Fano Manifolds, Spring 2018
Fifth problem set. Cones of Del Pezzo surfaces.

To solve the problems in this series we will need the following characterization of Mori extremal rays of smooth surfaces (this was partially done in the previous series). You may assume this theorem.
Let S be a smooth surface and $\gamma = \mathbb{R}_{\geqslant 0}[C] \prec \mathcal{C}(S) \subset N_1(S)$ be an extremal ray generated by the class of a rational curve C, $0 < -K_S \cdot C \leqslant 3$. If $\rho(S) = \dim N_1(S) = 1$, then $S \simeq \mathbb{P}^2$. Otherwise one of the following holds:

(i) $-K_S \cdot C = 2$ and the contraction of the ray $\varphi_\gamma : S \to B$ is a $\mathbb{P}^1$-bundle over a smooth curve B, or

(ii) $-K_S \cdot C = 1$ and the contraction of the ray $\varphi_\gamma : S \to S'$ is a blow-down of C to a smooth point on a smooth surface S' (Castelnuovo theorem).

From now on we assume that S is a Del Pezzo surface, that is $-K_S$ is ample. We will consider dual (closed polyhedral) cones $\mathcal{C} = \mathcal{C}(S)$ and $\mathcal{C}^\vee$ in $N^1(S) = N_1(S)$. By $\Gamma(S)$ and, respectively, $\Delta(S)$ we denote the intersection of these cones with the affine hyperplane $\{u \in N(S) : -K_S \cdot u = 1\}$. Duality of cones implies duality of polytopes $\Gamma(S)$ and $\Delta(S)$.
For more information you may read Stalij's MSc Thesis or notes of a related talk in Gdansk.

1. Let $\phi : S \to B$ be a surjective morphism from S onto a smooth curve B. Prove that $B \simeq \mathbb{P}^1$.

   (a) Prove that if S has an extremal contraction of type (i) above then $\rho(S) = 2$.

   (b) Prove that for $\rho(S) \geqslant 3$ the vertices of the polytope $\Gamma(S)$ are classes of extremal curves of type (ii).

   (c) Prove that vertices of $\Delta(S)$ which lie on the boundary of the cone $\mathcal{P}^+$ are associated to contractions to $\mathbb{P}^1$, we will call them of type 1.

2. Let $\varphi : S \to S'$ be a blow-down of a (-1)-curve, that is a contraction of type (ii). Prove that S' is Del Pezzo. Use the adjunction formula.

   (a) Prove that $\varphi^*(\mathcal{C}(S')^\vee)$ is a facet of $\mathcal{C}(S)^\vee$ hence $\Delta(S')$ may be identified with a facet (codimension 1 face) of $\Delta(S)$.

(b) Prove that vertices of $\Delta(S)$ which lie inside the cone $\mathcal{P}^+$ are associated to contractions to $\mathbb{P}^2$ which are inverse of blow-ups of a number of points in $\mathbb{P}^2$. We will call these vertices of $\Delta(S)$ of type two.

3. Suppose that $\rho = \rho(S) \geqslant 4$. Prove that the facets of $\Gamma(S)$ are of two types:

   (a) Simplicial, dual to to vertices of $\Delta(S)$ of type 2.

   (b) Cross-polytopes with vertices generated by classes of (-1)-curves $C_1, C_1', \ldots, C_{\rho-2}, C'_{rho-2}$ satisfying relations

   $$C_1 + C_1' \equiv \cdots \equiv C_{\rho-2} + C'_{\rho-2}$$

   These are the facets dual to vertices of Δ of the first type.

4. Draw polytopes $\Delta(S)$ and $\Gamma(S)$ for $\rho(S) \leqslant 3$. Prove that these are the only possibilities. For every polytope $\Delta(S)$ with $r = \rho(S) - 1$ define a polynomial

   $$P_r(x, y) = \sum a_i x^i + b x^r y$$

   where b is the number of vertices of type 1 and a_i is the number of codimension i faces of $\Delta(S)$ except the vertices of type 1. Prove that:

   (a) $P_2(x, y) = 1 + 3x + x^2 + 2x^2 y$

   (b) $P_3(x, y) = 1 + 6x + 9x^2 + 2x^3 + 3x^3 y$

5. Prove that polynomials P_r satisfy the following equations:

   - $\partial_x P_r(x, 0) = \partial_x P_r(0, 0) \cdot P_{r-1}(x, 0)$
   - $2(r-1) \cdot \partial_y P_r(1, 0) = \partial_x P_r(0, 0) \cdot \partial_y P_{r-1}(1, 0)$
   - $P_r(-1, 1) = (-1)^r$