let $X$ be a toric variety defined by a fan $\Delta$, $D_\psi$ be a $T$-invariant Cartier divisor defined by a support function $\psi$. Suppose that $|\Delta|$ is convex and $\mathcal{O}(D_\psi)$ ample. Then for any $T$-invariant smooth divisor $D$ of $X$, there is an integer $k_0$ such that for any $k \geq k_0$, the restriction map:*

$$\check{H}^0(X, \mathcal{O}(kD_\psi)) \to \check{H}^0(D, \mathcal{O}(kD_\psi)_{|D})$$

*is surjective.*

**Proof.** It suffices to prove that $\check{H}^1(X, \mathcal{O}(kD_\psi - D)) = 0$. If $\chi$ is a support function defining $D$, since $\psi$ is strictly convex, there exists $k_0$ such that for any $k \geq k_0$, $k\psi - \chi$ is convex. Corollary 4.3 gives the result. □

We end this lecture by the following exercise, for people having basic knowledge of direct images of sheaves:

**Exercise 4.1.** Let $\Delta'$ be a refinement of a fan $\Delta$, giving a birational proper map $f : X' \to X$. Show that $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and $R^i f_*(\mathcal{O}_{X'}) = 0$ for all $i > 0$. *(Hint: since the assertion is local, one can suppose that $\Delta$ is a single cone, therefore $|\Delta'|$ is convex. Then apply Corollary 4.3.)*

By taking $X'$ to be a resolution of singularities (Lecture 6), this exercise says that every toric variety $X$ has *rational singularities.*

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