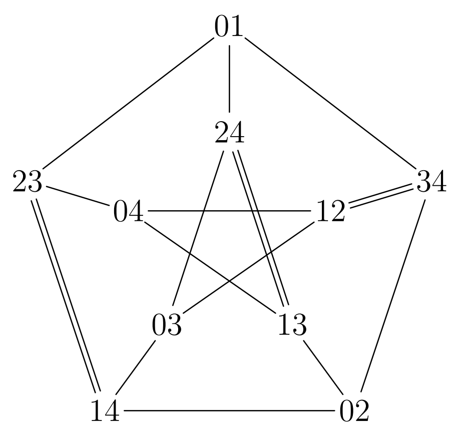
Fano Manifolds, Spring 2018
Nineth problem set.

Recall that a Del Pezzo surface S_r with $r = \rho(S) - 1$ is obtained by blowing up r points on $\mathbb{P}^2$.
The Petersen graph describes the incidence of (-1)-curves on S_4 with 10 vertices standing for the curves and edges for their intersection. The curves E_{0i} may be interpreted as the exceptional curves of the blow-up of a $\mathbb{P}^2$ at four points numbered by $i = 1, \ldots, 4$ and then the curve E_{ij} is the strict transform of a line **not** passing through i-th and j-th point. Using this notation we can encode five conic pencils on S_4 associated to contractons to $\mathbb{P}^1$. Each pencil contains three reducible fibers which can be represented by three edges in this graph (two lines plus their point of intersection). Three double edges in the graph below stand for such a pencil. Thus the 15 edges of the graph are divided into 5 classes.

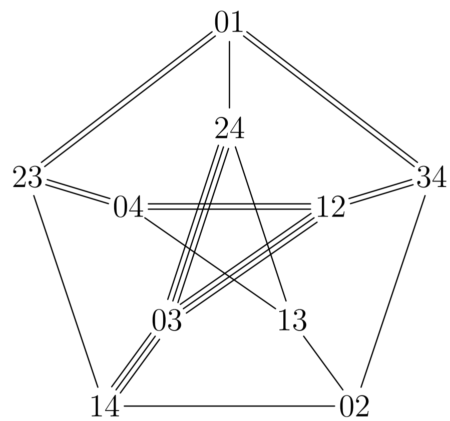


According to the following notation (-1) curves on S_4 will be denoted by E_{ij} where $0 \le i < j \le 4$.

1. Let $-K = -K_{S_4}$ be the anticanonical divisor on S_4
   (a) Prove that $-2K$ is linearly equivalent to $\sum E_{ij}$.
   (b) Prove that the following reducible curves are the only members of $|-K|$ which are sums of curves E_{ij}:
   - "5-cycles" of length 5, for exmple $E_{01}+E_{23}+E_{04}+E_{12}+E_{34}$ depicted below by double edges

- "forks" i.e. three curvs meeting fourth one with multiplicity 2, for example $2E_{03} + E_{12} + E_{14} + E_{24}$ depicted below by triple edges

(c) Prove that $|-K|$ can be generated by six 5-cycles.



2. Let $G(2,5) \subset \mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V)$ denote grassmanian of planes in a linear space V of dimension five with coordinates v_i, $i = 0, \ldots, 4$. By x_{ij}, for $0 \leq i < j \leq 4$, we denote coordinates in $\mathbb{P}(\bigwedge^2 V)$ and we degine points $e_{ij} = \{[x_{rs}] \in \mathbb{P}(\bigwedge^2 V) : x_{rs} = 0 \text{ for } (r,s) \neq (i,j)\}$

   (a) Check that the form $\omega = \sum x_{ij} v_i \wedge v_j$ is primitive if and only if $\omega \wedge \omega = 0$

   (b) Conclude that the ideal of $G(2,5)$ is generated by five quadrics:

$$I_{G(2,5)} = (x_{ij}x_{rs} - x_{ir}x_{js} + x_{is}x_{jr} : 0 \leq i < j < r < s \leq 4)$$

   (c) Let us construct a graph $\mathcal{G}$ with vertices $e_{ij} \in G(2,5)$, as above; two such vertices are joint by an edge in $\mathcal{G}$ if the lie passing through e_{ij} and e_{rs} is contained in $G(2,5)$. Prove that the complement of the graph $\mathcal{G}$ in the complete graph on 10 vertices e_{ij} is the Petersen graph with vertices numbered as above.

   (d) Prove that the graph $\mathcal{G}$ is 1-skeleton of the polytope $\Gamma(S_4)$ introduced in previous series of problems.