Existence of solutions in fully anisotropic and inhomogeneous Musielak-Orlicz space

Ying Li

Shanghai University

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MIMUW Warsaw

Based on joint work with Iwona Chlebicka, Arttu Karppinen and Bartosz Budnarowski.

The talk is based on the following papers:

- Iwona Chlebicka, Arttu Kappinen, Ying Li, *A direct proof of existence of weak solutions to fully anisotropic and inhomogenous elliptic problems.* (Submitted)
- Bartosz Budnarowski, Ying Li, *Existence of renormalized solutions to fully anisotropic and inhomogenous elliptic problems.* (Submitted)

Outline

Definition of N-function

A function $M(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}$ is called an N-function if

- it is a Carathéodory function satisfying $M(x,0)=0$;
- it is a convex function with respect to ξ;
- $M(x,\xi)=M(x,-\xi)$ for a.e. $x\in\Omega$;
- there exist two convex functions $m_1,m_2:[0,\infty)\to[0,\infty)$ such that

$$\lim_{s\to 0+}\frac{m_1(s)}{s}=0=\lim_{s\to 0+}\frac{m_2(s)}{s}\quad\text{and}\quad\lim_{s\to\infty}\frac{m_1(s)}{s}=\infty=\lim_{s\to\infty}\frac{m_2(s)}{s},$$

and for a.e. $x\in\Omega$

$$m_1(|\xi|)\le M(x,\xi)\le m_2(|\xi|).$$

Musielak-Orlicz space

Suppose $\Omega \in \mathbb{R}^n$.

- For an N-function we define the general Musielak–Orlicz class $\mathcal{L}_M(\Omega)$ as the set of all measurable functions $\xi : \Omega \to \mathbb{R}^n$ satisfying

$$\int_\Omega M(x, \xi(x))\, dx < \infty\,.$$

- $L_M(\Omega)$ are defined as sets of functions $\xi : \Omega \to \mathbb{R}^n$ satisfying

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for some $\lambda \in \mathbb{R}$.

- $E_M(\Omega)$ are defined as sets of functions $\xi : \Omega \to \mathbb{R}^n$ satisfying

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Complementary function

Complementary (conjugate, Legendre's transform) function M^* to an N-function M if defined by

$$M^*(x,\eta) := \sup_{\xi\in\mathbb{R}^n} [\xi\cdot\eta - M(x,\xi)] \quad \text{for any } \eta\in\mathbb{R}^n \text{ and a.e. } x\in\Omega.$$

- M^* is an N-function.

The Fenchel–Young inequality reads

$$\xi\cdot\eta \le M(x,\xi) + M^*(x,\eta) \quad \text{for all } \xi,\eta\in\mathbb{R}^n \text{ and a.e. } x\in\Omega.$$

Notation

$$V_0^1 L_M(\Omega) = \{u\in W_0^{1,1}(\Omega) : \nabla u \in L_M(\Omega)\}.$$

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Some properties of Musielak-Orlicz Space

- $E_M(\Omega;\mathbb{R}^n) \subset \mathcal{L}_M(\Omega;\mathbb{R}^n) \subset L_M(\Omega;\mathbb{R}^n)$
  Without growth conditions on M the inclusions are proper!
- The space $E_M(\Omega;\mathbb{R}^n)$ is the closure in L_M-norm of the set of bounded functions.
- $(E_M(\Omega;\mathbb{R}^n))^* = L_{M^*}(\Omega;\mathbb{R}^n)$ and $(E_{M^*}(\Omega;\mathbb{R}^n))^* = L_M(\Omega;\mathbb{R}^n)$ but no other duality relations are expected.
- Both are equipped with Luxemburg norm

$$\|\xi\|_{L_M(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega M\left(x, \frac{\xi(x)}{\lambda}\right) dx \leq 1\right\}.$$

- If $M \in \Delta_2$, then $L_M(\Omega;\mathbb{R}^n) = E_M(\Omega;\mathbb{R}^n)$.

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Δ_2 condition

Definition of Δ_2-condition

We say that an N-function $M : \Omega \times [0, \infty) \to \mathbb{R}$ *satisfies* Δ_2*-condition if there exists a constant* $c > 0$ *and* $h \in L^1(\Omega)$*,*$h \geq 0$*,such that*

$$M(x, 2s) \leq cM(x, s) + h(x).$$

Important! $M, M^* \in \Delta_2 \Longleftrightarrow L_M$ *is reflexive and separable.*

But, in our paper, we do not control the growth of M with respect to the second variable by any kind of doubling condition or a power function.

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Framework

The leading part of the operator satisfies general conditions settling the problem in the framework of fully anisotropic and inhomogeneous Musielak-Orlicz space.

- general growth – when the power function governing the growth of the operator is substituted by an N-function $M(x, \xi) = M(|\xi|)$, which do not necessarily satisfy the so-called Δ_2-condition (being a necessary condition for an Orlicz space L_M to be reflexive);
- inhomogeneity – when the growth of the operator could be controlled by an x-dependent function e.g. $M(x, \xi) = |\xi|^{p(x)}$ (which results in the lack of the density of smooth functions in $L^{p(\cdot)}$, if $p(\cdot)$ is not regular enough);
- anisotropy – when the growth of the operator is governed by a function depending on the full vector of ξ, not just its length $|\xi|$.

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Anisotropy

A function M which admits a decomposition

$$M(x,\xi)=\sum_{i=1}^{n}M_i(x,|\xi_i|),\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n,M_i:\Omega\times\mathbb{R}\to[0,\infty),$$

is called **orthotropic function**.

- $M(x,\xi)=\sum_{i=1}^{n}|\xi_i|^{p_i}$,

They have monotonicity property: if
$\xi=(\xi_1,\cdots,\xi_n),\eta=(\eta_1,\cdots,\eta_n),|\xi_i|\le|\eta_i|,$ then $M(x,\xi)\le M(x,\eta)$.
(But, it not true in general!)
The family of fully anisotropic function is far more robust!

Essentially fully anisotropic: *if there exists no linear invertible map $T:\mathbb{R}^n\to\mathbb{R}^n$ such that*

$$M(x,T(\xi_1,\cdots,\xi_n))=\sum_{i=1}^{n}M_i(x,|\xi_i|)$$

for some Young functions $M_i:\Omega\times\mathbb{R}\to[0,\infty)$. [Chlebicka,Nayar, MMAS 2021].

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If we project it onto a 2-Dimension plane based on the growth, we will see

Figure: isotropic

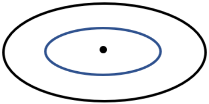

Figure: orthotropic

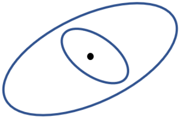

Figure: anisotropic

- **Isotropic**: $M(x,\xi) = M(x,|\xi|)$. Rely on the length of $|\xi|$.
- **Orthotropic**: $M(x,\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}$. Described by its behavior in each direction separately.
- **Essentially fully anisotropic**: It's impossible to indicate the direction of the quickest growth.
  (The direction of the quickest growth man change on each level set.)

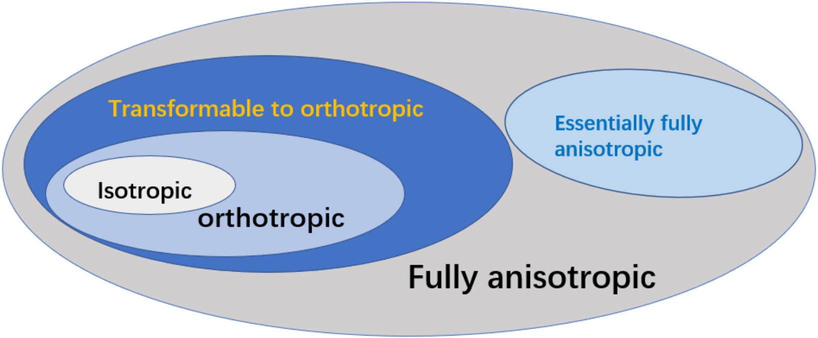

Figure: Venn diagram

Modular density

Under no such control on the growth, in the case of

- **Classical Orlicz-Sobolev space**: Gossez, (Studia Math.1982)
  Smooth functions are dense only with respect to modular topology (not in norm).
  Anisotropic: [Alberico, Chlebicka, Cianchi, Zatorska-Golstein, CalcVar2018]
- **Musielak-Orlicz-Sobolev space**: To get modular density of smooth function in a Musielak-Orlicz-Sobolev space, one need to assume that there is a condition balancing the behaviour of M with respect to its variable.
  Ahmida, Borowski, Chlebika, Gwiazda, Miasojedow, Skrzeczkowski, Świerczewska-Gwiazda, Wróblewska-Kamińska, Youssfi, Zatorska-Golstein...

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Modular Convergence

Definition (Modularly convergence)

A sequence $\{\xi_n\}_{n=1}^{\infty}$ converges modularly to ξ in $L_M(\Omega)$, which we denote as $\xi_i \xrightarrow{M} \xi$, if

$$\int_\Omega M\left(x, \frac{\xi_i - \xi}{\lambda}\right) dx \xrightarrow{n\to\infty} 0$$

for some $\lambda > 0$.

- If $\xi_n \xrightarrow{M} \xi$ in $L_M(\Omega)$ then, up to a subsequence, $\xi_n \xrightarrow{n\to\infty} \xi$ in $\sigma(L_M, L_{M^*})$.
- Let X and Y be subsets of $L^1(\Omega)$ not necessarily related by duality. We say $f_n \to f$ for $\sigma(X, Y)$ if

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Quick Review-Just mention a few

The difficulty caused by the lack of reflexivity of L_M under non-doubling regime was avoided by the idea of the complementary systems in Orlicz–Sobolev spaces. Contributions in this direction were initiated by Donaldson

T.Donaldson. *Nonlinear elliptic boundary value problems in Orlicz – Sobolev spaces.* In: Journal of Differential Equations 10.3 (1971), pp. 507 – 528.

and continued by Gossez, Mustonen and Tienari,

J. Gossez. *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients.* In: Transactions of the American Mathematical Society 190 (1974), pp. 163 – 205.

J. Gossez. *Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems.* In: Nonlinear analysis, function spaces and applications (1979), pp. 59 – 94.

V. Mustonen and M. Tienari. *On monotone-like mappings in Orlicz – Sobolev spaces.* In: Mathematica Bohemica 124.2-3 (1999), pp. 255 – 271.

Quick Review-Just mention a few

- For analysis of problems in anisotropic Orlicz spaces governed by possibly fully anisotropic modular function, which is independent of the spacial variable:

A. Alberico, I. Chlebicka, A. Cianchi, A. Zatorska-Golstein. *Fully anisotropic elliptic problems with minimally integrable data*. In: Calc. Var. Partial Differential Equations 58:186 (2019).

G. Barletta and A. Cianchi. *Dirichlet problems for fully anisotropic elliptic equations*. In: Proc. Roy. Soc. Edinburgh Sect. A 147.1 (2017), pp. 25 – 60.

I. Chlebicka and P. Nayar. *Essentially fully anisotropic Orlicz functions and uniqueness to measure data problem*. In: Math. Methods Appl. Sci. 45.14 (2022), pp. 8503 – 8527

Quick Review-Just mention a few

- Existence to problems that are in the same time of general growth, inhomogeneous, and fully anisotropic were studied in:

A. Denkowska, P. Gwiazda, and P. Kalita. *On renormalized solutions to elliptic inclusions with nonstandard growth*. In: Calc. Var. Partial Differential Equations 60.1 (2021), 21:52.

I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. *Existence of renormalized solutions to elliptic equation in Musielak-Orlicz space* . In: Journal of Differential Equations 264.1 (2018), pp. 341 – 377.

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- Anisotropic problems with lower-order terms are less understood – we can only refer to:

A. DiCastro. *Anisotropic elliptic problems with natural growth terms*. In: Manuscripta Math 135.3-4 (2011), pp. 521 – 543.

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Main results-First one

In this talk, the first result I will introduce is the existence of weak solutions for the following problem:

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x,\nabla u)+\Phi(u)\big)+b(x,u)=\operatorname{div}F & \text{in} \quad \Omega, \\ u(x)=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n > 1$.

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The First Result-Existence of weak solution

Vector field $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(A1) $\mathcal{A}$ is a Carathéodory's function;

(A2) [**Gowth and coercivity condition**] $\mathcal{A}(x, 0) = 0$ for almost every $x \in \Omega$ and there exists an N-function $M : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and constants $c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, c_3^{\mathcal{A}}, c_4^{\mathcal{A}} > 0$ such that for all $\xi \in \mathbb{R}^n$ we have

$$\mathcal{A}(x, \xi) \cdot \xi \geq M(x, c_1^{\mathcal{A}} \xi) - h_1(x)$$

and

$$c_2^{\mathcal{A}} M^*(x, c_3^{\mathcal{A}} \mathcal{A}(x, \xi)) \leq M(x, c_4^{\mathcal{A}} \xi) + h_2(x),$$

where M^* is the conjugate to M and $h_1, h_2 \in L^1(\Omega)$;

(A3) [**Monotone condition**] For all $\xi, \eta \in \mathbb{R}^n$ and for almost every $x \in \Omega$ we have

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- In the case p-growth. $M = c|\xi|^p$, $(A2)$ directly imply

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Existence of weak solutions

Moreover, we assume that

(P) $\Phi : \mathbb{R} \to \mathbb{R}^n$ is bounded and continuous;

(b) $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function, which is nondecreasing with respect to the second variable, and such that $b(\cdot, s) \in L^1(\Omega)$ and $b(\cdot, s)\,\mathrm{sign}(s) \geq 0$ for every $s \in \mathbb{R}$.

- *Let $\Phi : \mathbb{R} \to \mathbb{R}^n$ be continuous and belong to $L^\infty(\Omega, \mathbb{R}^n)$. Let $u \in W_0^{1,1}(\Omega)$. Then*

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Theorem (I. Chlebicka, A. Kappinen, Y.Li, submitted)

Let $\Omega \in \mathbb{R}^n$. N-function M is regular enough so that the set of smooth functions is dense in $V_0^1 L_M(\Omega)$ in the modular topology. Assume further that $F \in E_{M^*}(\Omega)$, $\mathcal{A}$ satisfies assumptions (A1), (A2) and (A3), Φ satisfies (P), and b satisfies (b). Then there exists at least one weak solution to the problem (1). *Namely, there exists a function $u \in V_0^1 L_M(\Omega)$ satisfying*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v + \Phi(u) \cdot \nabla v + b(x,u) v \, dx = \int_\Omega F \cdot \nabla v \, dx$$

for all $v \in V_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.

Proposition

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- The set of smooth functions is dense in $V_0^1 L_M(\Omega)$ in the modular topology can be ensured by the Balance condition (B).

Condition (B). Given an N-function $M : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ suppose there exists a constant $C_M > 1$ such that for every ball $B \subset \Omega$ with $|B| \leq 1$, every $x \in B$, and for all $\xi \in \mathbb{R}^n$ such that $|\xi| > 1$ and $M(x, C_M\xi) \in [1, \frac{1}{|B|}]$ there holds $\sup_{y\in B} M(y,\xi) \leq M(x, C_M\xi)$.

Theorem (Borowski-Chlebicka, J. Funct. Anal.(2022))

Assume that Ω is a Lipschitz domain and M is an N-function satisfying the Balance condition (B). Then for any $\phi \in V_0^1 L_M(\Omega)$, there exists a sequence $\{\phi_\delta\}_{\delta>0} \in C_c^\infty(\Omega)$ satisfying $\phi_\delta \to \phi$ in $L^1(\Omega)$ and $\nabla\phi_\delta \xrightarrow{M} \nabla\phi$. Additionally, if ϕ is bounded, then $\|\phi_\delta\|_{L^\infty(\Omega)} \leq C(\Omega)\|\phi\|_{L^\infty(\Omega)}$ for every $\delta > 0$.

- In our proof it only used to ensure the density of smooth functions.
- See also [Borowski-Chlebicka-Miasojedow, In arXiv:2210.15217].

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Examples

The following N-functions satisfy the balance condition (B).

1. **Variable exponent case**: $M(x,\xi) = |\xi|^{p(x)}$, where $p(x) : \Omega \to [p^-, p^+]$ is log-Hölder continuous and $1 < p^- \leq p(\cdot) \leq p^+ \leq \infty$;
2. **Double phase case**: $M(x,\xi) = |\xi|^p + a(x)|\xi|^q$, with $1 < p \leq q < \infty$, $0 \leq a \in C^{0,\alpha}(\Omega)$, $\alpha \in (0,1], \frac{q}{p} \leq 1 + \frac{\alpha}{n}$;
3. **Anisotropic variable case**: $M(x,\xi) = \sum_{i=1}^n |\xi_i|^{p_i(x)}$, where $p_i(x) : \Omega \to [p_i^-, p_i^+]$ are log-Hölder continuous and $1 < p_i^- \leq p_i(\cdot) \leq p_i^+ \leq \infty$;
4. **Anisotropic double phase case**: $M(x,\xi) = \sum_{i=1}^n (|\xi_i|^{p_i} + a_i(x)|\xi_i|^{q_i})$, where $1 < p_i \leq q_i < \infty$, $0 \leq a_i \in C^{0,\alpha_i}(\Omega)$, $\alpha_i \in (0,1]$, and $\frac{p_i}{q_i} \leq 1 + \frac{\alpha_i}{n}$;

For the proof, see [Borowski-Chlebicka, J. Funct. Anal.(2022)]

Some Remarks

- Our results cover among others problems with anisotropic polynomial, Orlicz, variable exponent, and double phase growth.
- Our result is valid in the case of bounded data. In fact, for each $g \in L^\infty(\Omega)$, we know that there exists $F : \Omega \to \mathbb{R}^n$, such that $g = \operatorname{div} F$ and $F \in E_{M^*}(\Omega)$.

* For the case $\Phi \equiv 0$ and $b \equiv 0$, see [Gwiazda, Minakowski & Wróblewska-Kamińska, CEJM(2012)].
    * The main idea in their paper is to introduce a regularised problem with solutions in the classical Orlicz–Sobolev space, make use of the theory of pseudo-monotone operators, and pass to the limit.

Our result is the first direct proof even in the homogeneous case ($M(x, \xi) = M(|\xi|)$) even in the absence of the lower-order terms.

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Sketch of Proof

- We first discuss finite dimensional approximations of our problem (1) and their solutions, called Galerkin solutions. The weak solutions of the problem (1) is found as a limit of subsequence of the Galerkin solutions when the dimension of the approximating problem is increased. We divide our proof into 4 steps.
  - Step 1: Existence of Galerkin solution
  - Step 2: A priori estimate
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Existence of Galerkin Solution

Step 1 **Existence of Galerkin Solution**

Since $C_c^\infty(\Omega)$ is separable and dense in $C_c^1(\Omega)$ we can extract a sequence of $\{\varphi_i\}_{i=1}^\infty \subset C_c^\infty(\Omega)$ such that $\overline{\text{span}\{\varphi_1, \varphi_2, \dots\}}^{C_c^1} = C_c^1(\Omega)$. We denote the finite dimensional spaces as $V_n := \text{span}\{\varphi_1, \dots \varphi_n\}$.

Lemma (Existence of Galerkin solutions)

For every $n \in \mathbb{N}$, there exists a function $u_n \in V_n$, is called a Galerkin solution satisfying

$$\int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla\varphi + \Phi(u_n) \cdot \nabla\varphi + b(x, u_n)\varphi \, dx = \int_\Omega F \cdot \nabla\varphi \, dx. \qquad (2)$$

holds for every $\varphi \in V_n$.

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Uniform boundedness of Galerkin solutions

Step 2 **A priori estimate** Testing the equation by u_n, we obtain

$$\int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla u_n + \Phi(u_n) \cdot \nabla u_n + b(x, u_n) u_n \, dx = \int_\Omega F \cdot \nabla u_n \, dx$$

$\Longrightarrow$ there exists a constant C independent of n such that for every Galerkin solution u_n it holds

$$\int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla u_n \, dx \le C; \quad \|\nabla u_n\|_{L_M(\Omega)} \le C; \quad \int_\Omega b(x, u_n) u_n \, dx \le C.$$

$\Longrightarrow$ There exists a function $u \in W_0^{1,1}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,1}(\Omega), \quad \nabla u_n \overset{*}{\rightharpoonup} \nabla u \quad \text{for } \sigma(L_M, E_{M^*}),$$

and there exists a function $h \in L_{M^*}(\Omega)$ such that

$$\mathcal{A}(x, \nabla u_n) \overset{*}{\rightharpoonup} h \quad \text{for} \quad \sigma(L_{M^*}, E_M).$$

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Sketch of proof

Step 4 **($h = \mathcal{A}(x, \nabla u)$ almost everywhere).**
Let $w \in L^\infty(\Omega, \mathbb{R}^n)$ be arbitrary. By (A3) we have

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} \int_\Omega (\mathcal{A}(x, \nabla T_k(u_n)) - \mathcal{A}(x, w)) \cdot (\nabla T_k(u_n) - w)\, dx \\
&= \lim_{n\to\infty} \Big(\int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla T_k(u_n)\, dx - \int_\Omega \mathcal{A}(x, \nabla T_k(u_n)) \cdot w\, dx \\
&\qquad - \int_\Omega \mathcal{A}(x, w) \cdot \nabla T_k(u_n)\, dx + \int_\Omega A(x, w) \cdot w\, dx \Big) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
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Recent Results-Existence and Uniqueness of Renormalized solutions

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in a fully anisotropic and inhomogeneous Musielak-Orlicz space.

* $\Phi:\mathbb{R}\to\mathbb{R}^n$ is a Lipschitz continuous function.

* *As we consider problems with data of low integrability, it is reasonable to work with renormalized solutions.*

Joint work with Bartosz Budnarowski.

Bartosz Budnarowski, Ying Li, *Existence of renormalized solutions to fully anisotropic and inhomogenous elliptic problems.* (Submitted)

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