Fano Manifolds, Spring 2018
First problem set. Intersection of divisors and curves.

We work over an algebraically closed field, usually $\mathbb{C}$.
Review the definition of Weil and Cartier divisors, $WDiv(X)$ and $CDiv(X)$, as well as of class group $\mathrm{Cl}(X)$ and Picard group $\mathrm{Pic}(X)$ of a variety X, see e.g. [Hartshorne, II.6]. If D is a Cartier (Weil) divisor on X, then $\mathcal{O}(D) = \mathcal{O}_X(D)$ denotes the associated invertible (respectively reflexive) sheaf on X.

1. Algebraic lemmata. Let $k(X)$ be a field of dimension (trancendence degree) 1 over k with discrete valuation μ whose valuation ring is A. By the theory of valuation we know that A is a regular local ring with maximal ideal $\mathfrak{m}_A$ generated by t. Consider finite extension $k(X) \subset k(Y)$ of degree d. Let $B \subset k(Y)$ be the integral closure of A in $k(Y)$ and let $\mathfrak{n}_B = \mathfrak{m}_A \cdot B = t \cdot B$.
   (a) Prove that B is integrally closed in $k(Y)$, hence normal, hence regular, hence every maximal ideal is generated by a single element.
   (b) Prove that B is torsion free over A and therefore it is a free module over A of rank d, use Nakayama lemma, see e.g. [Atiyah-Macdonald, ch. 7, exercise 15]. Conclude that $B/\mathfrak{n}_B$ is of dimension d over $A/\mathfrak{m}_A = k$
   (c) Take the minimal primary decomposition $\mathfrak{n}_B = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. Prove that $\mathfrak{m}_i = \sqrt{\mathfrak{q}_i}$ are different maximal ideals.
   (d) Prove that $\dim_k B/\mathfrak{q}_i$ is equal to the valuation of t in the local ring $B_{\mathfrak{m}_i}$.
   (e) Use the Chinese remainder theorem to prove that $\dim_k B/\mathfrak{q}_1 + \cdots + \dim_k B/\mathfrak{q}_r = d$.

2. Degree homomorphism. Let C be a complete normal (hence smooth) curve over the field k. We define the degree map $\deg_C : WDiv(C) \to \mathbb{Z}$, such that for $D = \sum a_i p_i$ with $p_i \in C$, $a_i \in \mathbb{Z}$, we set $deg_C(D) = \sum_i a_i$.
   (a) Let $\phi : C_1 \to C_2$ be a finite morphism of smooth complete curves of degree $d = [k(C_1) : k(C_2)]$. Use the previous exercise to prove that for every $D \in WDivC_2$ we have $\deg_{C_1}(\phi^*(D)) = d \cdot \deg_{C_2}(D)$, [Hartshorne, II.6.9]
   (b) Given a non-constant rational function $f \in k(C)$ on a normal complete curve C. Prove that the finite extension of fields $k(f) \subseteq k(C)$ extends to a finite morphism $\widehat{f} : C \to \mathbb{P}^1$. Conclude that $\deg_C(div(f)) = 0$

(c) Conclude that $\deg_C$ descends to a homomorphism $\deg_C \operatorname{Pic}(C) = \operatorname{Cl}(C) \to \mathbb{Z}$.

3. Characterisation of $\mathbb{P}^1$. In the situation of the previous exercise prove that $\deg : \operatorname{Pic}(X) \to \mathbb{Z}$ is not an isomorphism unless $X = \mathbb{P}^1_k$. Hint: consider a divisor $D = p_0 - p_\infty \in WDiv(X)$, where $p_0 \neq p_\infty$ are two points.

4. The first Chern class on complex manifolds. Let X be a complex manifold with Euclidian topology and the structural sheaf $\mathcal{O}_X$ of holomorphic functions.

   (a) Prove that the following sequence of sheaves with second arrow $f \to \exp(2\pi i \cdot f)$ is exact

   $$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

   The boundary morphism $\operatorname{Pic}(X) = H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ is called the first Chern class, we denote it c_1.

   (b) Prove that the first Chern class for $X = \mathbb{P}^1_{\mathbb{C}}$ is the degree homomorphism. You may use direct Čech cohomology calculations with a covering in which every intersection of sets is contractible. For example in $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with non-homogeneous coordinate z consider a triangulation by the following simplices $\Delta_0 = \{|z| \geq 1\}$ and $\Delta_j = \{|z| \leq 1, Arg(z) \in [2\pi(j-1)/3, 2\pi j/3]\}$, for $j = 1,\ 2,\ 3$. Next, define covering $\mathcal{U} = \{U_j\}$ with $U_j = \mathbb{P}^1_{\mathbb{C}} \setminus \Delta_j$.

      i. Prove that $H^p(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$ for $p = 0,\ 2$ and it is zero otherwise.

      ii. Calculate the boundary map $H^1(\mathcal{U}, \mathcal{O}^*) \to H^2(\mathcal{U}, \mathbb{Z})$ for the cocycle $\sigma \in \prod \mathcal{O}^*(U_{ij})$ such that $\sigma_{0j} = z^d$ and $\sigma_{ij} = 1$ for $i,\ j = 1,\ 2,\ 3$. Remember to fix a branch of log on each of the sets U_{ij}.

   (c) Prove that $deg_C = c_1$ for any projective curve C. Use the fact that, for a finite morphism $\phi : C_1 \to C_2$ the map $\phi^* : H^2(C_2, \mathbb{Z}) \to H^2(C_1, \mathbb{Z})$ is the multiplication by degree of the map.

5. Numerically equivalent divisors and 1-cycles. Let X be a complete (e.g. projective) variety and $C \subset X$ a curve. Take its normalization $f_C : \widehat{C} \to C \subset X$. For any Cartier divisor D on X we define $D \cdot C = \deg_{\widehat{C}} f_C^* D$. We say that D_1 is numerically equivalent to D_2, denoted $D_1 \equiv D_2$ if $D_1 \cdot C = D_2 \cdot C$ for every curve C on X.

(a) Prove that if $c_1(D) = 0$ then $D \equiv 0$. Derive from this that numerical equivalence is well defined on $\mathrm{Pic}(X)$.

(b) Let $N^1(X)_{\mathbb{Q}} = (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{Q}$ and similarly define $N^1(X)_{\mathbb{R}}$. Prove that these are finite dimensional vector spaces, use the fact that X is of type of a finite simplicial complex.

(c) Dually we define

$$N_1(X)_{\mathbb{Q}} = \left(\left\{\sum a_i C_i : C_i \subset X, a_i \in \mathbb{Z}\right\} / \equiv\right) \otimes \mathbb{Q}$$

Prove that $(D, C) \mapsto D \cdot C$ induces a perfect pairing

$$N^1(X)_{\mathbb{Q}} \times N_1(X)_{\mathbb{Q}} \to \mathbb{Q}$$

6. Riemann-Roch theorem on a smooth surface. We will use the weak form of RR theorem for curves: if $\widehat{C}$ is a smooth curve and D a Cartier divisor on $\widehat{C}$ then

$$\chi(\widehat{C}, \mathcal{O}(D)) = \deg D + \chi(\mathcal{O}_{\widehat{C}})$$

(a) Prove that RR formula holds for a possibly non-normal curve C with normalization $\widehat{C} \to C$. Show that $h^1(\mathcal{O}_C) - h^1(\mathcal{O}_{\widehat{C}})$ is the length of the cokernel of the normalization $\mathcal{O}_C \to f_*\mathcal{O}_{\widehat{C}}$.

(b) Prove the following version of RR for smooth surfaces: if D a Cartier divisor and C a curve on a smooth surface S then

$$\chi(S, \mathcal{O}(D)) = \chi(S, \mathcal{O}(D - C)) + \chi(\widehat{C}, f_C^*(D)) - h^1(\mathcal{O}_C) + h^1(\mathcal{O}_{\widehat{C}})$$

7. Intersection on a smooth surfaces. Let D_1, D_2 be Cartier divisors on a complete smooth surface S. We define the intersection $D_1 \cdot_S D_2$ on S as the coefficient with monomial $t_1 t_2$ in the Hilbert polynomial

$$\chi(X, \mathcal{O}(t_1 D_1 + t_2 D_2)) = \sum (-1)^i h^i(X, \mathcal{O}(t_1 D_1 + t_2 D_2))$$

(a) Prove that the above definition makes sense. Use the fact that every divisor is a difference of effective divisors.

(b) Prove that the intersection of divisors is symmetric and bilinear.

(c) Show that on a smooth surface the intersection of divisors coincides with intersection of divisors and 1-cycles.

(d) Prove that the intersection of curves on a smooth surface S is a symmetric bilinear form on $N^1(S) = N_1(S)$.

(e) Let $\beta : S' \to S$ be a blow-up with the exceptional divisor E. Prove that we have a decomposition $N^1(S') = \beta^* N^1(S) \oplus \mathbb{Q} \cdot E$ which is orthogonal in terms of the intersection form.

8. Examples. In this exercise, we consider the group $\mathrm{Pic}(S)$ with the intersection form (symmetric bilinear form) for each of the surfaces below. Prove that each of the *Pic* groups is a two-dimensional lattice $\mathbb{Z}^2$ and decide if for any two of them there exists an isomorphism which preserves the intersection form.

(a) $S = \mathbb{P}^1 \times \mathbb{P}^1$

(b) S is the blow up of $\mathbb{P}^2$ at one point

(c) First blow-up $\mathbb{P}^2$ at two different points; show that the strict transform of the line passing through them is a (-1) curve: then take the surface S which is obtained by contracting this curve.