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Abstract

In the paper we describe the construction of a smooth complete toric surface from an exceptional collection of line bundles on a smooth rational surface. In an attempt to understand the interrelation between these two surfaces we investigate the connection between an exceptional collection and the underlying variety. We recall the construction of a quiver and in a toric case give an explicit algorithm assigning to every point of a variety a module over it.

Słowa kluczowe

exceptional collections, quivers, toric varieties

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic geometry
14F (Co)homology theory
14F05 Sheaves, derived categories of sheaves and related constructions

Tytuł pracy w języku angielskim

Exceptional collections on toric varieties

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Introduction

Derived categories of coherent sheaves on algebraic varieties have been an object of vivid interest since their discovery in 1960's. An important question is how much geometry of the underlying variety do they carry. In [BO] Bondal and Orlov prove a theorem which reconstructs a variety from the derived category provided that its canonical bundle is either ample or anti-ample. A full exceptional collection is a tool that makes it possible to comprehend such an overwhelming category. If it exists, it is a sort of basis which reduces the study to simple combinatorial objects – quivers. The following question arises: how the variety itself can be seen by means of the quiver?

We address this question for smooth complete toric varieties and exceptional collections of line bundles on them. This paper describes an explicit algorithm that assigns a module over the quiver to every point of a variety. The idea behind this construction is to to represent every point x by the skyscraper $\mathbb{C}_x$. It is a coherent sheaf so it leads to a module over the quiver which can be described by homogeneous coordinates of x.

The motivation for this work was a paper by Hille and Perling [HP] about toric systems. It gives an algorithm assigning to every rational smooth surface X, together with an exceptional collection $\mathcal{E}$ of line bundles, a smooth complete toric surface $Y(\mathcal{E})$. Alexey Bondal's conjecture states that then $Y(\mathcal{E})$ is a degeneration of X. If one could find a degeneration encoded in the exceptional collection on X (which gives an exceptional collection on $Y(\mathcal{E})$) it would mean that the derived category of coherent sheaves carries a lot of information the geometry of the variety itself.

An important step in finding such a degeneration should be some kind of reconstruction of a variety from the exceptional collection or a quiver it gives. The results of Craw and Smith [CS] and also of Bergman and Proudfoot [BP] realise X as a moduli space of modules over the quiver. However, these constructions rely either on finding an appropriate stability condition or on further assumptions about the exceptional collection. The algorithm presented in the present paper makes it easier to check whether the given stability condition is convenient.

The example we treat is $\mathbb{P}^2$ blown up in two different points. There exists an exceptional collection on it which leads to $\mathbb{P}^2$ blown up in two infinitely close points. These varieties will accompany us during the exploration of the theory.

This paper is divided into three parts. In Chapter 1 the notion of an exceptional collection is recalled and some facts about toric surfaces are stated. Then the construction of Hille and Perling is described. Finally some calculations are presented. Chapter 2 is devoted to quivers. It starts with the basic definitions and then calculations for the main example are done.

Chapter 3 describes a construction of homogeneous coordinates on toric varieties (see further [CLS]). These coordinates are finally used to assign a module over a quiver to every point of a variety. As an example we present the modules corresponding to points of $\mathbb{P}^2$ blown up in two different points and in two infinitely close points.

Notation

Throughout this paper X, Y and Z are always smooth complete algebraic varieties defined over $\mathbb{C}$. Coherent sheaves on X form an abelian category $\mathcal{C}oh(X)$ whose derived category is $\mathcal{D}^b(X) = \mathcal{D}^b(\mathcal{C}oh(X))$. For $\mathcal{E}$ and $\mathcal{F}$ coherent sheaves on X the group of homomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ is sometimes denoted by $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$. Moreover, RHom is the functor between derives categories, i.e. $\mathrm{RHom}(\mathcal{E},\mathcal{F})$ is a complex of abelian groups such that $\mathrm{H}^i(\mathrm{RHom}(\mathcal{E},\mathcal{F})) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$.

For a smooth variety X, $K_X \in \mathrm{Pic}(X)$ is its canonical class. For any divisor $D \in \mathrm{Pic}(X)$, $\chi(D) = \sum(-1)^i \mathrm{dim} H^i(X,\mathcal{O}(D))$ is the Euler characteristic of D.

For a projective plane $\mathbb{P}^2$, $H \in \mathrm{Pic}(\mathbb{P}^2)$ denotes the hyperplane class. If there is a sequence of blow-ups

$$X = X_t \xrightarrow{b_t} X_{t-1} \xrightarrow{b_{t-1}} \cdots \xrightarrow{b_1} X_1 \xrightarrow{b_0} \mathbb{P}^2,$$

then $R_i \in \mathrm{Pic}(X_i)$ denotes the exceptional divisor of the blow-up b_{i-1}. R_i's and H will be identified with their pullbacks in $\mathrm{Pic}(X)$ and thus will be treated as elements of the latter group.

If X is a toric variety then $T = (\mathbb{C}^*)^{\dim(X)}$ is the torus acting on it. $M = \mathrm{Hom}(T,\mathbb{C}^*)$ is the lattice of characters of T and $N = M\check{} = \mathrm{Hom}(\mathbb{C}^*,T)$. N and M are dual lattices so there is a natural pairing between them which will be denoted by $\langle \cdot , \cdot \rangle$. The variety X is determined by its fan $\Sigma \subset N$. Σ consists of cones σ. The one-dimensional cones are rays and the set of them is denoted by $\Sigma(1)$. Analogously the set of rays in a cone σ is $\sigma(1)$. On the other hand, the set of cones of maximal dimension is denoted by Σ_{max}. On X there are T-invariant divisors, they form an abelian group $\mathrm{Div}_T(X)$.

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Chapter 1

Exceptional collections and toric systems

1.1. Exceptional collections

Let X be an algebraic variety. The category $\mathcal{C}oh(X)$ of coherent sheaves on it is abelian. General theory, [GM, chapter III], gives a way to construct from any abelian category $\mathcal{A}$ a triangulated category $\mathcal{D}^b(\mathcal{A})$ – the bounded derived category. As stated before in the case of $\mathcal{D}^b(\mathcal{C}oh(X))$ it will be denoted by $\mathcal{D}^b(X)$.

Triangulated categories are very big and therefore hard to work with. Fortunately there exists some kind of basis – exceptional collections. In [B] Bondal introduces the following definitions.

Definition 1.1.1. *An* exceptional sheaf *is a coherent sheaf $\mathcal{E}$ on X such that* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) = \mathbb{C}$ *and* $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) = 0$ *for* $i > 0$.

Definition 1.1.2. An exceptional collection *is an ordered collection of exceptional sheaves* $(\mathcal{E}_0, \ldots, \mathcal{E}_n)$ *such that* $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) = 0$ *for all* k *and* $i > j$.

Definition 1.1.3. *An exceptional collection* $(\mathcal{E}_0, \ldots, \mathcal{E}_n)$ *is* full *if the smallest triangulated subcategory of* $\mathcal{D}^b(X)$ *containing it is* $\mathcal{D}^b(X)$.

Remark (cf. [RVdB])**.** The number of elements in a full exceptional collection of sheaves on X equals the rank of the Grothendieck group $K_0(X)$.

Example 1.1.4 (Exceptional collections on $\mathbb{P}^n$)**.** The projective space $\mathbb{P}^n$ can be viewed as a $\mathrm{Proj}(\mathrm{Sym}V)$, where $\mathrm{Sym}V$ is a symmetric power of a $(n+1)$-dimensional vector space V. The only non-zero cohomology groups of line bundles on $\mathbb{P}^n$ are $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \mathrm{Sym}^k(V^*)$ for $k \in \mathbb{N}$ and, by Serre duality, $\mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-k)) = \mathrm{Sym}^k(V)$. It follows that $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$ is an exceptional collection on $\mathbb{P}^n$. Note that $\mathrm{Sym}^0(V) = \mathbb{C}$.

The way the full exceptional collection generates $\mathcal{D}^b(X)$ is still complicated; one has to take all possible translations and cones. Nevertheless full exceptional collections are much simpler than the whole $\mathcal{D}^b(X)$ and much work was devoted to understanding them.

1.2. Toric surfaces

This section presents some facts about toric surfaces. More details and proofs can be found in [F2].

A smooth complete toric surface Y is determined by its fan, spanned by a collection of elements $v_1, \ldots, v_n \in N$. We enumerate v_i's counterclockwise and consider their indexes, i's, to be elements of $\mathbb{Z}/n\mathbb{Z}$. Then for every $i \in \mathbb{Z}/n\mathbb{Z}$, vectors v_i and v_{i+1} form an oriented basis of N. Moreover, for every such pair there exists no other v_k lying in the rational polyhedral cone generated by v_i and v_{i+1} in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. However, for every i there exists $a_i \in \mathbb{Z}$ such that $v_{i-1} + a_i v_i + v_{i+1} = 0$. The T-invariant divisors corresponding to v_i's are D_i's. Every D_i is isomorphic to $\mathbb{P}^1$ and the divisors D_i and D_j are either disjoint – when $|i-j| > 1$ or intersect transversely when $|i-j| = 1$. In the second case the intersection point is the fixed point of T-action associated to the cone spanned by v_i and v_j. Finally, the self intersection numbers are $D_i^2 = a_i$, where $v_{i-1} + a_i v_i + v_{i+1} = 0$.

Clearly, the integers a_i's allow to reconstruct v_i's up to an automorphism of N. However, not all sequences lead to a well-defined toric variety. The admissible ones are determined by the minimal model program for toric surfaces.

Theorem 1.2.1. *Every toric surface can be obtained by a finite sequence of equivariant blow-ups of $\mathbb{P}^2$ or some Hirzebruch surface $\mathbb{F}_a$.*

Equivariant means that the point blown up on Y is a fixed point of torus action. If $a_1, \ldots, a_n$ is the sequence of self-intersection numbers on Y and $\widetilde{Y}$ is a blow-up of Y in a point $x = D_i \cap D_{i+1}$ then the sequence for $\widetilde{Y}$ has the form $a_1, \ldots, a_{i-1}$, $a_i - 1$, -1, $a_{i+1} - 1$, $a_{i+2}, \ldots, a_n$, where i's are ordered cyclically. In this case the sequence for $\widetilde{Y}$ is called an *augmentation* of the sequence for Y. The sequence of self-intersection numbers for $\mathbb{P}^2$ is $1, 1, 1$ and for $\mathbb{F}_a$ it is $0, a, 0, -a$ and these two sequences determine all admissible ones. In particular, it implies

Proposition 1.2.2. *Let X be a smooth complete toric surface determined by self-intersection numbers $a_1, \ldots, a_n$. Then $\Sigma_{i=1}^n a_i = 12 - 3n$.*

1.3. Exceptional sequences and toric systems

In this section we present the algorithm of Hille and Perling described in their paper [HP].

Let X be a smooth, complete, rational surface and let $n-2$ be the rank of its Picard group. The Riemann-Roch theorem says that $\chi(D) = 1 + \frac{1}{2}D.(D - K_X)$ for any divisor $D \in \mathrm{Pic}(X)$. It follows that

$$\chi(D) + \chi(-D) = 2 + D^2,$$
$$\chi(D) - \chi(-D) = -K_X.D.$$

The following lemma will be useful later.

Lemma 1.3.1. *Let $D, E \in Pic(X)$ such that $\chi(-D) = \chi(-E) = 0$. Then*

(i) $\chi(D) = -K_X.D$;

(ii) $\chi(D) = D^2 + 2$;

(iii) $\chi(-D-E) = 0$ *iff* $D.E = 1$ *iff* $\chi(D) + \chi(E) = \chi(D+E)$.

Proof. The first two assertions follow from the Riemann-Roch theorem. In order to proof the last one it suffices to check that $\chi(-D-E) = +\frac{1}{2}(D+E).(D+E+K_X) = 1 + \frac{1}{2}D.(D+K_X) + 1 + \frac{1}{2}E.(E+K_X) - 1 + D.E = \chi(-D) + \chi(-E) - 1 + D.E$. For the last equivalence the proof is analogous. $\square$

For a smooth and complete X, let $E_1, \dots, E_n \in \mathrm{Pic}(X)$ be such divisors that $(\mathcal{O}_X(E_1), \dots, \mathcal{O}_X(E_n))$ is a full exceptional collection on X. Then, by definition, $\mathrm{H}^k(X, \mathcal{O}_X(E_i - E_j)) = \mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{O}_X(E_j), \mathcal{O}_X(E_i)) = 0$ for $k \geqslant 0$ and $j > i$. Let $A_i = E_{i+1} - E_i$ for $1 \leqslant i < n$ and $A_n = -K_X - \sum_{i=1}^{n-1} A_i = -K_X - E_n + E_1$. Then, for i's treated as elements of $\mathbb{Z}/n\mathbb{Z}$, the following holds

Lemma 1.3.2 (cf. [HP])**.** *(i)* $A_i.A_{i+1} = 1$ *for* $1 \leqslant i \leqslant n$;

(ii) $A_i.A_j = 0$ *for* $i \neq j$ *and* $\{i,j\} \neq \{k, k+1\}$ *for some* $1 \leqslant k \leqslant n$;

(iii) $\sum_{i=1}^{n} A_i = -K_X$.

Proof. The last part of the lemma follows from the definition of A_n.

It is easy to check that $\chi(-A_i) = 0$ for $1 \leqslant i \leqslant n$. For $i \neq n$ it follows immediately from definition of A_i's and for A_n one needs to use the equality $\chi(K_X + D) = \chi(-D)$. Thus in order to prove (i) and (ii) the Lemma 1.3.1 can be used.

For $n \notin \{i, i+1\}$ we have $\chi(-A_i - A_{i+1}) = \chi(E_i - E_{i+2}) = 0$. If $i = n-1$ then $\chi(-A_{n-1} - A_n) = \chi(E_{n-1} + K_X - E_1) = \chi(E_1 - E_n) = 0$ and an analogous equality holds for $i = n$. This proves part (i).

In order to prove (ii) for $|i-j| = 2$ it suffices to observe that $\chi(E_i - E_{i+2}) = \chi(E_{i+2} - E_{i+3}) = \chi(E_i - E_{i+3}) = 0$. Then, again by the lemma 1.3.1, $1 = (E_i - E_{i+2}).(E_{i+2} - E_{i+3}) = (-A_i - A_{i+1}).(-A_{i+2}) = A_i.A_{i+2} + 1$. It follows that for i such that $n \notin \{i, i+1, i+2\}$ $A_i.A_{i+2} = 0$. For the intersections $A_{n-2}.A_n$, $A_{n-1}.A_1$ and $A_2.A_n$ analogous tricks works

In order to complete the proof of (ii) one needs to show step by step that $A_i.A_j = 0$ for $|i-j| = 3, 4, \dots, \frac{n}{2}$. $\square$

Definition 1.3.3. *Let X be a smooth rational surface such that $rkPic(X) = n-2$. Then the set of n divisors on X is a* toric system *if it satisfies the conditions of Lemma 1.3.2.*

Let us see where such a name comes from. Consider a map $A : \mathrm{Pic}(X)^* \to \mathbb{Z}^n$ given by $A(D) = (A_1.D, \dots, A_n.D)$. It leads to a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)^* \xrightarrow{A} \mathbb{Z}^n \longrightarrow N \longrightarrow 0.$$

It turns out that $N = \mathbb{Z}^2$ and the images of the standard basis of $\mathbb{Z}^n$ under the quotient map define rays of a fan of a complete smooth toric surface Y. More precisely, the following holds

Theorem 1.3.4 (see also [HP], thm. 3.5)**.** *Let $A_1, \ldots, A_n \in Pic(X)$ be a toric system. Let N be as above and let $l_1, \ldots, l_n \in N$ be images of the standard basis of $\mathbb{Z}^n$. Then $N = \mathbb{Z}^2$ and $l_1, \ldots, l_n$ generate the fan of a smooth complete toric surface Y with T-invariant irreducible divisors $D_1, \ldots, D_n$ such that $D_i^2 = A_i^2$ for every $1 \leqslant i \leqslant n$. In particular, the lattices $Pic(X)$ and $Pic(Y)$ with the intersection forms can be identified.*

Proof. (after [HP]) As the rank of $\mathrm{Pic}(X)$ is $n-2$, for $n < 3$ there is nothing to prove. For $n = 3$ $\mathrm{Pic}(X) = \langle H \rangle$, where H is a positive generator, so $A_i = a_i H$. The condition *(i)* of Lemma 1.3.2 implies that all a_i's have to be either 1 or -1. Then the map A is given by $A(1) = (1,1,1)$ so clearly $N = \mathbb{Z}^2$, $l_1 = e_1$, $l_2 = e_2$ and $l_3 = -e_1 - e_2$. It follows that $Y = \mathbb{P}^2$.

For $n \geqslant 4$ we will show that $N = \mathbb{Z}^2$, vectors l_i and l_{i+1} form an oriented basis of N for every i and there is no other l_k in the rational polyhedral cone generated by l_i and l_{i+1}.

First thing to prove is that the set $\{A_j | j \neq i, i+1\}$ forms a basis of $\mathrm{Pic}(X)$ for every $i \in \mathbb{Z}/n\mathbb{Z}$. Then $N = \mathbb{Z}^2$ and $\{l_i,\, l_{i+1}\}$ is a base of N. Up to a cyclic renumbering it suffices to show that $A_1, \ldots, A_{n-2}$ is a basis of $\mathrm{Pic}(X)$. In fact for $i \leqslant n-2$ the divisors $A_1, \ldots, A_i$ generate a saturated subgroup of $\mathrm{Pic}(X)$ of rank i. For $i = 2$, the subgroup generated by A_1 and A_2 is of rank two because $A_1.A_2 = 1$, $A_1.A_n = 1$ and $A_2.A_n = 0$. If there existed $C \in \mathrm{Pic}(X)$ and $a_1, a_2 \in \mathbb{Z}$ such that $A_i = a_i C$ then the first condition would state that both a_i's are non-zero and the non-trivial intersection with A_n would force $A_n.C \neq 0$. But then a product of three non-zero integers $a_1 a_2 C.H$ would be zero. This contradiction shows that in fact $\langle A_1, A_2 \rangle \subset \mathrm{Pic}(X)$ is of rank two. The fact that it is saturated follows from integrality of intersection product in $\mathrm{Pic}(X)$.

Now the induction works. Let $i < n-2$. Then $A_1, \ldots, A_i$ generate a saturated subgroup of $\mathrm{Pic}(X)$ of rank i. If $B = \sum_{j=1}^{i} \alpha_j A_j$ then $B.A_{i+2} = 0$. But $A_{i+1}.A_{i+2} = 1$ which proves that $A_1, \ldots, A_{i+1}$ are linearly independent. The fact that the subgroup generated by them is saturated follows again from the integrality of intersection product.

Assuming for a while that l_i's form a fan of a smooth complete toric surface Y let us find self intersection numbers of T-invariant divisors on Y. As mentioned earlier if D_i is a divisor corresponding to l_i then $D_i^2 = a_i$, where $l_{i-1} + a_i l_i + l_{i+1} = 0$. In order to find a_i, consider the quotient $\mathrm{Pic}(X)/A_i^{\perp} = \mathbb{Z}$. Then A_{i-1} and A_{i+1} are identified with 1 and the image of A_i is $a_i = A_i.A_i$. We have the diagram, where vertical arrows are projections

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(X) \arrow[r, "A"] \arrow[d] & \mathbb{Z}^n \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Pic}(X)/A_i^{\perp} \arrow[r, "\widetilde{A}"] & \mathbb{Z}^3 \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}$$

which shows that $l_{i-1} + a_i l_i + l_{i+1} = 0$.

We already know that every pair l_i, l_{i+1} forms a basis of $N = \mathbb{Z}^2$. It suffices to check, that there is no l_k lying in a cone spanned by l_i and l_{i+1} and that all these pairs form an oriented basis of N. Let us choose an orientation of N given by the basis l_1, l_2. If there was a vector l_k in a cone spanned by l_i and l_{i+1} or if any pair l_i, l_{i+1} would give a reverse orientation of N, then moving from l_1 to l_n in a direction compatible with the chosen orientation would lead to more than one rotations around the origin; let's say r of them. Every rotation gives a set of rays in N which can be filled up to a fan of a smooth complete toric surface. Assume that this procedure requires adding n' rays. Let a_i' be the new intersection numbers. Then the Proposition 1.2.2 tells us that $\sum a_i' = 12r - 3(n+n')$. On the other hand $10-n = K_x^2 = (\sum A_i)^2 = \sum a_i + 2(n-1)$ which implies that $\sum a_i = 12 - 3n$. But passing from the first set of rays to the second one required adding n' rays, so $\sum a_i' = \sum a_i - 3n'$. All these equalities say that $12-3n = 12r-3n$ so r=1. $\square$

It is natural to ask what are the exceptional collections of line bundles on a rational surface X. Hille and Perling in [HP] give an algorithm for finding some of them provided the sequence of blow-ups is fixed

$$X = X_t \xrightarrow{b_t} X_{t-1} \xrightarrow{b_{t-1}} \cdots \xrightarrow{b_0} X_0.$$

Here X_0 equals either $\mathbb{P}^2$ or some Hirzebruch surface $\mathbb{F}_a$.

This sequence leads to a basis of $\mathrm{Pic}(X)$. If $X_0 = \mathbb{P}^2$ then, as described in the Introduction, $H, R_1, \ldots, R_{t+1}$ form an orthogonal basis:

$$H^2 = 1,\ R_i^2 = -1,\ H.R_i = 0 \text{ for all } i \text{ and } R_i.R_j = 0 \text{ for } i \neq j.$$

In the case when $X_0 = \mathbb{F}_a$ for some $a \geqslant 0$ let $\{P, Q\}$ be the basis of $\mathrm{Pic}(\mathbb{F}_a)$ such that $P^2 = 0$, $Q^2 = -a$ and $P.Q = 1$. Then an analogous procedure to the one for $\mathbb{P}^2$ leads to a basis $P, Q, R_1, \ldots, R_t$ of $\mathrm{Pic}(X)$ such that $R_i^2 = -1$, $R_i.P = 0 = R_i.Q$ for all i's and $R_i.R_j = 0$ for $i \neq j$.

A full exceptional collection on X_0 gives a toric system. By imitating the augmentation procedure described in the Section 1.2 a toric system on X can be obtained. Moreover, under some assumptions it comes from a full exceptional collection on X. More precisely, let $\mathcal{A} = A_1, \ldots, A_k$ be a toric system on X_{i-1} ($k = i+2$ for $X_0 = \mathbb{P}^2$ and $k = i+3$ for $X_0 = \mathbb{F}_a$). Then it can be checked that for any l the sequence $A_1, \ldots, A_{l-2}, A_{l-1} - R_i, R_i, A_l - R_i, A_{l+1}, \ldots, A_k$ is a toric system on X_i. It is called an *augmentation* of $\mathcal{A}$. Here, as before the divisors on X_{i-1} are identified with their pullbacks via b_i.

A full exceptional collection on $\mathbb{P}^2$ is $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ which gives a toric system H, H, H. On a Hirzebruch surface $\mathbb{F}_a$ for any $s \in \mathbb{Z}$ the toric system $P, sP+Q, P, -(a+s)P+Q$ comes from a full exceptional collection, [HP, prop. 5.2].

Definition 1.3.5. Standard toric systems *are the toric systems on* $\mathbb{P}^2$ *and* $\mathbb{F}_a$ *described above.* A standard augmentation *is a toric system on a smooth complete rational surface* X *that is an augmentation of a standard toric system.*

In order to determine which standard augmentations come from exceptional collections one has to define a partial order on the set $\{R_1, \ldots, R_t\}$. Geometrically, its a partial order such that $R_j \succeq R_i$ if the point x_j blown up by b_j lies on R_i – the exceptional divisor of b_{i-1}. However, this relation wouldn't be transitive, hence the

Definition 1.3.6 (cf. [HP]). *Assume $i, j > 0$ and denote by x_i and x_j the points on X_{i-1} and X_{j-1} respectively, which are blown up by the maps b_i and b_j. Then $\succeq$ is a partial order on the set $\{R_1, \ldots, R_t\}$ such that $R_i \succeq R_i$ for every i and $R_j \succeq R_i$ iff $j > i$ and $b_i \circ \ldots \circ b_{j-1}(x_j) = x_i$.*

Definition 1.3.7. *A standard augmentation is* admissible *if it contains no element of the form $R_i - \sum_{j \in S} R_j$ such that $R_j \preceq R_i$ for some $j \in S$.*

Hille and Perling prove in [HP] the following

Proposition 1.3.8. *Every standard augmentation comes from an exceptional collection on X iff it is admissible.*

Remark. If $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is an exceptional collection of line bundles, then for any line bundle $\mathcal{L}$ also $(\mathcal{E}_1 \otimes \mathcal{L}, \ldots, \mathcal{E}_n \otimes \mathcal{L})$ is exceptional. One can thus assume that any exceptional collection begins with $\mathcal{O}$.

Example 1.3.9 (Toric systems on $\mathbb{P}^2$ blown up in 2 different points). Let X be $\mathbb{P}^2$ blown up in two points and let $X \xrightarrow{b_1} bl(\mathbb{P}^2) \xrightarrow{b_0} \mathbb{P}^2$ be a sequence of blowing ups. Then there are essentially two possible toric systems on X

$$\mathcal{E}_1 = H - R_1, R_1, H - R_1 - R_2, R_2, H - R_2,$$

$$\mathcal{E}_2 = H - R_1, R_1 - R_2, R_2, H - R_1 - R_2, H.$$

The partial order on $\{R_1, R_2\}$ is trivial so both are admissible. It means that ($\mathcal{O}_X$, $\mathcal{O}_X(H - R_1)$, $\mathcal{O}_X(H)$, $\mathcal{O}_X(2H - R_1 - R_2)$, $\mathcal{O}_X(2H - R_1)$) and ($\mathcal{O}_X$, $\mathcal{O}_X(H - R_1)$, $\mathcal{O}_X(H - R_2)$, $\mathcal{O}_X(H)$, $\mathcal{O}_X(2H - R_1 - R_2)$) are full exceptional collections on X, what can be also checked by direct calculations. The toric surfaces they give are the following.

In the case of $\mathcal{E}_1$ the map from $\mathrm{Pic}(X)$ to $\mathbb{Z}^5$ is given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

If the basis of N is chosen to consist of v_1 and v_5 (l_i's are images of the standard basis of $\mathbb{Z}^5$ under the quotient map) then $v_2 = -v_5$, $v_3 = -v_1 - v_5$ and $v_4 = -v_1$. It gives the fan

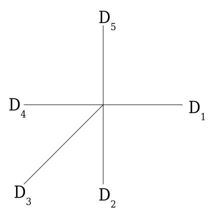


Figure 1.1: The fan of $\mathbb{P}^2$ blown up in two different points

so the toric variety $Y_1 = Y(\mathcal{E}_1)$ is again X.

For the toric system $\mathcal{E}_2$ the corresponding matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}$$

If again one choose v_1, v_5 to be a basis of N then $v_2 = -v_5$, $v_3 = -v_1 - 2v_2$ and $v_4 = -v_1 - v_2$. Then the fan is

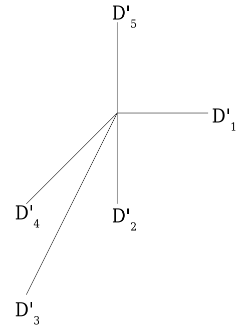


Figure 1.2: The fan of $\mathbb{P}^2$ blown up in two infinitely close points

so the toric variety $Y_2 = Y(\mathcal{E}_2)$ is $\mathbb{P}^2$ blown up in two infinitely close points.

Recall, that if $\mathcal{A} = A_1, \ldots, A_s$ is a toric system on X, then the collection it comes from is $(\mathcal{O}_X,\ \mathcal{O}_X(A_1),\ \mathcal{O}_X(A_1 + A_2), \ldots, \mathcal{O}_X(A_1 + \ldots + A_{s-1}))$. On the other hand the procedure of assigning a toric surface to a toric system identifies $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y(\mathcal{A}))$. Therefore in a natural way it leads to a collection of line bundles on $Y = Y(\mathcal{A})$, namely $(\mathcal{O}_Y,\ \mathcal{O}_Y(D_1), \ldots, \mathcal{O}_Y(D_1 + \ldots + D_{s-1}))$.

Let's calculate what sequences of sheaves are obtained on Y_i's defined in the Example 1.3.9.

In $\mathrm{Pic}(Y_1)$ the following equalities hold $D_1 = D_3 + D_4$ and $D_5 = D_2 + D_3$. Moreover, $D_2 = R_1$ is the first exceptional divisor and $D_4 = R_2$ the second one. The class of hyperplane $H = D_2 + D_3 + D_4$. Then

$$
\begin{gathered}
\mathcal{O}(D_1) = \mathcal{O}(D_3 + D_4) = \mathcal{O}(H - R_1), \\
\mathcal{O}(D_1 + D_2) = \mathcal{O}(D_2 + D_3 + D_4) = \mathcal{O}(H), \\
\mathcal{O}(D_1 + D_2 + D_3) = \mathcal{O}(D_2 + 2D_3 + D_4) = \mathcal{O}(2H - R_1 - R_2), \\
\mathcal{O}(D_1 + D_2 + D_3 + D_4) = \mathcal{O}(D_2 + 2D_3 + 2D_4) = \mathcal{O}(2H - R_1).
\end{gathered}
$$

Thus the collection on Y_1 is the collection that led to the construction of this variety.

In the case of Y_2 one collection is distinguished – $(\mathcal{O}_{Y_2},\ \mathcal{O}_{Y_2}(D'_1), \ldots, \mathcal{O}_{Y_2}(D'_1 + \ldots + D'_4))$. In $\mathrm{Pic}(Y_2)$ there is $D'_1 = D'_3 + D'_4$ and $D'_5 = D'_2 + 2D'_3 + D'_4$ so this collection becomes $(\mathcal{O}_{Y_2},\ \mathcal{O}_{Y_2}(D'_3 + D'_4),\ \mathcal{O}_{Y_2}(D'_2 + D'_3 + D'_4),\ \mathcal{O}_{Y_2}(D'_2 + 2D'_3 + D'_4),\ \mathcal{O}_{Y_2}(D'_2 + 2D'_3 + 2D'_4))$.

Chapter 2

Quivers and endomorphisms algebras

A full exceptional collection of coherent sheaves on an algebraic variety X allows us to assign to X a combinatorial object, which under some assumptions encodes information about X itself and the category $\mathcal{D}^b(X)$.

2.1. Quivers

In this section we define quivers and modules over them. For further information see [B].

A *quiver* Δ is a set consisting of vertices, denoted by p_i, and arrows between them. A *finite quiver* is such a quiver that the set of arrows and vertices is finite. Every arrow can be understood as a simple path from one vertex to another and every vertex can be viewed as a constant path, denoted also by p_i. In general a *path* is a sequence of arrows in which the tail of each following arrow coincides with the head of the previous one. A concatenation defines the *composition of paths*. Thus, the set $\mathbb{C}\Delta$ of formal linear combinations of elements of Δ is an algebra with respect to composition of paths. If the tail of β doesn't coincide with the head of α then $\beta \circ \alpha = 0$.

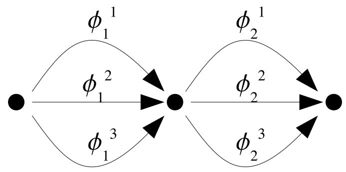
If $S \subset \mathbb{C}\Delta$ is any subset then a *quiver with relations* is the quotient algebra of the path algebra by the ideal generated by S. This notion allows to consider two paths from one vertex to another, consisting of different arrows, equal.

Let A be the path algebra of a quiver Δ with relations S; $A = \mathbb{C}\Delta/(S)$. The vertices $p_1, \ldots, p_n$ of Δ are orthogonal projections in A i.e. $p_k^2 = p_k$, $p_k \circ p_m = 0$ for $k \neq m$ and id$= p_1 + \ldots + p_n$.

A vector space V over $\mathbb{C}$ is a *left A-module* if there exists a left action $A \times V \to V$ of A on V. Since id$= p_1 + \ldots + p_n$ we have $V = \bigoplus_{i=1}^n p_i V$. This equality allows us to consider V as a set of vector spaces $V_i := p_i V$ assigned to every vertex of Δ. Then for every $i \neq j$, A_{ij} i.e. a space of all paths between p_i and p_j gives a map $A_{ij} \otimes V_i \to V_j$. Moreover, all the natural diagrams are commutative. Left A-modules will also be called *representations* of A.

The notion of considering an A-module as a set of n vector spaces leads to a definition of a *dimension vector*, which for a module V equals $(\mathrm{dim}V_1, \dots, \mathrm{dim}V_n) \in \mathbb{Z}^n$. A set of all representations of Δ is clearly a disjoint sum of representations with different dimension vectors.

Example 2.1.1 (Quivers S_n)**.** The quiver S_n has n vertices $\{q_1, \dots, q_n\}$ and between two adjacent vertices there are n arrows $\phi_i^k : q_i \to q_{i+1}$ $(i = 1, \dots, n-1;\ k = 1, \dots, n)$. The relations are $\phi_{i+1}^k \circ \phi_i^l = \phi_{i+1}^l \circ \phi_i^k$. Quiver S_3 is shown on the figure:



Right modules over A, the path algebra over the quiver Δ, correspond to representations of Δ^{opp} i.e. a quiver obtained from Δ by reversing directions of all arrows.

An *ordered quiver* is a quiver with linear ordering of its vertices and such that the tail of every arrow has smaller index than its head.

2.2. Equivalence of derived categories

An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ of sheaves on an algebraic variety X leads to a quiver with vertices corresponding to $\mathcal{E}_i$'s and arrows between them given by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)$,i.e. there are $\mathrm{dimHom}(\mathcal{E}_i, \mathcal{E}_j)$ arrows from the vertex i to the vertex j. It is of course a quiver with relations because one map can be presented as a composition in many ways. On the other hand the condition of being exceptional assures that this quiver is ordered. However, in the category $\mathcal{C}oh(X)$ of coherent sheaves on X there exist also Ext^i functors providing information about $R\mathrm{Hom}$'s between $\mathcal{E}_i$ and $\mathcal{E}_j$ in the derived category $\mathcal{D}^b(X)$. These information are not seen by the structure of a quiver. Hence, a

Definition 2.2.1. A strong exceptional collection *is an exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ on X such that $Ext^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $k > 0$ and all i, j.*

It follows that if a collection is a strong exceptional collection then the ordered quiver described at the beginning of this section encodes all information about morphisms between $\mathcal{E}_i$'s.

Let $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{E}_i$ be the direct sum of sheaves in the strong exceptional collection and let Δ be a quiver associated to this collection. Then the algebra $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ of endomorphisms of the sheaf $\mathcal{E}$ is the path algebra of the quiver; $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) = A = \mathbb{C}\Delta/(S)$ (Proposition 3.3 in [CS]).

Even more is true. In [B] Bondal has shown

Theorem 2.2.2. *Let $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ be a full, strong, exceptional collection of coherent sheaves on a smooth variety X. Then $\mathcal{D}^b(X)$ the derived category of coherent sheaves on X is equivalent to the bounded derived category $\mathcal{D}^b(\text{mod-}A)$ of finite-dimensional right modules over $A = End_{\mathcal{O}_X}(\bigoplus \mathcal{E}_i)$.*

Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(\text{mod-}A)$ denote the functor inducing equivalence. Then for a coherent sheaf $\mathcal{F}$ on X treated as an element of $\mathcal{D}^b(X)$

$$\Phi(\mathcal{F}) = \mathrm{RHom}(\mathcal{E}, \mathcal{F}).$$

Example 2.2.3 (Quivers associated to $\mathbb{P}^n$)**.** The Example 1.1.4 says that the collection $(\mathcal{O}, \dots, \mathcal{O}(n))$ on $\mathbb{P}^n$ is strong and exceptional. The quiver Δ associated to it has thus n vertices and there are n arrows between every two adjacent vertices. Relations say that the arrows are commutative i.e. $\phi^k_{i+1}\phi^l_i = \phi^l_{i+1}\phi^k_i$. It follows that the obtained quiver is S_n.

2.3. Exceptional collections on $\mathbb{P}^2$ blown up in two points

The Example 1.3.9 gives exceptional collections of $\mathbb{P}^2$ blown up in two different points and on $\mathbb{P}^2$ blown up in two infinitely close points. The aim of this section is to understand how their quivers look like.

2.3.1. Cohomology of line bundles on toric surfaces

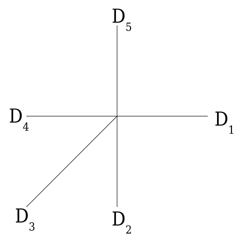
In order to check whether these collections are strong and draw the quivers one needs to be able to calculate $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}(D_1), \mathcal{O}(D_2))$ for $i \in \{0, 1, 2\}$ and divisors D_1, D_2. However, the sheaf $\mathcal{O}(D_1)$ is locally free, so (cf. [Har, prop. 6.7])

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}(D_1), \mathcal{O}(D_2)) = \mathrm{H}^i(X, \mathcal{O}(D_2 - D_1)).$$

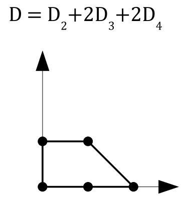
Thus, the question reduces to calculating cohomology groups of line bundles on toric surfaces.

The book [F2, sections 3.4 and 3.5] gives an easy algorithm for finding the dimension of $\mathrm{H}^0(Z, \mathcal{O}_z(D))$ for a toric surface Z and a divisor D on it. Namely, for T-invariant divisors $D_1, \dots, D_s$ on Z, let $\{v_i\}_{i=1,\dots,s} \subset N$ be the generators of rays. To a divisor $D = \sum_{i=1}^s a_i D_i$ there is associated a polyhedron $P_D = \{u \in M_{\mathbb{R}} | \langle u, v_i \rangle \geqslant -a_i \text{ for all } i\}$ in M. The dimension of $\mathrm{H}^0(Z, \mathcal{O}_Z(D))$ is then equal to the number of lattice points in P_D; $\dim \mathrm{H}^0(Z, \mathcal{O}_Z(D)) = |(M \cap P_D)|$.

Example 2.3.1 (Cohomology of line bundles)**.** Let again X be $\mathbb{P}^2$ blown up in two points. Its fan is on the picture. The rays' generators are: $v_1 = (1, 0)$, $v_2 = (0, -1)$, $v_3 = (-1, -1)$, $v_4 = (-1, 0)$ and $v_5 = (0, 1)$.



Take for example $D = D_2 + 2D_3 + 2D_4$. Then the polyhedron $P_D \subset \mathbb{Z}^2$ is



There are five lattice point in it so dim $\mathrm{H}^0(X, \mathcal{O}(D)) = 5$.

An easy way to compute the dimensions of the first and second cohomology of a line bundle on Z is given by the Serre duality and the Riemann-Roch theorem. Serre duality states that $\mathrm{H}^2(Z, \mathcal{O}_Z(D)) = \mathrm{H}^0(Z, \mathcal{O}_Z(K_Z - D))$ for $K_Z = -\sum_{i=1}^{s} D_i$ the canonical divisor on Z. Then the Riemann-Roch theorem allows us to determine the dimension of $\mathrm{H}^1(Z, \mathcal{O}_Z(D))$.

2.3.2. Calculating quivers

Algorithm described above makes it possible not only to draw the quivers but also to find relations in them. Consider an example.

Example 2.3.2 (Finding relations between paths)**.** Let X be equal to $\mathbb{P}^2$ blown up in two points and let $(\mathcal{O}, \mathcal{O}(H - R_1), \mathcal{O}(H - R_2), \mathcal{O}(H))$ be the collection of line bundles on X. This collection isn't full but is strong and exceptional and the quiver associated to it has one relation. The aim of this example is to find this relation and to illustrate how in general relations in quivers can be determined.

There are four elements in the collection so the quiver Δ will have four vertices – p_1, p_2, p_3 and p_4. In order to find the arrows between them we have to calculate dimensions of Hom-spaces. Recall, that for line bundles $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(E_1), \mathcal{O}(E_2)) = \mathrm{H}^0(X, \mathcal{O}(E_1 - E_2))$.

$\mathrm{H}^0(X, \mathcal{O}(H - R_1)) = \mathbb{C}^2$ and the lattice points in the polytope P_{H-R_1} are $(0,0)$ and $(0,1)$. Thus, the quiver Δ has two arrows between vertices p_1 and p_2 – say α_1 and α_2.Also $\dim(\mathrm{H}^0(X, \mathcal{O}(H - R_2))) = 2$ this time with lattice points inside the polytope being $(0,0)$ and $(1,0)$. Then β_1 and β_0 are the two arrows between p_1 and p_3. Moreover, $\dim(\mathrm{H}^0(X, \mathcal{O}(R_1 - R_2))) = 0$, $\dim(\mathrm{H}^0(X, \mathcal{O}(R_1))) = 1$ and $\dim(\mathrm{H}^0(X, \mathcal{O}(R_2))) = 1$ with only $(0,0)$ in the non-empty polytopes. The quiver Δ is then

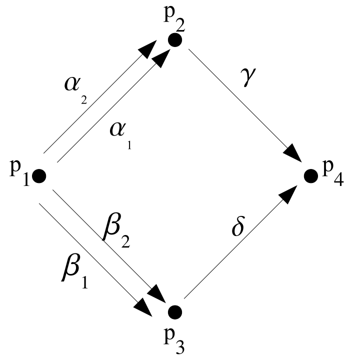
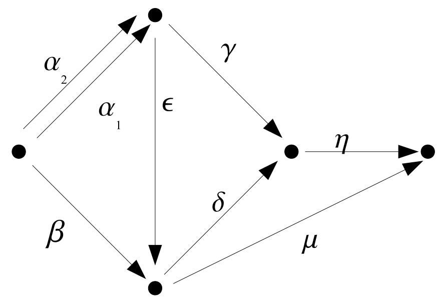


Figure 2.1: Relations in a quiver

$\mathrm{Dim}(\mathrm{X}^0(Z, \mathcal{O}(H))) = 3$ but in Δ there are four paths from p_1 to p_4, namely : $\gamma\alpha_1$, $\gamma\alpha_2$, $\delta\beta_1$ and $\delta\beta_2$. Therefore two of them has to be identified. The lattice points in P_H are $(0,0)$, $(0,1)$ and $(1,0)$. The concatenation of arrows corresponds to adding the lattice points. Therefore, the lattice coordinates of $\gamma\alpha_1$ are $(0,0)+(0,0) = (0,0)$. The remaining paths have coordinates respectively $(0,1)$, $(0,0)$ and $(1,0)$. It follows that $\gamma\alpha_1 = \delta\beta_1$ as ways from p_1 to p_4. The quiver Δ is then the quiver on the figure 2.3.2 with one relation

$$\gamma\alpha_1 = \delta\beta_1.$$

Coming back to the Example 1.3.9 and $\mathbb{P}^2$ blown up in two points. Direct calculations show that the collection ($\mathcal{O}$, $\mathcal{O}(H - R_1)$, $\mathcal{O}(H)$, $\mathcal{O}(2H - R_1 - R_2)$, $\mathcal{O}(2H - R_1)$) on X is strong and its quiver is:



As on the figure 2.3.2 the first vertex of the quiver is the one on the left, the second one - the one on the top and then the remaining ones in order from left to right.

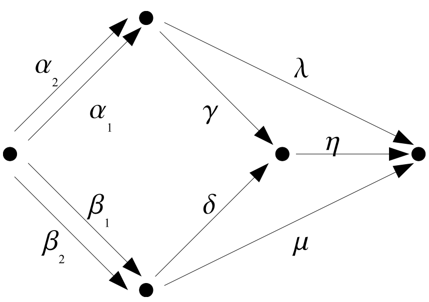
The relations are

$$\gamma\alpha_1 = \delta\beta,$$

$$\beta\mu = \eta\gamma\alpha_2,$$

$$\mu\epsilon\alpha_1 = \eta\delta\epsilon\alpha_2.$$

Also the second collection on X, that is ($\mathcal{O}$, $\mathcal{O}(H - R_1)$, $\mathcal{O}(H - R_2)$, $\mathcal{O}(H)$, $\mathcal{O}(2H - R_1 - R_2)$) is strong. Its quiver is



with relations

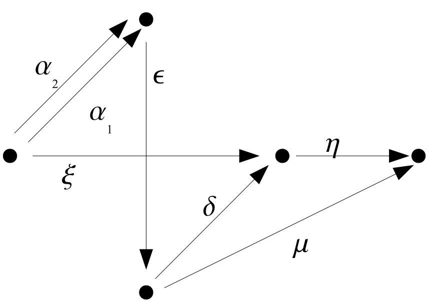
$$\gamma\alpha_1 = \delta\beta_1,$$

$$\lambda\alpha_2 = \mu\beta_2,$$

$$\lambda\alpha_1 = \eta\delta\beta_2,$$

$$\mu\beta_1 = \eta\gamma\alpha_2.$$

The Example gave also collections of sheaves on toric varieties obtained from toric systems. In the case of the first collection, the toric variety Y_1 was equal X and the collection remained unchanged. The second collection gave Y_2 equal to $\mathbb{P}^2$ blown up in two infinitely close points. In the basis of T-invariant divisors associated to rays of its fan the obtained collection is ($\mathcal{O}$, $\mathcal{O}(D_1')$, $\mathcal{O}(D_1'+D_2')$, $\mathcal{O}(D_1'+D_2'+D_3')$, $\mathcal{O}(D_1'+D_2'+D_3'+D_4')$). It is a full, exceptional collection on Y_2 but it isn't strong. There is one one-dimensional Ext-space; $\mathrm{Ext}^1(\mathcal{O}(D_1),\mathcal{O}(D_1+D_2))=\mathbb{C}$. Nevertheless it still makes sense to draw a quiver obtained from this collection which turns out to be



with one relation

$$\mu\epsilon\alpha_1 = \eta\delta\epsilon\alpha_2.$$

Chapter 3

Points of X and modules over a quiver

For a smooth algebraic variety X and a strong full exceptional collection of coherent sheaves $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ there is a quiver Δ, which encodes the structure of $A = \mathrm{End}(\mathcal{E})$ for $\mathcal{E} = \bigoplus \mathcal{E}_i$. The Theorem 2.2.2 allows one to assign to every sheaf $\mathcal{F}$ on X a complex of modules over Δ, i.e. an element of $\mathcal{D}^b(\text{mod-}A)$.

Let x be any point of X, Then $\mathbb{C}_x$ – the skyscraper sheaf of x is coherent. In this section we will calculate what complex of modules is associated to it.

The Proposition 6.7 of [Har] states that for any locally free $\mathcal{F}$ sheaf of finite rank on X and any sheaf $\mathcal{G}$

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = \mathrm{H}^i(X, \mathcal{F}\check{\ } \otimes \mathcal{G}),$$

where $\mathcal{F}\check{\ } = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf.

Thus, for a line bundle $\mathcal{L}$ on X, $\mathrm{Ext}^i(\mathcal{L}, \mathbb{C}_x) = \mathrm{H}^i(X, \mathcal{L}\check{\ } \otimes \mathbb{C}_x) = \mathrm{H}^i(X, \mathbb{C}_x)$ is non-zero only when $i = 0$ and then is equal to $\mathbb{C}$.

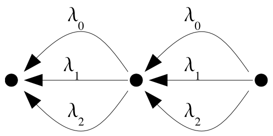
A complex C of objects of an abelian category with only one non-zero homology group is quasi-isomorphic to the complex $\mathrm{H}^*(C)$. It follows that the complex $\mathrm{RHom}(\mathcal{L}, \mathbb{C}_x)$ is quasi-isomorphic to a complex consisting only of $\mathbb{C}$ in its zeroth grading.

To sum up, let $(\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n))$ be an exceptional collection on X consisting of line bundles and let Δ be the quiver obtained from it. Let $p_1, \ldots, p_n$ be the vertices of Δ and let $A = \mathbb{C}\Delta/(S)$ be its path algebra for some relations $S \subset \mathbb{C}\Delta$. The functor Φ of the Theorem 2.2.2 associates to every point x in X, by means of $\Phi(\mathbb{C}_x)$, a module with dimension vector $(1, \ldots, 1)$, i.e. a representation of a quiver having over every vertex $p \in \Delta$ a one-dimensional vector space V_p. It remains to determine how $A_{i,j}$'s, paths from p_i to p_j, act on these spaces. An arrow $\alpha \in A_{ij}$ determines a linear map $V_{p_i} \to V_{p_j}$. Thus, after choosing bases in V_{p_i} for all $1 \leqslant i \leqslant n$, every arrow α should be labelled with a complex number λ_α representing the matrix of linear map from $\mathbb{C}$ to $\mathbb{C}$.

However, if one changes the bases of V_{p_i}'s then also λ_α's change. Thus, two labellings of arrows of Δ represent the same module if the differ by the action of $H = \mathrm{GL}(1, \mathbb{C})^n$. It is easy to see that for the described action an element $(h_1, \ldots, h_n)$ of H acts on λ_α for $\alpha \in A_{ij}$ by $\lambda_\alpha \to h_i h_j^{-1} \lambda_\alpha$.

Remark. The Theorem 2.2.2 is about right modules over a quiver. As mentioned in the Section 2.1 they correspond to left modules over an opposite quiver i.e. a quiver with all arrow reversed. Thus, the modules corresponding to points will be drawn as representations of the opposite quivers.

Example 3.0.3 (Modules corresponding to points of $\mathbb{P}^2$)**.** The exceptional collection on $\mathbb{P}^2 = \mathbb{P}(V)$ is $(\mathcal{O},\ \mathcal{O}(1),\ \mathcal{O}(2))$. Since $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(1)) = V^*$, choosing the basis of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(1))$ is the same as choosing coordinates on V. If $x \in \mathbb{P}^2$ can be written in this coordinates as $[\lambda_0 : \lambda_1 : \lambda_2]$, then a representation of S_3^{opp} assigned to x has dimension vector (1,1,1) and the following arrows:



Now, the goal is to generalise this example to the cases of any toric surface X with an exceptional collection of line bundles on it. This is done in two steps. T-invariant divisors label arrows of a quiver and give homogeneous coordinates on X.

3.1. T-invariant divisors and arrows in a quiver

Let $\mathcal{O}(E_i)$ and $\mathcal{O}(E_j)$ be two elements of an exceptional collection. The paths between their vertices in the quiver correspond to $\mathrm{H}^0(X, \mathcal{O}_X(E_j - E_i))$. On the other hand $\dim(\mathrm{H}^0(X, \mathcal{O}_X(D))$ is the number of effective divisors linearly equivalent to D. If X is a toric variety such that the rays' generators span N there is a short exact sequence

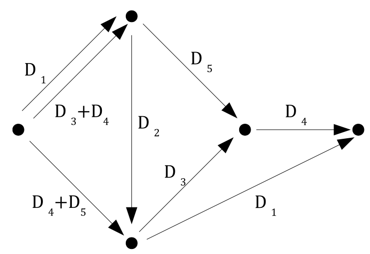
$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

Thus $\dim(\mathrm{H}^0(X, \mathcal{O}_X(D)))$ is the number of effective T-invariant divisors linearly equivalent with D.

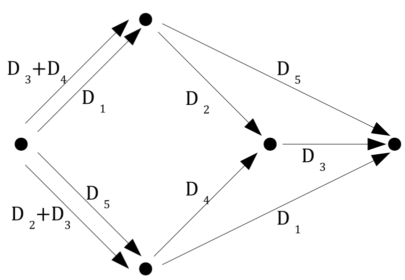
Hence, for an exceptional collection of line bundle on a toric variety X every arrow can be labelled with an effective T-invariant divisor.

Example 3.1.1 (Labelling quivers with divisors)**.** In section 2.3.2 three quivers were calculated. Let's see what are the divisors corresponding to arrows in them.

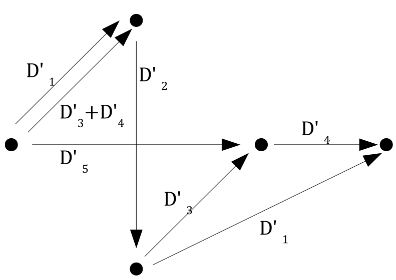
In the first case X was equal to $\mathbb{P}^2$ blown up in two points with T-invariant divisors $D_1, \ldots, D_5$ and a collection $(\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1+D_2), \mathcal{O}(D_1+D_2+D_3), \mathcal{O}(D_1+D_2+D_3+D_4))$. Moreover, in $\mathrm{Pic}(X)$ there were two relations $D_1 = D_3 + D_4$ and $D_5 = D_2 + D_3$ (compare the figure 1.3.9) The labelling of a quiver is then



The second collection considered was again on X but this time of the form $(\mathcal{O}, \mathcal{O}(D_3+D_4), \mathcal{O}(D_2+D_3), \mathcal{O}(D_2+D_3+D_4), \mathcal{O}(D_2+2D_3+D_4))$. Its quiver is



The third collection was on Y_2 equal $\mathbb{P}^2$ blown up in two infinitely close points. The T-invariant divisors were again called $D_1, \ldots, D_5$ but the relations in $\mathrm{Pic}(Y_1)$ were slightly different: $D_1 = D_3 + D_4$ and $D_5 = D_2 + 2D_3 + D_4$ (see the figure 1.3.9). The quiver of a collection $(\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1+D_2), \mathcal{O}(D_1+D_2+D_3), \mathcal{O}(D_1+D_2+D_3+D_4))$ is then



Notice that labelling arrows with T-invariant divisors gives another way to find relations in the path algebra.

3.2. Homogeneous coordinates on toric varieties

Let X be a toric variety with a strong, full, exceptional collection of line bundles $(\mathcal{O}(E_1)\ldots, \mathcal{O}(E_n))$ and let Δ be the quiver associated to this collection. Then every arrow in Δ is an effective T-invariant divisor. There is a way to define coordinates on X depending on its invariant divisors. Hence, every point $x \in X$ gives a module over Δ and the complex numbers labelling arrows correspond to these coordinates.

The following definitions and propositions are generally known and here are cited after [CLS] where also all proofs can be found.

3.2.1. Preliminaries about geometric quotients

Consider an algebraic group G acting on a variety X. G acts on X *algebraically* if the action $(g, x) \to gx$ induces a morphism

$$G \times X \to X.$$

In order to get the best properties of a quotient map a good categorical quotient is defined.

Definition 3.2.1 (see also Definition 5.0.5 in [CLS]). *Let G act on X and let $\pi : X \to Y$ be a morphism constant on G-orbits. π is a* good categorical quotient *if:*

(i) for every open $U \subset Y$ the natural map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism

$$\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\pi^{-1}(U))^G;$$

(ii) for every closed and G-invariant $W \subset X$, $\pi(W) \subset Y$ is closed;

(iii) for every closed, disjoint and G-invariant $W_1, W_2 \subset X$, $\pi(W_1)$ and $\pi(W_2)$ are disjoint in Y.

A good categorical quotient is often denoted by $\pi : X \to X//G$. The map π has some properties analogous to quotients in topology.

Theorem 3.2.2 (cf. Theorem 5.0.6 in [CLS]). *Let $\pi : X \to X//G$ be a good categorical quotient. Then*

1. *Given any diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Z \\ & X//G \arrow[ur, dotted, "\widetilde{\phi}"'] & \end{tikzcd}$$

   where ϕ is a morphism such that $\phi(gx) = \phi(x)$ for $g \in G$ and $x \in X$, there exists a unique morphism $\widetilde{\phi}$ making the diagram commutative;

2. *π is surjective;*

3. *a subset $U \subset X//G$ is open iff $\pi^{-1}(U) \subset X$ is;*

4. *if $U \subset X//G$ is open and non-empty then $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a good categorical quotient;*

5. *for $x, y \in X$*

$$\pi(x) = \pi(y) \Longleftrightarrow \overline{Gx} \cap \overline{Gy} \neq \emptyset.$$

The best quotients are those where points are orbits. For good categorical quotients this condition is equivalent to the fact that orbits are closed.

Proposition 3.2.3 (see also Proposition 5.0.8 in [CLS]). *Let $\pi : X \to X//G$ be a good categorical quotient. Then the following are equivalent:*

1. *all G-orbits are closed in X;*
2. *for $x, y \in X$*
$$\pi(x) = \pi(y) \Longleftrightarrow x \text{ and } y \text{ lie in the same } G\text{-orbit};$$
3. *π induces a bijection between G-orbits in X and $X//G$.*

Definition 3.2.4. *A good categorical quotient is called a* geometric quotient *if it satisfies the conditions of the Proposition 3.2.3.*

A good geometric quotient is denoted by X/G.

3.2.2. Toric varieties as geometric quotients

Let X be a toric variety with the fan $\Sigma \subset N$. Assume moreover, that $\Sigma(1)$ spans N and that Σ is simplicial i.e. for every cone $\sigma \in \Sigma$ its minimal generators are linearly independent. Then the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

after applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot\,, \mathbb{C}^*)$ gives

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow N \longrightarrow 0.$$

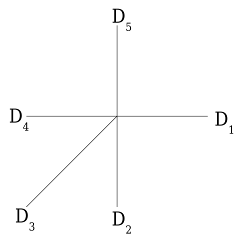
The group $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}^*)$ can be written as a subgroup of $(\mathbb{C}^*)^{\Sigma(1)}$ by $G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_\rho t_\rho^{\langle e_i, v_\rho\rangle} = 1 \text{ for } 1 \leqslant i \leqslant n\}$. Here $e_1, \ldots, e_n$ is a basis of M and $v_\rho \in N$ are the rays' generators.

G acts on $\mathbb{C}^{\Sigma(1)}$. After removing an exceptional set V from $\mathbb{C}^{\Sigma(1)}$ we will get $(\mathbb{C}^{\Sigma(1)} \setminus V)/G = X$.

The set V is defined as a zero set of an ideal $B(\Sigma)$. Let $S = \mathbb{C}[x_\rho | \rho \in \Sigma(1)]$ be the coordinate ring of $\mathbb{C}^{\Sigma(1)}$. For every cone $\sigma \in \Sigma$ let $x^{\check{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \in S$. Then the *irrelevant ideal* $B(\Sigma)$ is defined as $B(\Sigma) = \langle x^{\check{\sigma}} | \sigma \in \Sigma_{\max}\rangle \subset S$.

It remains to construct a map from $\mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$ to X that would be constant on G-orbits and therefore would define a geometric quotient. The first step is to define a toric structure on $\mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$. Let $\{e_\rho | \rho \in \Sigma(1)\}$ be the standard basis of $\mathbb{Z}^{\Sigma(1)}$. For each $\sigma \in \Sigma$ define the cone $\widetilde{\sigma} = \mathrm{Cone}(e_\rho | \rho \in \sigma(1)) \subset \mathbb{R}^{\Sigma(1)}$. These cones form a fan $\widetilde{\Sigma} \subset \mathbb{Z}^{\Sigma(1)} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{\Sigma(1)}$ which is the fan of $\mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma))$. Moreover, the map $e_\rho \to v_\rho$ defines a map of lattices $\mathbb{Z}^{\Sigma(1)} \to N$ compatible with both fan structures. Thus, it gives a map $\pi : \mathbb{C}^{\Sigma(1)} \setminus V(B(\Sigma)) \to X$ which is a geometric quotient (see also Theorem 5.1.10 of [CLS]).

Example 3.2.5 (Homogeneous coordinates on $\mathbb{P}^2$ blown up in two points). Recall that if X is $\mathbb{P}^2$ blown up in two points, then its fan is the following.



X should be then a geometric quotient of $\mathbb{C}^5 \setminus V$ by a group G. By definition

$$G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_\rho t_\rho^{\langle e_i, v_\rho \rangle} = 1 \text{ for } 1 \leqslant i \leqslant n\} =$$

$$= \{(t_1, \ldots, t_5) \mid t_1 t_3^{-1} t_4^{-1} = 1,\ t_5 t_2^{-1} t_3^{-1} = 1\} = \{(\mu\gamma, \lambda, \mu, \gamma, \lambda\mu)\}_{\lambda, \mu, \gamma \in \mathbb{C}^*}.$$

The set V is the zero set of an ideal

$$B(\Sigma) = \langle x_2 x_3 x_4,\ x_3 x_4 x_5,\ x_1 x_4 x_5,\ x_1 x_2 x_5,\ x_1 x_2 x_3 \rangle.$$

Hence,

$$V = \{x_1 = 0,\ x_3 = 0\} \cup \{x_1 = 0,\ x_4 = 0\} \cup \{x_2 = 0,\ x_4 = 0\} \cup \{x_2 = 0,\ x_5 = 0\} \cup \{x_3 = 0,\ x_5 = 0\}.$$

In other words, to every point $x \in X$ one can assign five complex numbers $(x_1, \ldots, x_5)$ such that

$$x_i = 0 \Rightarrow x_{i+2}\, x_{i+3} \neq 0,$$

where i is treated as an element of $\mathbb{Z}/5\mathbb{Z}$.

Moreover, points $(x_1, \ldots, x_5)$ and $(x'_1, \ldots, x'_5)$ are identified if there exist $\lambda, \mu, \gamma \in \mathbb{C}^*$ such that

$$x_1 = \mu\gamma x'_1,$$

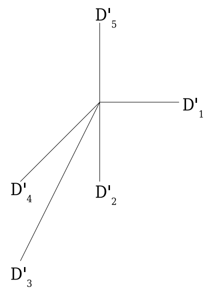
$$x_2 = \lambda x'_2,$$

$$x_3 = \mu x'_3,$$

$$x_4 = \gamma x'_4,$$

$$x_5 = \lambda\mu x'_5.$$

Example 3.2.6 (Homogeneous coordinates on $\mathbb{P}^2$ blown up in two infinitely close points)**.** If Y is $\mathbb{P}^2$ blown up in two infinitely close points, then its fan is



The associated group G is

$$G = \{(t_1, \dots, t_5) \in (\mathbb{C}^*)^5 \mid t_1 t_3^{-1} t_4^{-1} = 1,\ t_5 t_2^{-1} t_3^{-2} t_4^{-1} = 1\} =$$

$$= \{(\mu\gamma, \lambda, \mu, \gamma, \lambda\mu^2\gamma)\}_{\lambda,\,\mu,\,\gamma \in \mathbb{C}^*}.$$

The set V depends only on rays lying in common maximal cones, therefore it is the same as in the previous example.

Hence, to every point $y \in Y$ one can assign five complex numbers $(y_1, \dots, y_5)$ such that

$$y_i = 0 \Rightarrow y_{i+2}\, y_{i+3} \neq 0,$$

where again i is treated as an element of $\mathbb{Z}/5\mathbb{Z}$

Moreover, points $(y_1, \dots, y_5)$ and $(y'_1, \dots, y'_5)$ are identified if there exist $\lambda, \mu, \gamma \in \mathbb{C}^*$ such that

$$y_1 = \mu\gamma y'_1,$$

$$y_2 = \lambda y'_2,$$

$$y_3 = \mu y'_3,$$

$$y_4 = \gamma y'_4,$$

$$y_5 = \lambda\mu^2\gamma y'_5.$$

3.3. Modules corresponding to points

Let again X be a toric surface with a fan Σ. Let $D_1, \dots, D_n$ be T-invariant divisors corresponding to the rays of Σ. The number n of these divisors is equal to the rank of the Picard group $\mathrm{Pic}(X)$ increased by 2. On the other hand this is the rank of the Grothendieck group $K_0(X)$ (see [F1, Prop. 18.3.2] and [Hil, Thm. 2.2]). Thus, by the remark in the Section 1.1, n is the number of sheaves in any full exceptional collection on X.

Let now $(\mathcal{O}(L_1), \dots, \mathcal{O}(L_n))$ be such a collection consisting of line bundles and let Δ be its quiver. The previous sections allow us assign to every point $x \in X$ its coordinates $(x_1, \dots, x_n) \in \mathbb{C}^n \setminus V$, for some exceptional set V, up to an action of a $(n-2)$-dimensional torus $G \subset (\mathbb{C}^*)^n$. These coordinates come from $D_1, \dots, D_n$. On the other hand, the T-invariant divisors label the arrows of Δ. A module M over Δ with a dimension vector $(1, \dots, 1)$ is determined by a labelling arrows of Δ with complex numbers. Two such labellings define the same module if they differ by an action of a n-dimensional torus $H = (\mathbb{C}^*)^n$.

A point $\bar{x} = (x_1, \dots, x_n) \in \mathbb{C}^n \setminus V$ associated to $x \in X$ determines a labelling of the arrows of Δ, i.e. a labelling by an effective divisor $\sum_{i=1}^n b_i D_i$ is understood as a labelling by $\prod_{i=1}^n x_i^{b_i} \in \mathbb{C}$. This induces an action of G on the labellings of Δ. If this action is compatible with the action of H, i. e. if from the fact that two labellings differ by the action of G it follows that they differ by the action of H, then there is a well-defined map from X to modules over Δ with the dimension vector $(1, \dots, 1)$.

Theorem 3.3.1. *Let X be a toric surface with the simplicial fan $\Sigma \subset N$ such that $\Sigma(1)$ spans N. Let also $(\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n))$ be a full exceptional collection on X with a quiver Δ. Then labelling arrows of Δ with homogeneous coordinates of X extends to a well-defined map which assigns to every point x of X an isomorphism class of a module M_x over Δ with dimension vector $(1, \ldots, 1)$.*

Remark. Craw and Smith in [CS] study the moduli space $\mathcal{M}$ of modules over a quiver. $\mathcal{M}$ contains only modules which don't have some "forbidden" submodules; so called semistable ones. However, the construction presented in the above theorem may assign to a point x of X a module which isn't semistable. Thus, the map from X to $\mathcal{M}$ may not be everywhere defined. This problem is addressed in Proposition 4.1 of [CS] which gives conditions under which the map form a projective toric variety X to the moduli space $\mathcal{M}$ of modules over a quiver is regular. Moreover, Theorem 5.4 of the same paper says when it is an isomorphism.

Proof. As mentioned before it suffices to check whether the actions of G and H on labellings of Δ are compatible.

H changes the bases in vector spaces, so it acts in the same way on any paths between two different vertices. Let us first check that it is also true for G.

The linear equivalence between T-invariant divisors is given by the pairing between M and N - the characters' lattice and its dual. This can be seen from the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

The T-invariant divisors correspond to the rays of the fan Σ of X. If $v_{\rho_1}, \ldots, v_{\rho_n} \in N$ are the generators of the rays, then the above map from M to $\mathrm{Div}_T(X)$ is given by $m \rightarrow \langle m, v_{\rho_1} \rangle D_1 + \ldots + \langle m, v_{\rho_n} \rangle D_n$. Hence, the linear equivalences between T-invariant divisors are given by

$$\langle e_i, v_{\rho_1} \rangle D_1 + \ldots \langle e_i, v_{\rho_n} \rangle D_n \approx 0 \quad \forall i,$$

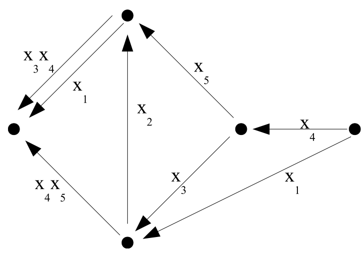
where e_i's form a basis of M.

Now, recall that $G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_\rho t_\rho^{\langle e_i, v_\rho \rangle} = 1 \text{ for } 1 \leqslant i \leqslant n\}$. Together with the previous paragraph it tells us that the action of G on $\mathrm{Div}_T(X)$ preserves linear equivalences, i.e. G acts in the same way on linearly equivalent divisors. On the other hand, the paths between two vertices p_i and p_j in Δ correspond to T-invariant effective divisors linearly equivalent with $E_j - E_i$. It follows that G acts in the same way on all paths between two given vertices.

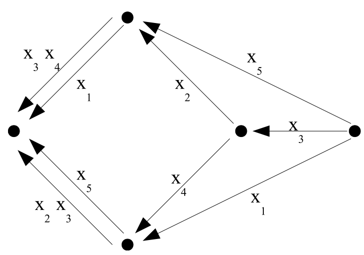
Let $p_1, \ldots, p_n$ be vertices of Δ. For any $g \in G$ we will find $h = (h_1, \ldots, h_n) \in H$ which acts on labellings of arrows the same as g. We may assume that Δ is connected, otherwise the algorithm described below should be repeated for all its connected components. Put $h_1 = 1$ and choose any path $p_1 \rightarrow p_{j_1} \rightarrow \ldots \rightarrow p_{j_k}$ in Δ. Then the values of h_i are determined by the action of g for every i such that p_i is a vertex of this path. In order to see that notice, that since h_1 is fixed, h_{j_1} can take only one possible value and the same is true for all the following vertices in this path. On the other hand every vertex p_i of Δ can be reached by a path that starts in a vertex p_j such that h_j is already determined (one possible way to see that is to recall that the Depth-first search algorithm visits every vertex of an oriented graph). Thus, the value of h_j is also determined. The element $h = (h_1, \dots, h_n)$ does not depend on the paths chosen because G and H act in the same way on any paths between given vertices. Thus, for any $g \in G$ we have found $h \in H$ which acts on labellings in the same way as g does. $\square$

Let us look at examples. The easiest case of $\mathbb{P}^2$ was already described. Before proceeding to other examples recall, that all quivers are drawn with reverse orientation due to the fact that M_x is a right module over Δ.

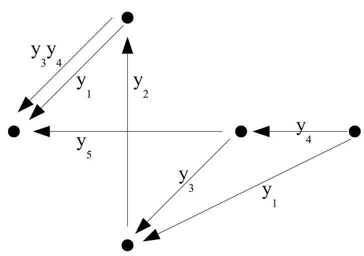
Example 3.3.2 (Modules corresponding to points of $\mathbb{P}^2$ blown up in two points)**.** All the calculations were made before, now it suffices to change the labelling by divisors to the labelling by homogeneous coordinates $(x_1, \dots, x_5)$. In the case of the quiver of $(\mathcal{O}, \mathcal{O}(H - R_1), \mathcal{O}(H),$ $\mathcal{O}(2H - R_1 - R_2), \mathcal{O}(2H - R_1))$ the module corresponding to $x = (x_1, \dots, x_5)$ is



and in the case of $(\mathcal{O}, \mathcal{O}(H - R_1), \mathcal{O}(H - R_2), \mathcal{O}(H), \mathcal{O}(2H - R_1 - R_2))$ it is



Example 3.3.3 (Modules corresponding to point of $\mathbb{P}^2$ blown up in two infinitely closed points)**.** The collection on $\mathbb{P}^2$ blown up in two infinitely close points considered before is $(\mathcal{O},$ $\mathcal{O}(D_1'), \mathcal{O}(D_1' + D_2'), \mathcal{O}(D_1' + D_2' + D_3'), \mathcal{O}(D_1' + D_2' + D_3' + D_4'))$. Then a module corresponding to $y = (y_1, \dots, y_5)$ is the following.



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