Fano Manifolds, Spring 2018
Fourth problem set. Contractions of curves.

---

1. Let S be a smooth projective surface. Hodge index theorem says that the intersection form on $N(S) = N^1(S) = N_1(S)$ is of index $(+, -, \cdots, -)$. Using the Hodge index theorem prove the following:

   (a) the set $\mathcal{P} = \{u \in N(S) : u^2 > 0\}$ has two components;

   (b) the closure (standard topology) of one of the components of $\mathcal{P}$, call it $\mathcal{P}^+$, contains the cone $\mathcal{A}$ generated by the classes of ample divisors;

   (c) we have the inclusion of convex cones:

   $$\mathcal{A} \subseteq \mathcal{C}^\vee \subseteq \overline{\mathcal{P}}^+ \subseteq \mathcal{C}$$

   where $\mathcal{C} = \mathcal{C}(S)$ is the Mori cone of effective 1-cycles and $\mathcal{C}^\vee$ its dual.

   In what follows we will use Kleiman theorem which says $\overline{\mathcal{A}} = \mathcal{C}^\vee$.

2. A curve C on a variety X is extremal if from the numerical equivalence $C \equiv a_1C_1 + a_2C_2$, with $a_1, a_2 > 0$ and C_1, C_2 curves it follows that C, C_1 and C_2 are numerically proportional, that is $C \equiv b_1C_1 \equiv b_2C_2$. Prove that if C is extremal on a surface S and $C^2 > 0$ then $\rho(S) = \dim N^1(S) = 1$.

3. Let S be a smooth algebraic surface with $\rho(S) > 1$. Suppose that the curve C is extremal and $-K_S \cdot C = 2$.

   (a) Prove that $C \simeq \mathbb{P}^1$ and $C^2 = 0$, use the adjunction formula for the normalization $\widehat{C} \to C$.

   (b) Prove that the linear system $|\mathcal{O}(C)|$ is base-point free of dimension 1 hence it defines a morphism $\pi : S \to B$ onto a smooth curve B, which contract C to a point.

   (c) Prove that all fibers of π are rational curves numerically equivalent to C, conclude that $\rho(S) = 2$.

   (d) Suppose that $-K_S$ is ample. Prove that either $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or S is a blow-up of $\mathbb{P}^2$ at one point. Use the next exercise.

4. Let S be a smooth algebraic surface with $\rho(S) > 1$. Suppose that the curve C is extremal and $-K_S \cdot C = 1$.

(a) Prove that $C \simeq \mathbb{P}^1$ and $C^2 = -1$, use the adjunction formula for the normalization $\widehat{C} \to C$.

(b) From this point on assume that $-K_S$ is ample so that, by Mori theorem, $\mathcal{C}(S)$ is rational polyhedral; prove that $C^\perp \cap \mathcal{C}^\vee = \{u \in \mathcal{C}^\vee : u \cdot C = 0\}$ is a facet (codimension 1 face) of $\mathcal{C}^\vee$.

(c) Prove that there exists a divisor D on S such that $[D]$ is in the relative interior of $C^\perp \cap \mathcal{C}^\vee$ and $D^2 > 0$; conclude that if $D \cdot C' = 0$ for some curve $C' \subset S$ then $C' = C$.

(d) Prove that for $m \gg 0$ the class $mD - K_X$ is in the interior of the cone $\mathcal{C}^\vee$, conclude that the function $h^0(S, \mathcal{O}(mD))$ grows like degree two polynomial with leading term $(D^2/2)m^2$.

(e) Prove that for $m \gg 0$ for any curve $C' \subset S$ we have $H^1(S, \mathcal{O}(mD - C')) = 0$. Use Kleiman's theorem and Kodaira vanishing.

(f) Prove that for $m \gg 0$ the linear system $|\mathcal{O}(mD)|$ is base-point-free and the associated map $\phi : S \to S'$ contracts C to a point and does not contract any other curve.