**Fano Manifolds, Spring 2018**
**Third problem set.** Various problems.

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1. Addendum to the first problem sheet. Suppose that $\phi : X \to Y$ is a morphism of projective varieties.

   (a) Prove that $\phi$ induces homomorphisms $\phi_* : N_1(X) \to N_1(Y)$ and $\phi^* : N^1(Y) \to N^1(X)$ which agrees with the intersection of 1-cycles and divisors; that is $\phi_*(C) \cdot D = C \cdot \phi^*(D)$.

   (b) Prove that $\phi_*$ is surjective and $\phi^*$ is injective if $\phi$ is surjective.

   (c) Show that injectivity of $\phi$ does not imply injectivity of $\phi_*$.

2. A $\mathbb{Q}$-Cartier divisor $D$ on a variety $X$ is nef if $C \cdot D \geqslant 0$ for every curve $C$. A variety $X$ is uniruled if there exists a dominating rational map $\psi : M \times \mathbb{P}^1 - \to X$, where $\dim M = \dim X - 1$. Prove that if $X$ is uniruled then $K_X$ is not nef. Use the ramification formula.

3. Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to M$ be a $\mathbb{P}^1$-bundle. Suppose that $\pi$ admits a section $\sigma : M \to Y$, $\pi \circ \sigma = id_M$. Let us assume that there exists a surjective morphism $\psi : Y \to X$ of projective varieties, such that $\psi \circ \sigma(M)$ is a point. Prove that then $\dim N^1(X) = 1$.

4. Let $S$ be a smooth projective surface such that $-K_S \cdot C \geqslant 3$ for every curve $C \subset S$. Prove that $S \simeq \mathbb{P}^2$.

   (a) Use Mori theorem of the existence of rational curves to see that there exists a rational curve $C \subset S$ such that $-K_X \cdot C = 3$.

   (b) Use Mori-type arguments on $Mor((\mathbb{P}^1, p), (S, x))$ to prove that there exists a map like in the previous problem, hence $\dim N^1(S) = 1$.

   (c) Prove that $-K_S \equiv 3C$ and conlude that $S \simeq \mathbb{P}^2$.

5. Let $C \subset S$ be a curve on a smooth surface. Consider the normalization $f : \widehat{C} \to C \subset X$.

   (a) Suppose that $C$ is defined locally by zeroes of regular functions $f_i$. Prove that differentials $df_i$ determine an injection $\mathcal{O}_C(-C) \to \Omega_{S|C}$.

(b) Prove that there an exact sequence of sheaves on $\widehat{C}$

$$0 \to f^*\mathcal{O}_S(-C) \to f^*\Omega_S \to \Omega_{\widehat{C}}$$

with torsion cokernel of the right-hand-side map.

(c) Prove that

$$g(\widehat{C}) \leqslant \frac{1}{2}(K_S + C) \cdot C + 1$$

with equality if and only if $C$ is smooth.

6. Del Pezzo surfaces. Consider the Veronese map $\nu : \mathbb{P}^2 \to \mathbb{P}^9$ defined by the anticanonical system $|\mathcal{O}(3)|$. The image will be called $S_0$. For $r = 1, \ldots, 6$ we define $S_r$ as the image of the projection

$$\pi_r : \mathbb{P}^{9-r+1} \supset S_{r-1} - \to S_r \subset \mathbb{P}^{9-r}$$

from a general point $x \in S_{r-1}$.

(a) Prove that $S_r$ is smooth and embedded in $\mathbb{P}^{9-r}$ via the anticanonical system $|-K_{S_r}|$.

(b) Prove that $S_r$ is obtained from $S_{r-1}$ by the blow-up at $x \in S_{r-1}$.

(c) Prove that $(-K_{S_r})^2 = 9 - r$.