Fano Manifolds, Spring 2018
Eight problem set.

Recall that a Del Pezzo surface S with $r = \rho(S) - 1$ is obtained by blowing up r points on $\mathbb{P}^2$. We will assume that $r \geqslant 3$. We proved that in this case the cone $\mathcal{C}(S_r)$ is generated by classes of (-1) curves. We will identify spaces $N_1(S_r) = N^1(S_r)$ and call it $N(S_r)$. We use the lattice $\operatorname{Pic} S_r \subset N(S_r)$, by l_0 we denote the classes of a (general) line on $\mathbb{P}^2$ (pulled back to S_r) and by $l_1 \dots, l_r$ the classes of exceptional curves. The present series of problems is inspired by Manin's book *Cubic forms* Chapter IV

1. Consider a linear subspace $\Lambda_r \subset N(S_r)$ orthogonal to the anticanonical class $-K_{S_r}$.
   (a) Prove that $l_0 - l_1 - l_2 - l_3$ and $l_1 - l_2, l_1 - l_2, \dots, l_1 - l_r$ make a basis of Λ_r. Prove that each of these vectors have intersection (-2).
   (b) By Q_r denote the lattice in Λ_r generated by the classes enumerated above and by $Q_r^\vee$ we denote the dual lattice $Q_r^\vee = \{w \in \Lambda_r : (w, v) \in \mathbb{Z} \ \forall\ v \in Q_r\}$. Prove that $Q_r^\vee / Q_r \simeq \mathbb{Z}_{9-r}$.

2. Elements of $v \in Q_r$ such that $(v, v) = -2$ will be called roots; their set is denoted by R_r.
   (a) Let $v = al_0 + \sum_1^r b_i l_i$ be a root, prove that $3a = \sum_1^r b_i$ and $a^2 - \sum_1^r b_i^2 = -2$
   (b) Prove that, up to permutation of positive i's and changing signs of a and b_i's there are only the following solutions of the above equations:
      i. $l_i - l_j,\ i \neq j,\ i, j > 0$
      ii. $l_0 - l_1 - l_2 - l_3$
      iii. $2l_0 - l_1 - l_2 - l_3 - l_4 - l_5 - l_6$, if $r \geqslant 6$
      iv. $3l_0 - 2l_1 - l_2 - l_3 - l_4 - l_5 - l_6 - l_7 - l_8$, if $r = 8$
   (c) Using the above presentation of roots find its number; these should be as follows for the respective r's:

r	3	4	5	6	7	8
$\lvert R_r \rvert$	8	20	40	72	126	240

3. We will try to recalculate the number of roots using polytopes $\Gamma(S_r)$ introduced in the previous set of problems. Let us recall that we have calculated face polynomials for polytopes $\Gamma(S_r)$:

$$
\begin{aligned}
P_3 &= 1+6x+9x^2+2x^3+3x^3y \\
P_4 &= 1+10x+30x^2+30x^3+5x^4+5x^4y \\
P_5 &= 1+16x+80x^2+160x^3+120x^4+16x^5+10x^5y \\
P_6 &= 1+27x+216x^2+720x^3+1080x^4+648x^5+72x^6+27x^6y \\
P_7 &= 1+56x+756x^2+4032x^3+10080x^4+12096x^5+6048x^6+ \\
&\quad +756x^7+126x^7y
\end{aligned}
$$

   (a) Prove that every root v can be presented as a difference of two non-meeting (-1) curves on S_r, say $v=\gamma_1-\gamma_2$, where each γ_i is a class of a (-1) curve, a vertex in the polytope $\Gamma(S_r)$. Prove that there is an edge of $\Gamma(S_r)$ containing γ_1 and γ_2, we denote it $\langle\gamma_1,\gamma_2\rangle \prec \Gamma$

   (b) Prove that the above presentation is non-unique, that is for $r \leqslant 6$ if $\gamma_1-\gamma_2=\gamma_3-\gamma_4$, for $(\gamma_1,\gamma_2)\neq(\gamma_3,\gamma_4)$, then $\langle\gamma_1,\gamma_2\rangle$ and $\langle\gamma_3,\gamma_4\rangle$ are edges in in a cross polytope, a facet of $\Gamma(S_r)$ dual to type 1 vertex of $\Delta(S_r)$. Conclude that the number of presentations of a root in R_r as edges of $\Gamma(S_r)$ is equal to the number of vertices of type 1 in $\Delta(S_{r-2})$ plus 1.

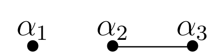
   (c) Find a similar condition for $r=7,8$

   (d) Using the above information recalculate the number of roots.

4. For $r=3,\ldots,6$ find roots in R_r consisting of vectors $\alpha_1,\ldots,\alpha_r$ such that $(\alpha_i,\alpha_j)=1$ if only if the respective $\bullet$'s are connected in one of the Dynkin diagrams; otherwise $(\alpha_i,\alpha_j)=0$

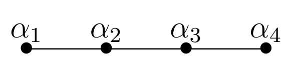
   (a) for example, for $r=3$, take $\alpha_1=l_0-l_1-l_2-l_3$, $\alpha_2=l_1-l_2$, $\alpha_3=l_2-l_3$

$A_1\times A_2$:



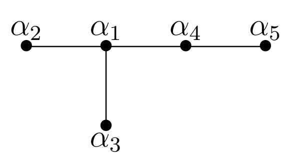
   (b)

A_4 :



   (c)

D_5 :



(d)