Fano Manifolds, Spring 2018
Sixth problem set.

We assume that S is a Del Pezzo surface, that is $-K_S$ is ample. We will consider dual (closed polyhedral) cones $\mathcal{C} = \mathcal{C}(S)$ and $\mathcal{C}^\vee$ in $N^1(S) = N_1(S)$. By $\Gamma(S)$ and, respectively, $\Delta(S)$ we denote the intersection of these cones with the affine hyperplane $\{u \in N(S) : -K_S \cdot u = 1\}$. Duality of cones implies duality of polytopes $\Gamma(S)$ and $\Delta(S)$.
Before solving problems from this series we will have to complete the previous series. In fact the last exercise which we did not solve contained a misprint, now corrected.

1. For every polytope $\Delta(S)$ with $r = \rho(S) - 1$ define a polynomial

$$P_r(x, y) = \sum a_i x^i + b x^r y$$

where b is the number of vertices of type 1 and a_i is the number of codimension i faces of $\Delta(S)$ except the vertices of type 1. Prove that polynomials P_r satisfy the following equations:

- $\partial_x P_r(x, 0) = \partial_x P_r(0, 0) \cdot P_{r-1}(x, 0)$
- $2(r-1) \cdot \partial_y P_r(1, 0) = \partial_x P_r(0, 0) \cdot \partial_y P_{r-1}(1, 0)$
- $P_r(-1, 1) = (-1)^r$

2. Let S be a Del Pezzo surface which has two extremal contractions of fiber type, $\varphi_i : S \to \mathbb{P}^1$, with fibers f_i, $i = 1,\ 2$.

   (a) Prove that $\operatorname{Pic} S = \mathbb{Z}[f_1] \oplus \mathbb{Z}[f_2]$.

   (b) Prove that $f_1 \cdot f_2 = 1$ and $-K_S \equiv 2f_1 + 2f_2$ or $f_1 \cdot f_2 = 2$ and $-K_S = f_1 + f_2$.

   (c) Prove that in the former case $S = \mathbb{P}^1 \times \mathbb{P}^1$ while in the latter we have a double cover $S \to \mathbb{P}^1 \times \mathbb{P}^1$.

3. Double cover of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : S_4 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover which is a Del Pezzo surface. Assume moreover that $S_4 \neq \mathbb{P}^1 \times \mathbb{P}^1$.

   (a) Prove that π is branched along a divisor B which is of bi-degree $(2, 2)$; B is an eliptic curve and each of the projections $B \to \mathbb{P}^1$ has four ramification points.

(b) Prove that each of projections $S_4 \to \mathbb{P}^1$ has four fibers which are unions of (-1)-curves. Conclude that $\rho(S_4) = 6$.

(c) Prove that the anticanonical divisor $-K_{S_4}$ determines an embedding into $\mathbb{P}^4$ and S_4 becomes an intersection of two quadrics.

(d) Prove that S_4 is blow-up of $\mathbb{P}^2$ in five points, no two of them on a line not all of them on a conic.

4. Let S_5 be the blow-up of 4 points no three of them on a line.

(a) Prove that the four points are base point locus of a pencil of conics which defines a rational map $\mathbb{P}^2 - \to \mathbb{P}^1$ and a regular map $S_5 \to \mathbb{P}^1$.

(b) Prove that as the graph of the rational map S_5 is a divisor in $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree $(2, 1)$.

(c) Prove that there are ten (-1)-curves on S_5 and their incidence is the Petersen graph.

(d) Prove that S_4 from the previous problem contains sixteen (-1)-curves whose incidence is the Clebsch graph.