Algebraic Geometry, Fall 2016

Hand-outs, set 1
Review on Čech cohomology.
Readings: see e.g. Ravi Vakil notes, chapter 18.
Suppose that X is a topological space. In what follows, we assume that X is quasi-compact or paracompact. Actually, you may assume X to be a variety over an algebraically closed field k (always algebraically closed!).
We choose $\mathcal{U} = (U_i)_{1=1}^r$ a finite open covering of X. For any set of indexes $I \subset \{1, \ldots, r\}$ of cardinality $1 \leq |I| \leq r$ we set $U_I = \bigcap_{i \in I} U_i$.
For $\mathcal{U}$ as above and $\mathcal{F}$, a sheaf of abelian groups over X we define the Čech complex

$$0 \longrightarrow \prod_{|I|=1} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=2} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=3} \mathcal{F}(U_I) \longrightarrow \cdots$$

with differentiation map $\delta^p : \prod_{|I|=p} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=p+1} \mathcal{F}(U_I)$ which to $\sigma = (\sigma_{i_1 < \cdots < i_p}) \in \prod_{|I|=p} \mathcal{F}(U_I)$ associates $\sigma' = (\sigma'_{i_0 < \cdots < i_p}) \in \prod_{|I|=p+1} \mathcal{F}(U_I)$ such that

$$\sigma'_{i_0 < \cdots < i_p} = \sum_{j=0}^{p} (-1)^j \sigma_{i_0 < \cdots \widehat{i_j} \cdots < i_p}$$

where $\widehat{i_j}$ means omission of the index and the right-hand summands are restricted to $U_{i_0} \cap \cdots \cap U_{i_p}$. We define the p-th Čech cohomology as the cohomology of the above complex, that is

$$H^p(\mathcal{U}, \mathcal{F}) = \ker \delta^{p+1} / \operatorname{im} \delta^p$$

For an inscribed covering $\mathcal{U}' \preceq \mathcal{U}$ we have the restriction map $H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}', \mathcal{F})$ and the Čech cohomology is defined as the direct (injective) limit. We denote it $\check{H}^p(X, \mathcal{F})$ or just $H^p(X, \mathcal{F})$
Notation, convention: the elements of $\prod_{|I|=p+1} \mathcal{F}(U_I)$ are called cochains, the elements in $\ker \delta$ cocycles and in $\operatorname{im} \delta$ coboundaries. Sometimes it is convenient to write the set of indexes I not in the increasing order: if we change the order then we have to change the sign of σ, that is $\sigma_{i_1 \cdots i_p} = \operatorname{sgn}(\pi) \cdot \sigma_{\pi(i_1) \cdots \pi(i_p)}$
Properties of this construction; do it yourself version.

- Check that the Čech complex is a complex, that is $\delta^{p+1} \circ \delta^p = 0$.

- Using the Čech complex show the following general nonsense properties of Čech cohomology

  1. $H^0(X, \mathcal{F}) = \mathcal{F}(X)$
  2. $H^p(X, \cdot\,)$ is a covariant functor from category of sheaves of abelian groups over X to category of abelian groups.
  3. For an exact sequence of sheaves of abelian groups

     $$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

     we have an exact sequence of groups

     $$0 \longrightarrow H^0(X,\mathcal{A}) \longrightarrow H^0(X,\mathcal{B}) \longrightarrow H^0(X,\mathcal{C}) \longrightarrow H^1(X,\mathcal{A}) \longrightarrow$$

     $$H^1(X,\mathcal{B}) \longrightarrow H^1(X,\mathcal{C}) \longrightarrow \cdots\cdots$$

  where the connecting homomorphism $H^p(X,\mathcal{C}) \to H^{p+1}(X,\mathcal{A})$ comes from snake-lemma construction used to a diagram consisting of three Čech complexes and the maps coming from maps of the sheaves which you may assume make short exact sequences. Checking exactness is tedious work but the idea is simple: divide the diagram to maps of two consecutive exact sequences and use snake lemma to produce connecting homomorphism for cohomology.

- Acyclic resolution. Suppose that we have a long exact sequence of sheaves of abelian groups on a topological space X

  $$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}^0 \longrightarrow \mathcal{B}^1 \longrightarrow \mathcal{B}^2 \longrightarrow \cdots \longrightarrow \mathcal{B}^m \longrightarrow 0$$

  and assume that all non-zero cohomology of sheaves $\mathcal{B}^i$ vanish (such $\mathcal{B}^i$ is called acyclic). Divide the above sequence into short sequences and prove that cohomology of $\mathcal{A}$ is equal to cohomology (kernel divided by the image of the respective arrow) of the following complex of global sections of the above sequence

  $$0 \longrightarrow \mathcal{B}^0(X) \longrightarrow \mathcal{B}^1(X) \longrightarrow \mathcal{B}^2(X) \longrightarrow \cdots \longrightarrow \mathcal{B}^m(X) \longrightarrow 0$$

- A sheaf of rings $\mathcal{A}$ over a topological space admits a partition of unity if for every finite (for simplicity) covering (U_i) there exist sections $f_i \in \mathcal{A}(X)$, each f_i having support inside U_i (which means that there exists a closed set $K_i \subset U_i$ such that $f_{i|V} = 0$ for every $V \subset X \setminus K_i$) such that $\sum_i f_i = 1$. A sheaf $\mathcal{F}$ of $\mathcal{A}$ modules is called fine; below, we assume $\mathcal{F}$ is fine.

  1. Let $\sigma \in \prod_{|I|=p+1} \mathcal{F}(U_I)$ be in the kernel of δ^{p+1}. We define $\widehat{\sigma} \in \prod_{|I|=p} \mathcal{F}(U_I)$ by the formula (think why does it make sense)
  $$\widehat{\sigma}_{i_1 \cdots i_p} = \sum_i f_i \sigma_{i, i_1 \cdots i_p}$$
  Prove that $\delta^p(\widehat{\sigma}) = \sigma$. (Note that our convention involves sign change.)
  2. Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.
  3. If X is a differentiable manifold use differentials forms to construct an acyclic resolution of a locally constant sheaf $\mathbb{R}_X$ and prove that its Čech cohomology is equal to de Rham cohomology of X: $H^p(X, \mathbb{R}_X) = H^r_{DR}(X)$.

- Calculating cohomology of constant sheaf on simplex or $\mathbb{P}^n$. In this exercise $A = k$ and $A^n_I = k$ (k is your favorite field) if $n \geq 0$ and $\emptyset \neq I \subseteq \{0, \ldots, n\}$. Moreover, we assume that $A^n_I = 0$ if I is not a subset of $\{0, \ldots, n\}$. We define maps:

  i. $\delta^0 : A \to \prod_{|I|=1} A^n_I$ is the diagonal map
  $$k \ni t \to (t, \ldots, t) \in k^{n+1}$$
  ii. $\delta^p : \prod_{|I|=p} A^n_I \to \prod_{|I|=p+1} A^n_I$ is the usual differential, which we write as follows (remember the sign convention)
  $$\delta(\sigma)_I = \sum_{i \in I} \operatorname{sgn}(i, I \setminus \{i\}) \cdot \sigma_{I \setminus \{i\}}$$

  The main task of this exercise is to prove that the following sequence (call it $\clubsuit^n$) is exact
  $$0 \longrightarrow A \xrightarrow{\delta^0} \prod_{|I|=1} A^n_I \xrightarrow{\delta^1} \cdots \longrightarrow \prod_{|I|=n} A^n_I \xrightarrow{\delta^n} \prod_{|I|=n+1} A^n_I \longrightarrow 0$$

1. Prove exactness of $\clubsuit^n$ for $n = 0$ and $n = 1$.
2. Define the the following maps:
   (a) $\alpha^{n-1} : A_I^{n-1} \to A_{I\cup\{n\}}^n$ is the identity
   (b) $\beta^{n-1} : A_I^n \to A_I^{n-1}$ is identity if $n \notin I$ or zero otherwise

   Prove that the following sequence is exact

$$0 \longrightarrow \prod_{|I|=p-1} A_I^{n-1} \xrightarrow{\alpha^{n-1}} \prod_{|I|=p} A_I^n \xrightarrow{\beta^{n-1}} \prod_{|I|=p} A_I^{n-1} \longrightarrow 0$$

   where α and β stand for the products of the respective maps.
3. Prove that the following diagram commutes

$$\begin{array}{ccccc}
\prod_{|I|=p-1} A_I^{n-1} & \longrightarrow & \prod_{|I|=p} A_I^n & \longrightarrow & \prod_{|I|=p} A_I^{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{|I|=p} A_I^{n-1} & \longrightarrow & \prod_{|I|=p+1} A_I^n & \longrightarrow & \prod_{|I|=p+1} A_I^{n-1}
\end{array}$$

   where the vertical arrows are differentials and the horizontal sequences are from the previous point.
4. Prove that α's, β's (vertical arrows) and δ's (horizontal arrows) in this table commute and use induction to prove exactness of $\clubsuit^n$

$$\begin{array}{ccccccccc}
 & & A & \longrightarrow & \cdots & \longrightarrow & \prod_{|I|=n-1} A_I^{n-1} & \longrightarrow & \prod_{|I|=n} A_I^{n-1} \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
A & \longrightarrow & \prod_{|I|=1} A_I^n & \longrightarrow & \cdots & \longrightarrow & \prod_{|I|=n} A_I^n & \longrightarrow & \prod_{|I|=n+1} A_I^n \\
\downarrow & & \downarrow & & & & \downarrow & & \\
A & \longrightarrow & \prod_{|I|=1} A_I^{n-1} & \longrightarrow & \cdots & \longrightarrow & \prod_{|I|=n} A_I^{n-1} & &
\end{array}$$