Fano Manifolds, Spring 2018
Second problem set. Ruled surfaces.

We work over an algebraically closed field, usually $\mathbb{C}$.
Review the definition of Weil and Cartier divisors, $WDiv(X)$ and $CDiv(X)$, as well as of class group $\mathrm{Cl}(X)$ and Picard group $\mathrm{Pic}(X)$ of a variety X, see e.g. [Hartshorne, II.6]. If D is a Cartier (Weil) divisor on X, then $\mathcal{O}(D) = \mathcal{O}_X(D)$ denotes the associated invertible (respectively reflexive) sheaf on X.

Please read Chapter 2 of Reid *Chapters on Surfaces*. For more information about ruled surfaces see also [Hartshorne, V.2].

For a rank $r+1$ vector bundle $\mathcal{E}$ (or a locally free sheaf) on a variety X we take $\mathbb{P}(\mathcal{E}) = Proj_X(Sym\mathcal{E})$ with the sheaf $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the projection $\pi : \mathbb{P}(\mathcal{E}) \to X$ which is $\mathbb{P}^r$ bundle such that $\pi_* \mathcal{O}(1) = \mathcal{E}$.

1. With the notation introduced above prove the following
   (a) For any line bundle $\mathcal{L}$ over X we have $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \simeq \mathbb{P}(\mathcal{E})$ with
   $$\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{L})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{L}$$
   (b) Suppose that X is smooth (or, at least, $-K_X$ is Cartier) then $\mathbb{P}(\mathcal{E})$ has the same feature and
   $$\mathcal{O}(-K_{\mathbb{P}(\mathcal{E})}) = \pi^*(\mathcal{O}(-K_X) \otimes \det \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r+1)$$
2. Sections of projective bundles. With the notation introduced above consider a surjective map of locally free sheaves $\mathcal{E} \to \mathcal{F}$ over a variety X.
   (a) Prove that we have an induced embedding $\mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$ which commutes with projections onto X.
   (b) Given a projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ with a section $\sigma : X \to \mathbb{P}(\mathcal{E})$ (that is, $\pi \circ \sigma = id_X$) a line bundle $\mathcal{L} = \sigma^*\mathcal{O}_X(1)$ prove that there exists a surjective morphism $\mathcal{E} \to \mathcal{L}$ such that the induced morphism $X = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{E})$ coincides with σ.
   (c) Prove that given a non-zero section $s \in \mathrm{H}^0(X, \mathcal{E})$ it determines a divisor $D_s \subset \mathbb{P}(\mathcal{E})$ such that $\pi : D_s \to X$ is generically a $\mathbb{P}^{r-1}$ bundle with r-dimensional fibers over the set of zeroes of s.

3. Calculate Čech cohomology of $\mathcal{O}(a)$ on $\mathbb{P}^1$. Prove that $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(a)) = \max(0, a+1)$ and $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(a)) = \max(0, -a-1)$ Use the standard affine covering $\mathcal{U}$ of $\mathbb{P}^1$ consisting of two copies of $\mathbb{A}^1$, U_0 and U_∞ with coordinates z and z^{-1}, respectively. Then

   (a) the sheaf $\mathcal{O}(a)$ is associated to Cartier divisor $(U_0, z^a), (U_\infty, 1)$ hence given in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}^*)$ by the 1-cocycle $g_{\infty 0} = z^{-a}$ (we use right-to-left notation $g_{\infty 0} f_0 = f_\infty$);

   (b) for $a \geqslant 0$ a basis of sections of $\mathcal{O}(a)$ is given by 0-cocycle $(U_0, z^t), (U_\infty, z^{t-a})$ with $t = 0, \ldots, a$

   (c) for $a \leqslant -2$ a basis of cohomology $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}(a))$ is given by cocycles $(U_0 \cap U_\infty, z^t)$ with $0 > t > -a$, where the trivialization of $\mathcal{O}(a)$ on $U_0 \cap U_\infty$ comes from embedding $U_0 \cap U_\infty \hookrightarrow U_0$.

4. All bundles over $\mathbb{P}^1$ are decomposable. Prove that every locally free sheaf $\mathcal{E}$ over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_r)$ with $a_0 \leqslant \cdots \leqslant a_r$. Use induction and the previous exercise and the following steps:

   (a) For every $\mathcal{E}$ over $\mathbb{P}^1$ we can choose $d \in \mathbb{Z}$ such that $\mathcal{E}(d) = \mathcal{E} \otimes \mathcal{O}(d)$ has a section but $\mathcal{E}(d-1)$ has no section.

   (b) If $s \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{E})$ is a nowhere vanishing section of a locally free sheaf $\mathcal{E}$ then the cokernel $\mathcal{E}' = \mathcal{E}/\mathcal{O}$ is locally free.

5. Intersection on a ruled surface. Let $\pi : S = \mathbb{P}(E) \to C$ be a ruled surface over a (smooth) curve C; that is $\mathcal{E}$ is of rank 2.

   (a) Prove that, possibly twisting $\mathcal{E}$ by a line bundle, we may assume that $\mathrm{H}^0(C, \mathcal{E}) \neq 0$ and $\mathrm{H}^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for every line bundle $\mathcal{L}$ which is of degree < 0.

   (b) Prove that every section s of $\mathcal{E}$ vanishes nowhere so the resulting cokernel of $s : \mathcal{O} \to \mathcal{E}$ is a line bundle. By C_0 we denote the associated section of the ruling π.

   (c) Prove that $d = \deg(\det \mathcal{E})$ satisfies inequality $d \leqslant g(C)$ and if $d < 0$ then C_0 is unique.

   (d) Prove that $(\operatorname{Pic} S/\equiv) = \mathbb{Z}[f] \oplus \mathbb{Z}[C_0]$ where $\equiv$ denotes numerical equivalence, $[f]$ is the class of a fiber of π and $[C_0]$ is the class of a section of the ruling π associated to the cokernel of a section $\mathcal{O} \hookrightarrow \mathcal{E}$.

(e) Prove that $f^2 = 0$, $f \cdot C_0 = 1$ and $C_0^2 = d$.

6. Breaking lemma. Let $\pi : S = \mathbb{P}(E) \to C$ be a ruled surface over a curve C. Let $\psi : S \to \mathbb{P}^N$ be a morphism which does not contract any fiber of π. (Contracted means mapped to a point.) Prove that one of the following holds:

   (a) the morphism ψ is finite-to-one, that is ψ has no positive dimensional fiber, or

   (b) there exists a unique curve $C_0 \subset S$ which is contracted by ψ and this curve is the unique section of π from the previous problem such that $C_0^2 = d < 0$, or

   (c) the image of the morphism ψ is a rational curve and $d = 0$.