Fano Manifolds, Spring 2018
Seventh problem set.

Recall that for a Del Pezzo surface S with $r = \rho(S) - 1$ we consider dual polytopes $\Gamma(S)$ and, respectively, $\Delta(S)$ we denote the intersection of $\mathcal{C}(S)$ and $\mathcal{C}(S)^\vee$ with the affine hyperplane $\{u \in N(S) : -K_S \cdot u = 1\}$. We define a polynomial $P_r(x, y) = \sum a_i x^i + b x^r y$ where b is the number of vertices associated to contractions to $\mathbb{P}^1$ (type 1) and a_i is the number of codimension i faces of $\Delta(S)$ except the vertices of type 1. The polynomials P_r satisfy the following equations:

(i) $\partial_x P_r(x, 0) = \partial_x P_r(0, 0) \cdot P_{r-1}(x, 0)$

(ii) $2(r-1) \cdot \partial_y P_r(1, 0) = \partial_x P_r(0, 0) \cdot \partial_y P_{r-1}(1, 0)$

(iii) $P_r(-1, 1) = (-1)^r$

We know polynomials P_2 and P_3.

1. Calculate polynomials P_r for $r = 4, \ldots, 7$. Note that relation (iii) is useless for $r = 4$ and 6. However, it can be applied to $r = 5$ and 7, see Stalij's MSc Thesis. Use this information to calculate the number of (-1) curves on the respective Del Pezzo
2. Cubic surface. Prove that the folowing conditions are equivalent:
   - S is a Del Pezzo surface with $\rho(S) = 7$.
   - S is a Del Pezzo surface with $-K_S^2 = 3$.
   - S is blow-up of $\mathbb{P}^2$ at 6 points, no three of them on a line, on five of them on a conic.
   - S is isomorphic to a smooth cubic surface in $\mathbb{P}^3$.
3. Find 27 lines on Fermat cubic given by equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbb{P}^3$. `https://math.stackexchange.com/questions/1281981/27-lines-on-fermat-surface`
4. Double cover of $\mathbb{P}^2$. Prove that the folowing conditions are equivalent:
   - S is a Del Pezzo surface with $\rho(S) = 8$.
   - S is a Del Pezzo surface with $-K_S^2 = 2$.

- S is isomorphic to a double cover of $\mathbb{P}^2$ ramified along a smooth curve of degree 4.

5. It is known that a smooth quartic in $\mathbb{P}^2$ admits 28 bitangent lines, see e.g. `https://en.wikipedia.org/wiki/Bitangents_of_a_quartic`. Suppose that $S \to \mathbb{P}^2$ is a double cover from the previous problem.

   (a) Prove that the inverse image of each of the bitangent lines consists of two (-1) curves on S.

   (b) Calculate bitangent lines of the Klein quartic given by equation $x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = 0$.

6. This problem emerged in a discussion during the last problem session. Let $X \subset \mathbb{P}^N$ be a (irreducible possibly singular) subvariety which is *not* contained in any linear subspace of $\mathbb{P}^N$. Prove that $\deg X \geqslant \operatorname{codim} X + 1$. Cut X by a general hyperplane and use induction with respect to $\dim X$. More on low degree projective varieties can be found a paper by by Eisenbud and Harris On varieties of minimal degree