Algebraic Signatures Enriched by Dependency Structure

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Abstract. Classical single-sorted algebraic signatures are defined as sets of operation symbols together with arities. In their many-sorted variant they also list sort symbols and use sort-sequences as operation types. An operation type not only indicates sorts of parameters, but also constitutes dependency between an operation and a set of sorts. In the paper we define algebraic signatures with dependency structures described as well-founded strict orders. We argue that the most natural morphisms between such structures are p-morphisms (short for pseudo-epimorphisms) and we prove that all together they constitute a category. In model-theory structures like $\langle W, R \rangle$, where W is a set and $R \subseteq W \times W$ is a transitive relation, are called transitive Kripke frames [Seg70]. Part of our result is a definition of a construction of non-empty products in the category of transitive Kripke frames. We prove that in general not all such products exist, but when the class of relations is limited to well-founded strict orders, the category has all products of non-empty families of objects. We also show the existence of equalizers and its cocompleteness.

1 Introduction

Classical single-sorted algebraic signatures are defined as sets of operation symbols together with arities. In their many-sorted variant they also list sort symbols and use sort-sequences as operation types. One should notice that an operation type not only indicates sorts of parameters, but also constitutes dependency between an operation and a set of sorts. Informally, one can reason that an operation cannot be defined unless all sort carriers from its type are present in the model.

In architectural approach to system specification [ST97], a signature represents a software module interface. The whole system (or, to be precise, its model) is obtained as a series of applications of so-called generic modules [BST99] also known as constructor implementations [ST88]. Modules are put together and constitute a whole only if all required parameter-modules are instantiated. Clearly this reveals a dependency relation between modules and, as a consequence, between operation symbols they define.

Our work on architectural models led us to a need of dependency structures put directly on operation symbols right in signatures. The idea is to extend the classical many-sorted signatures by explicitly defining the dependency of sorts and operations. Models over such signatures are the same as in the standard framework.

In the paper we define algebraic signatures with dependency structures described as well-founded strict orders. We argue that the most natural morphisms between such structures are p-morphisms (short for pseudo-epimorphisms) and we prove that all together they constitute a category. Unfortunately the category lacks the final object. However, we successfully show the existence of all products of non-empty families of objects and the presence of all equalizers. We also prove its cocompletness.

In model-theory structures like $\langle W, R\rangle$, where W is a set and $R \subseteq W \times W$ is a relation, are called Kripke frames. It is standard to define categories of Kripke frames and p-morphisms [Seg70]. Part of our result is a definition of a construction of non-empty products in the category of transitive Kripke frames. We prove that in general not all such products exist, but when the class of relations is limited to well-founded strict orders, the category has all products of non-empty families of objects. We also show the existence of equalizers and its cocompletness.

Results presented in the paper are part of the ongoing work on the use of signature fragments to specify architecture of generic software modules.

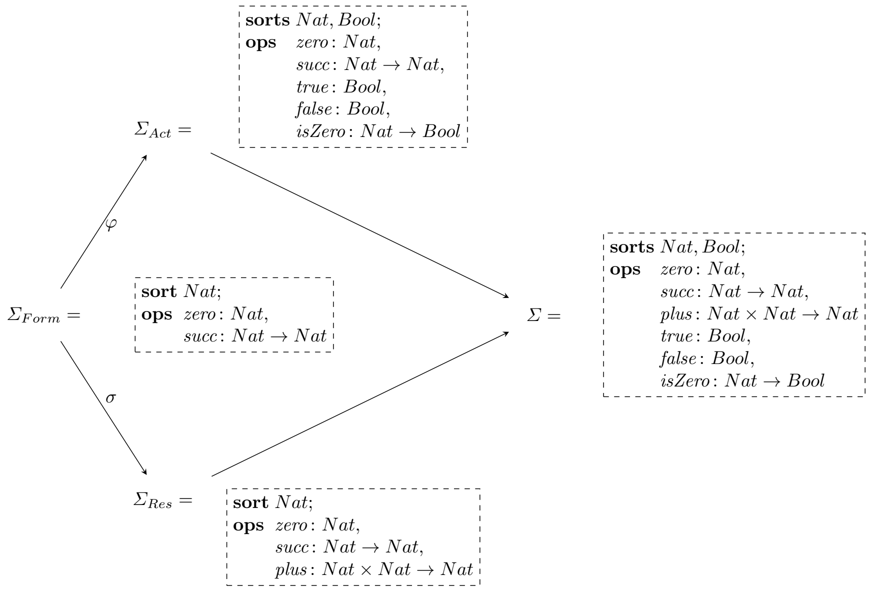
The paper is organized as follows. First in Sect. 2 we present the motivation to our work. In Sect. 3 we define the categories of dependency relations. In Sections 4 and 5 we analyze the existence of colimits in these categories and in Sect. 6 we define the category of signatures with dependency structure and prove its properties. Finally, Sect. 7 contains conclusion and future work. Proofs of most lemmas and theorems are in the Appendix A of the extended version of this paper [Mar10].

2 Motivation

Classical algebraic many-sorted signatures naturally contain dependence of operation symbols on their parameters' sorts. In the architectural specifications [BST99], signature of a generic module is an injective signature morphism $\sigma\colon \Sigma_{Form} \to \Sigma_{Res}$, where Σ_{Form} is a formal parameters signature and Σ_{Res} is a result signature. It renders a dependency between all symbols from the result signature and those from the parameters signature.

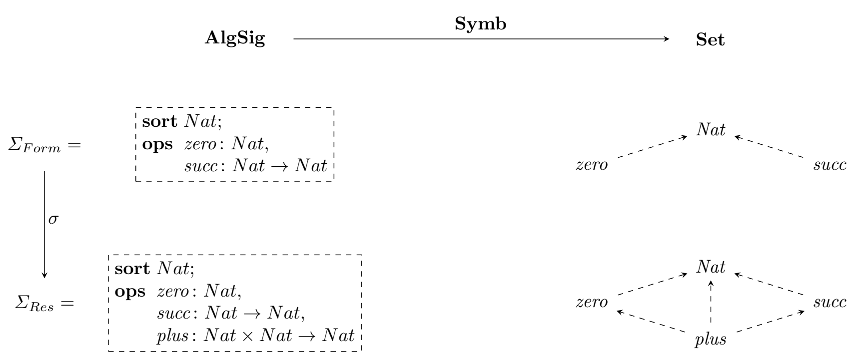
The above-described dependency is of the weak form, meaning that it is not required that actual implementation of result symbols uses the parameter symbols intrinsically. It rather conveys the negative information, leaving some symbols definitely independent of others. One may think of it as a of *potential dependency.*

The generic module application along a fitting morphism $\varphi\colon \Sigma_{Form} \to \Sigma_{Act}$ on the signature-level is simply the pushout of φ and σ. We notice that in the pushout signature the dependency of the result symbols on the parameter symbols is lost. Consider the following simple example.

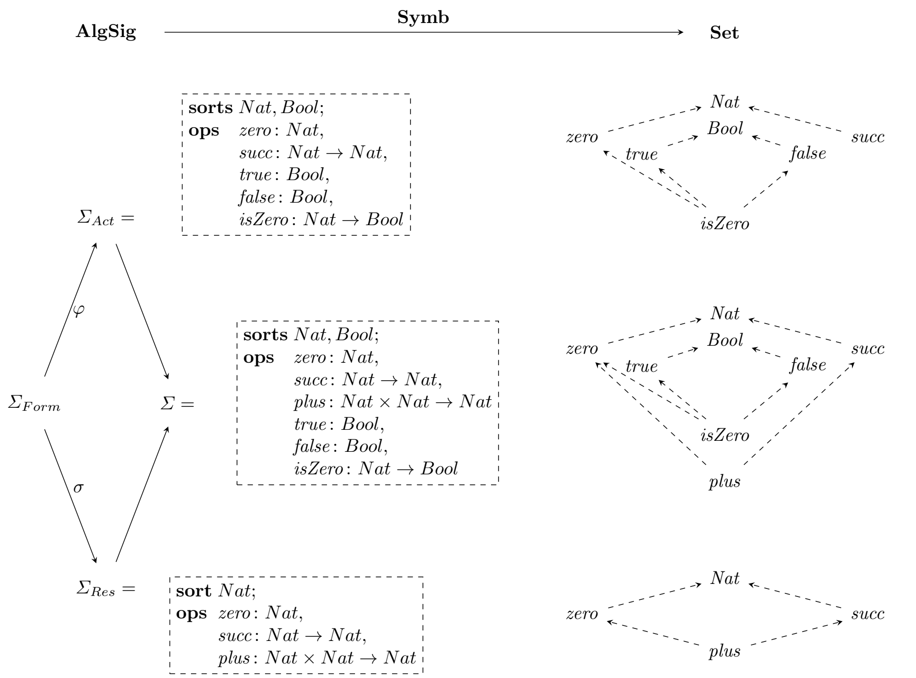


The construction signature σ defines an operation $plus\colon Nat \times Nat \to Nat$, provided it is given a sort Nat with operations $zero\colon Nat$ and $succ\colon Nat \to Nat$. The actual parameters signature Σ_{Act} enriches the formal parameters signature Σ_{Form} by several symbols like $Bool$, $true\colon Bool$ etc. The pushout signature contains all symbols together. However, the information about a potential dependency of the operation $plus$ on Nat, $zero$, $succ$ is lost.

Here comes the idea to *enrich the signatures by dependency structure* to explicitly show how symbols depend on other symbols. The diagram below shows how we imagine the dependency. It shall be a transitive relation (depicted as dashed lines) on set of symbols taken from the signature.



In the pushout, the dependencies should be preserved, as on the following diagram. It is visible that the operation *isZero* doesn't depend on *plus*. Neither the later depends on the former.



In our paper we try to answer the following questions.

1. What kind of dependency relation shall we use?
2. How to enrich the signatures by dependency structure?
3. How to define morphism between such enriched signatures?

The simple example given above already says something about the dependency relation – it needs to be transitive. We also have to make sure that our extension of signatures is defined as a category that keeps all important properties of the original category **AlgSig**. Namely, we require that morphisms shouldn't change (in)dependencies of symbols and that the category of enriched signatures have all pushouts and pullbacks.

3 Dependency relation

We start our work by investigating properties of several categories of sets ordered by various transitive relations. We consider morphisms that not only preserve the dependencies, but also weakly reflect their structure.

3.1 Category Rset↓ and its Subcategories

As we already mentioned in the previous section, we consider only transitive relations as candidates for the dependency relation.

Definition 1 (R-sets) *An* R-set[1] *is a pair* $\langle A, R_A \rangle$ *where* $R_A \subseteq A^2$ *is a* transitive *relation on a set* A*. In what follows we sometimes write* A_R *instead of* $\langle A, R_A \rangle$*. We may use the infix notation for* R_A *and for* $a_1, a_2 \in A_R$ *we may also write* $a_1 \, R \, a_2$ *instead of* $a_1 \, R_A \, a_2$*, when decorations are clear from the context.*

Definition 2 (Category Rset↓ of R-sets and P-morphisms) *A category* **Rset↓** *has* R-sets *as objects and* pseudo-epimorphisms*, or* p-morphisms*, as morphisms. A p-morphism is a function that preserves the relation* R *and weakly reflects R-set down-closures, i.e. a morphism* $f : \langle A, R_A \rangle \to \langle B, R_B \rangle$ *is a function* $m : A \to B$ *such that:*

1. *(monotonicity) for all* $a_1, a_2 \in A$*,* $a_1 \, R_A \, a_2$ *implies* $f(a_1) \, R_B \, f(a_2)$*.*
2. *(weakly reflected R-down-closures) for all* $a_2 \in A, b_1 \in B$*,* $b_1 \, R_B \, f(a_2)$ *implies that there exists* $a_1 \in A$*, that* $a_1 \, R_A \, a_2$ *and* $f(a_1) = b_1$*.*

Identities and composition are defined as expected.

Note that p-morphisms are such functions between ordered sets, that their graph is a bisimulation of orders seen as transitive systems.

Definition 3 (Sub R-set) *Given an R-set* A_R *and* $a \in A$*, its* closed down sub R-set induced by an element a *is defined as:*

$$A_R{}^a\!\downarrow = A'_R$$

where $A' = \{a' \in A \mid a' \, R_A \, a\} \cup \{a\}$ *and* $R_{A'} = R_A|_{A'}$*.*

It is important to notice that $a \in A_R{}^a\!\downarrow$, for any set A, $a \in A$ and a relation R.

Below we formalize several full subcategories of the category **Rset↓**. We think that they may be good candidates for the signature-symbols dependency orderings. In the following sections we are going to investigate their properties to choose the most appropriate one.

Definition 4 (Category Preord↓) *A category* **Preord↓** *of preorders and p-morphisms is a full subcategory of* **Rset↓** *where objects are preorders: an R-set* $\langle A, \leq_A \rangle$ *is a* **Preord↓***-object iff the relation* $\leq_A$ *is transitive and reflexive.*

Definition 5 (Category Soset↓) *A category* **Soset↓** *of strict orders and p-morphisms is a full subcategory of* **Rset↓** *where objects are strict orders, i.e. an R-set* $\langle A, <_A \rangle$ *is a* **Soset↓***-object iff the relation* $<_A$ *is transitive and asymmetric; thus, irreflexive.*

[1] In the model-theory the same mathematical structure is called a transitive Kripke frame [Seg70] thus, all results presented in this paper also concern transitive Kripke frames. The author would like to thank Bartek Klin for pointing out that fact.

Definition 6 (Category $\mathbf{Soset_{wf}}\!\downarrow$) *A category* $\mathbf{Soset_{wf}}\!\downarrow$ *of well-founded strict orders and p-morphisms is a full subcategory of* $\mathbf{Soset}\!\downarrow$*. Its objects are well-founded strict orders, i.e. strict orders without infinite descending chains.*

To safe space we don't discuss the possibility to have partial orders as a candidate for dependency relation. We checked that they are unacceptable.

3.2 R-multisets and Dependency Bisimulation

This section defines R-multisets that become handy when it comes to definition of products in the category $\mathbf{Rset}\!\downarrow$ and its subcategories (cf. Sect. 4). The reader may skip this section in the first reading.

The idea is to take an R-set and define a multiset of its elements without adding new dependencies; however, some original dependencies may be dropped.

Definition 7 (Labelled R-set) *A* labeled R-set *is a triple* $\langle A_R, P_R, \mu\rangle$ *where* A_R *and* P_R *are R-sets and* $\mu\colon A_R \to P_R$ *is a monotone labeling function.*

Definition 8 (Labelled R-set Isomorphism) *Two labeled R-sets* $\langle A_R, P_R, \mu\rangle$, $\langle A'_R, P_R, \mu'\rangle$ *are isomorphic iff there exists a bijection* $\tau\colon A \to A'$ *such that for all* $a \in A$, $\mu(a) = \mu'(\tau(a))$ *and for all* $a, a' \in A$ $a\, R_A\, a'$ *iff* $\tau(a)\, R_{A'}\, \tau(a)$.

Definition 9 (R-mset – R-multiset) *An* R-multiset, or R-mset, $[A_R, P_R, \mu]$ *is the isomorphism class*[2] *of a labeled R-set* $\langle A_R, P_R, \mu\rangle$.

Definition 10 (R-submultiset) *Given an R-mset* $[A_R, P_R, \mu]$ *and* $a \in A$*, its* R-submultiset $[A_R, P_R, \mu]^a\!\downarrow$ induced by a *is defined as an R-mset:*

$$[A_R, P_R, \mu]^a\!\downarrow = [{A_R}^a\!\downarrow, P_R, \mu|_{({A_R}^a\downarrow)}]$$

Lemma 11 *Given an R-mset* $[A_R, P_R, \mu]$ *and* $a, a' \in A$*, such that* $a' R a$*, it holds that*

$$[A_R, P_R, \mu]^{a'}\!\downarrow = ([A_R, P_R, \mu]^a\!\downarrow)^{a'}\!\downarrow$$

Having a regular multiset we can easily calculate the set of its distinct elements. The similar question can be asked with regard to R-multisets, but here the matter is to find a R-multiset of elements that have the distinct dependency structure. In the following definition we use the bisimulation to find the *kernel* of the given R-mset. It is going to play a crucial role in the definition of products in the category $\mathbf{Rset}\!\downarrow$ and its subcategories.

Definition 12 (Kernel Relation of R-mset) *Given an R-mset* $[A_R, P_R, \mu]$ *its* kernel relation $K([A_R, P_R, \mu]) \subseteq A^2$ *is the greatest* dependency bisimulation

[2] For technical convenience we will sometimes define concepts and constructions on R-msets by introducing them on representatives – leaving to the Reader the details of generalization to their isomorphism classes.

relation *on* A. *A dependency bisimulation is an equivalence relation* $\sim \subseteq A^2$*, such that for* $a_1, a_2 \in A$

$$\begin{aligned}
&\textit{if } a_1{\sim}a_2 \textit{ then } \mu(a_1) = \mu(a_2) \textit{ and} \\
&\qquad\qquad \textit{for all } a_1' \in A \textit{ such that } a_1' \, R \, a_1 \\
&\qquad\qquad\quad \textit{there exists } a_2' \in A,\ a_2' \, R \, a_2,\ a_1' \sim a_2' \\
&\qquad\qquad \textit{and for all } a_2' \in A \textit{ such that } a_2' \, R \, a_2 \\
&\qquad\qquad\quad \textit{there exists } a_1' \in A,\ a_1' \, R \, a_1,\ a_1' \sim a_2'
\end{aligned}$$

The family of dependency bisimulations is non-empty (it contains id_A*) and it is closed under the unions, hence the kernel relation exists for every R-mset* $[A_R, P_R, \mu]$*.*

4 Limits in Rset↓

In this section we try to find out whether the categories from Sect. 3.1 have enough limits. We are particularly interested in the existence of pullbacks, which we are going to define through equalizers and products of nonempty families of objects. However, to have a complete picture, we also look at the existence of final objects.

Theorem 13 *The category* **Preord↓** *has a singleton ordered by identity as its final object.*

Theorem 14 *The categories* **Soset↓** *and* $\mathbf{Soset_{wf}}$**↓** *do not have a final object.*

Proof: Since the relations in objects of **Soset↓** (and $\mathbf{Soset_{wf}}$**↓**) are irreflexive, their morphisms must not glue together any elements being in relation. Hence, if the final object existed, there would be an injective map from any ordinal (represented as an R-set with natural strict well-founded "ordering") into it. Hence, such a final object can not be a proper set. □

Conjecture 15 *The category* **Rset↓** *does not have a final object.*

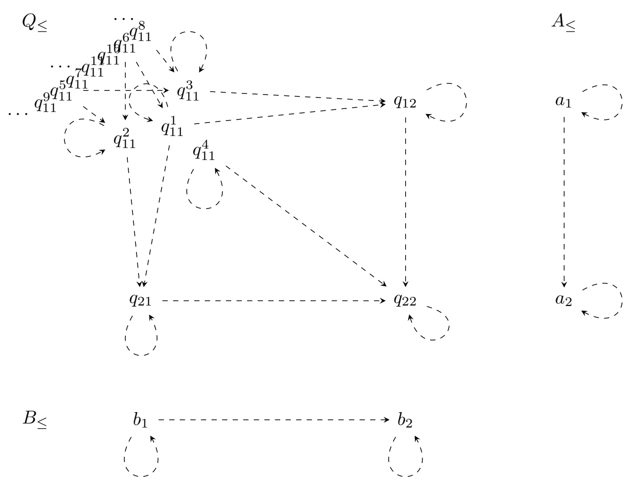
Only the category **Preord↓** proves to have the final object, but the lack of it does not disqualify the others. Importantly, we can prove that all above-defined categories have the equalizers.

Theorem 16 *The category* **Rset↓** *and its subcategories* **Preord↓***,* **Soset↓** *and* $\mathbf{Soset_{wf}}$**↓** *have all equalizers.*

The equalizers are the same as in **Set**. For the proof see App. A of [Mar10].

Now, we come to the most tricky part of the paper, namely products in R-sets. At first their definition seem obvious, but the deeper look unveils the quite surprising setting. Let us show it by an example in **Preord↓**. Let $A_{\leq} =$

$\langle \{a_1, a_2\}, a_1 \leq a_1, a_2 \leq a_2, a_2 \leq a_1 \rangle$ and $B_{\leq} = \langle \{b_1, b_2\}, b_1 \leq b_1, b_2 \leq b_2, b_2 \leq b_1 \rangle$ be two **Preord↓**-objects. Their product of $A_{\leq}$ and $B_{\leq}$ in **Preord↓** is $Q_{\leq}$ that may be depicted as below.



As expected there are $q_{22} = \langle a_2, b_2 \rangle$, $q_{12} = \langle a_1, b_2 \rangle$, $q_{21} = \langle a_2, b_1 \rangle$ in $Q_{\leq}$. They are in relation with themselves and also $q_{22} \leq_Q q_{21}$ and $q_{22} \leq_Q q_{12}$. There is also $q_{11}^1 = \langle a_1, b_1 \rangle$ that is in relation with all above mentioned elements. However, there are also infinity many distinct elements of $Q_{\leq}$, also, as q_{11}^1 projected to a_1 and b_1, marked above as $q_{11}^2, q_{11}^3, \ldots$ that differ on their dependency relations.

As we are going to prove later in Lemma 18, the following definition proposes the construction of the product. The only problem is that in some cases the structure it describes may fail to be a set.

Definition 17 (Product Candidate) *Given two* **Rset↓**-*objects* $\langle A, R_A \rangle$ *and* $\langle B, R_B \rangle$, *let us define a* product candidate *as a pair of a class and a relation* $\langle A \prod B, R_{A \prod B} \rangle$ *together with two functions* $\rho_A \colon A \prod B \to A$ *and* $\rho_B \colon A \prod B \to B$. *In Lemma 18 we prove that whenever* $A \prod B$ *happens to be a set, then* $\langle A \prod B, R_{A \prod B} \rangle$ *with projections* ρ_A *and* ρ_B *is a product of the given* **Rset↓**-*objects.*

First, let $\langle P, R_P \rangle$ *be a pair of the set* $P = A \times B$*, the product of* A *and* B *in* **Set** *together with projections* $\pi_A \colon P \to A$ *and* $\pi_B \colon P \to B$*, and a relation* $R_P \subseteq P^2$*:*

$$R_P = \{\langle p_1, p_2 \rangle \in P^2 \mid \pi_A(p_1)\, R_A\, \pi_A(p_2) \text{ and } \pi_B(p_1)\, R_B\, \pi_B(p_2)\}$$

Now, let the class[3] $A\prod B$ *be defined as:*

$$A\prod B = \bigcup_{p\in P} \{\langle [X_R, P_R, \mu], p\rangle \mid$$
$$[X_R, P_R, \mu] \textit{ is an R-mset such that:}$$
$$\bullet\ K([X_R, P_R, \mu]) = id_X,$$
$$\bullet\ \mu;\pi_A \textit{ and } \mu;\pi_B \textit{ are morphisms in } \mathbf{Rset}{\downarrow},$$
$$\bullet\ \textit{there exists } x\in X,\ \textit{such that } \mu(x) = p$$
$$\textit{and for all } x'\in X,\ \textit{if } x'\neq x \textit{ then } x' \, R_X \, x$$
$$\}$$

See Def. 9, 12 for the definition of R-msets and their kernels. The relation $R_{A\prod B}$ *is defined as follows:*

$$R_{A\prod B} = \{\langle\langle [X'_R, P_R, \mu'], p'\rangle, \langle [X_R, P_R, \mu], p\rangle\rangle \in (A\prod B)^2 \mid$$
$$\textit{there exists } x'\in X, \mu(x') = p', p' \, R \, p$$
$$\textit{and } [X'_R, P_R, \mu'] = [X_R, P_R, \mu]^{x'}{\downarrow}\}$$

Cf. Def. 10 for the definition of R-submultisets.
We define product-candidate projection functions $\rho_A\colon (A\prod B)\to A$ *and* $\rho_B\colon (A\prod B)\to B$ *as*

$$\rho_A(\langle [X_R, P_R, \mu], p\rangle) = \pi_A(p)$$
$$\rho_B(\langle [X_R, P_R, \mu], p\rangle) = \pi_B(p)$$

The class $A\prod B$ contains every element of P taken as many times, as there are distinct (wrt. the kernel relation) R-msets of elements lower than it wrt. R_P. These R-msets are subject to the requirement that their labellings composed with Cartesian product projections are p-morphisms. This makes them weakly reflect the R-down-closures of A_R and B_R.

The following lemma guarantees that once we show that the product candidate is a set, then it is indeed the product.

Lemma 18 *Consider two* **Rset**↓*-objects* A_R *and* B_R*, if* $A\prod B$ *is a set, then the product candidate* $\langle A_R\prod B_R, R_{A\prod B}\rangle$ *is their product in* **Rset**↓*.*

The proof is quite technical (see App. A of [Mar10]). The main point is to observe that when there is any other object with two projections to A_R and B_R, it may be seen as an R-multiset. We take its kernel (cf. Def. 12) and find that every element of the kernel (with its dependency structure) is also present in $A\prod B$. It allows us to define the unique morphism from the given object to $A\prod B$.

The products of n R-sets, for $n > 2$, if they exists, are defined following the same idea.

[3] In general, for a given nonempty set of labels P, the class of all P-labeled R-msets is a proper class.

Lemma 19 *Given two* $\mathbf{Rset}\downarrow$*-objects* $\langle A, A_R\rangle$ *and* $\langle B, R_B\rangle$*, if both* R_A *and* R_B *are reflexive / irreflexive / asymmetric / strict well-founded, then the relation* $R_{A\prod B}$ *from Def. 17 is also so.*

The proof is straightforward. For details see App. A of [Mar10].

Theorem 20 *The category* $\mathbf{Soset_{wf}}\downarrow$ *has all binary products. Moreover, given two* $\mathbf{Soset_{wf}}\downarrow$*-objects,* $A_<$*,* $B_<$*, their product is isomorphic to the product candidate from Def. 17.*

We prove that, given any $A_<, B_< \in \mathbf{Soset_{wf}}\downarrow$, the product candidate $A\prod B$ is a set. Then, by Lemma 18, we argue that $(A\prod B)_<$ is indeed the product of the given objects. The proof goes by induction. We bound the number of possible distinct structures labeled by every $p \in P_<$. Since dependency relations in question are well-founded, this is possible to do so. For details see App. A of [Mar10].

We cannot find the similar proof for $\mathbf{Rset}\downarrow$, $\mathbf{Preord}\downarrow$ and $\mathbf{Soset}\downarrow$, hence the following conjecture.

Conjecture 21 *The category* $\mathbf{Rset}\downarrow$ *and its subcategories* $\mathbf{Preord}\downarrow$*and* $\mathbf{Soset}\downarrow$ *do not have all binary products.*

5 Colimits in Rset↓

After the struggle with the limits in $\mathbf{Rset}\downarrow$ and its subcategories, colimits are easy and boring. Basically the colimits in these categories are the same as in **Set**. The only exception is $\mathbf{Soset}\downarrow$, which does not have all coequalizers.

Lemma 22 *Given two* $\mathbf{Rset}\downarrow$*-morphisms* $f, g\colon A_R \to B_R$*, their coequalizer in* **Set***,* $e\colon B \to C$*, is an* $\mathbf{Rset}\downarrow$*-morphism* $e\colon B_R \to C_R$ *where the relation* R_C *is defined simply as*

$$R_C = e(R_B)$$

Theorem 23 *The category* $\mathbf{Rset}\downarrow$ *and its subcategory* $\mathbf{Preord}\downarrow$ *have all coequalizers.*

In case of $\mathbf{Rset}\downarrow$ the proof is just a straight use of Lemma 22. For $\mathbf{Preord}\downarrow$ we show that e preserves the reflexivity of R_B (cf. App. A of [Mar10]). However, the irreflexivity of R_B does not cause the irreflexivity of $e(R_B)$, hence the following theorem.

Theorem 24 *The category* $\mathbf{Soset}\downarrow$ *does not have all coequalizers.*

The counterexample involves two functions $id\colon \mathbb{N} \to \mathbb{N}$ and $succ\colon \mathbb{N} \to \mathbb{N}$ and the inverse ordering of natural numbers. See App. A of [Mar10] for details.

It is enough to limit strict orders to well-founded orders to avoid counterexamples like the one presented above.

Theorem 25 *The category* $\mathbf{Soset_{wf}}\!\downarrow$ *has all coequalizers.*

The App. A of [Mar10] contains the proof. Finally, the easy part concerns coproducts.

Theorem 26 *The category* $\mathbf{Rset}\!\downarrow$ *and its subcategories* $\mathbf{Preord}\!\downarrow$,$\mathbf{Soset}\!\downarrow$ *and* $\mathbf{Soset_{wf}}\!\downarrow$ *have all coproducts*

The coproducts are same as in **Set**.

6 Algebraic Signatures with Dependent Symbols

The two previous sections left only one ordering suitable for a dependency relation, namely well-founded strict ordering category $\mathbf{Soset_{wf}}\!\downarrow$. All others probably don't have all products and $\mathbf{Soset}\!\downarrow$ does not have coequalizers.

In this section we formalize what we hand-waved in Sect. 2. We define the category of algebraic signatures and enrich it by the dependency structure.

Definition 27 (Algebraic Signatures) *We define a category* **AlgSig** *in the standard way – with objects being algebraic signatures defined as pairs of the form* $\Sigma = \langle S, \Omega_{S^+}\rangle$ *where* $S \in \mathbf{Set}$ *is a set of sorts,* S^+ *is a set of nonempty finite* S*-sequences and* $\Omega_{S^+} = \langle \Omega_e\rangle_{e\in S^+}$ *is an* S^+*-sorted set of operation names. Morphisms of* **AlgSig** *are pairs of the form* $\langle \sigma_S, \sigma_{\Omega_S}\rangle\colon \Sigma \to \Sigma'$ *where* $\sigma_S\colon S \to S'$ *and* $\sigma_{\Omega_S} = \langle \sigma_{\Omega_e}\colon \Omega_e \to \Omega'_{\sigma_S^+(e)}\rangle_{e\in S^+}$. *Identities and composition are defined as one may expect.*

Before we allow a general dependency structure o symbols, we define a functor that recognizes the basic dependency of operation symbols on sort symbols, as discussed in Sect. 2.

Definition 28 (SigSymb Functor) *Let* $\mathbf{SigSymb}\colon \mathbf{AlgSig} \to \mathbf{Soset_{wf}}\!\downarrow$ *be the functor that transforms algebraic signatures to* well-founded strict orders of signatures' symbols. *Given an algebraic signature* $\Sigma = \langle S, \Omega_{S^+}\rangle$ *we define*

$$\mathbf{SigSymb}(\Sigma) = \langle s \uplus (\biguplus_{e\in S^+} \{o : e | o \in \Omega_e\}), <_{\mathbf{SigSymb}(\Sigma)}\rangle$$

having operation symbols naturally dependent on sorts of their result and from their arities; i.e. for all $e \in S^+, e = \langle s_0 \ldots s_n\rangle, o \in \Omega_e$, *we have*

$$(s_k) <_{\mathbf{SigSymb}(\Sigma)} (o : e)$$

for all $0 < k < n$. *Given an algebraic signature morphism* $\sigma = \langle \sigma_s, \sigma_{\Omega_S}\rangle\colon \Sigma \to \Sigma'$, *we define a* $\mathbf{Soset_{wf}}\!\downarrow$*-morphism* $\mathbf{SigSymb}(\sigma)\colon \mathbf{SigSymb}(\Sigma) \to \mathbf{SigSymb}(\Sigma')$ *as*

$$\mathbf{SigSymb}(\sigma) = \sigma_s \uplus (\biguplus_{e\in S^+} \sigma'_{\Omega_e})$$

Where $\sigma'_{\Omega_e}(o : e) = \sigma_{\Omega_e}(o)$ *for* $o \in \Omega_e$. *By construction it meets both requirements from Def. 2.*

Of course the dependency may be forgotten, if one wishes.

Definition 29 (SetSymb Functor) *Let a functor that gives a set of signature's symbols,* $\mathbf{SetSymb}\colon \mathbf{AlgSig} \to \mathbf{Set}$*, be defined as* $\mathbf{SetSymb} = \mathbf{SigSymb}; \mathbf{U}_{\mathbf{Soset_{wf}}\downarrow}$ *where* $\mathbf{U}_{\mathbf{Soset_{wf}}\downarrow}\colon \mathbf{Soset_{wf}}\!\downarrow \to \mathbf{Set}$ *is the obvious forgetful functor.*

At the moment we have everything that we need to define the structures from the paper's title.

Definition 30 (Algebraic Signatures with Dependent Symbols) *Objects of a category* **AlgSigDep** *of* algebraic signatures with dependent symbols *are pairs*

$$\Sigma_< = \langle \Sigma, <_\Sigma \rangle$$

where $\Sigma \in \mathbf{AlgSig}$ *is an algebraic signature and* $<_\Sigma \subseteq \mathbf{SigSymb}(\Sigma) \times \mathbf{SigSymb}(\Sigma)$ *is such dependency relation that* $<_{\mathbf{SigSymb}(\Sigma)} \subseteq <_\Sigma$ *(cf. Def. 28) and*

$$\langle \mathbf{SetSymb}(\Sigma), <_\Sigma \rangle \in \mathbf{Soset_{wf}}\!\downarrow$$

Morphisms between algebraic signatures with dependent symbols $\Sigma_<, \Sigma'_< \in \mathbf{AlgSigDep}$ *are such algebraic signature morphisms* $\sigma\colon \Sigma \to \Sigma'$*, for which a function* $\mathbf{SigSymb}(\sigma)$ *seen as a morphism* $\mathbf{SigSymb}(\sigma)\colon \langle \mathbf{SetSymb}(\Sigma), <_\Sigma \rangle \to \langle \mathbf{SetSymb}(\Sigma'), <_{\Sigma'} \rangle$ *is a* $\mathbf{Soset_{wf}}\!\downarrow$*-morphism (cf. Def. 2 and Def. 6).*

Till the end of this section we define functors and present lemmas needed in the following sections.

Definition 31 (DepSymb Functor) *A functor* $\mathbf{DepSymb}\colon \mathbf{AlgSigDep} \to \mathbf{Soset_{wf}}\!\downarrow$ *is defined as* $\mathbf{DepSymb}(\Sigma_<) = \langle \mathbf{SetSymb}(\Sigma), <_\Sigma \rangle$*, for a signature with dependent symbols* $\Sigma_< \in \mathbf{AlgSigDep}$*, and* $\mathbf{DepSymb}(\sigma) = \mathbf{SetSymb}(\sigma)$*, for a signature morphism* $\sigma \in \mathbf{AlgSigDep}$.

Definition 32 (Symb Functor) *Let a functor that gives a set of symbols of the signature with dependent symbols,* $\mathbf{Symb}\colon \mathbf{AlgSigDep} \to \mathbf{Set}$*, be defined as* $\mathbf{Symb} = \mathbf{DepSymb}; \mathbf{U}_{\mathbf{Soset_{wf}}\downarrow}$ *where again* $\mathbf{U}_{\mathbf{Soset_{wf}}\downarrow}\colon \mathbf{Soset_{wf}}\!\downarrow \to \mathbf{Set}$ *is the obvious forgetful functor.*

Lemma 33 *The functor* $\mathbf{Symb}\colon \mathbf{AlgSigDep} \to \mathbf{Set}$ *is faithful.*

Definition 34 (Embedding AlgSig into AlgSigDep) *Category of algebraic signatures is naturally embeddable into the category of algebraic signatures with dependent symbols via the functor* $\mathbf{Dep}\colon \mathbf{AlgSig} \to \mathbf{AlgSigDep}$ *defined as* $\mathbf{Dep}(\Sigma) = \langle \Sigma, <_{\mathbf{DepSymb}(\Sigma)} \rangle$*, for a signature* $\Sigma \in \mathbf{AlgSig}$*, and* $\mathbf{Dep}(\sigma) = \sigma$*, for a signature morphism* $\sigma \in \mathbf{AlgSig}$.

Lemma 35 *It holds that* $\mathbf{SetSymb} = \mathbf{Dep}; \mathbf{Symb}$.

6.1 Reconstructing Signatures with Dependent Symbols

Converting a signature to a set of symbols should be complemented by an inverse operation. However, we can not just in an ad-hoc manner add a signature's structure to a given set. Instead, we propose a signature "reconstruction" given a function from an arbitrary set to an (ordered) set of signature's symbols.

Definition 36 (Signature Reconstruction) *Let us define a* signature with dependent symbols-reconstruction functor

$$\mathbf{Rec}\colon (\mathbf{Set} \downarrow \mathbf{Symb}) \to \mathbf{AlgSigDep}$$

where $(\mathbf{Set} \downarrow \mathbf{Symb})$ *is a comma category.*

- *Given a* $(\mathbf{Set} \downarrow \mathbf{Symb})$*-object, i.e. a function* $f\colon A \to \mathbf{Symb}(\Sigma_<)$*, for some* $\mathbf{AlgSigDep}$*-signature* $\Sigma_< = \langle\langle S, \Omega_{S^+}\rangle, <_\Sigma\rangle$ *and some set* $A \in \mathbf{Set}$*, we "reconstruct" the greatest signature derived from* $\Sigma_<$ *wrt.* f *that has symbols from* A
$$\mathbf{Rec}(f) = \langle\langle S^A, \Omega^A_{S^{A+}}\rangle, <_{\mathbf{Rec}(f)}\rangle$$
where $S^A = \{s \in A' \mid f(s) \in S\}$ *and for any* $e \in S^{A^+}$, $\Omega^A_e = \{o : e \mid o \in A'$ *and* $f(o) \in \Omega_{f(e)}\}$ *and* $A' = \{a \in A \mid f(A)_{<_\Sigma}{}^{f(a)}\!\!\downarrow\ \subseteq f(A)\}$ *and* $<_{\mathbf{Rec}(f)} = (<_\Sigma|_f)|_{A'}$.
- *For a* $(\mathbf{Set} \downarrow \mathbf{Symb})$*-morphism, i.e. a pair of morphisms* $\langle g\colon A_1 \to A_2, \sigma\colon \Sigma^1_< \to \Sigma^2_<\rangle$ *where* σ *is* $\mathbf{AlgSigDep}$*-morphism and* g *is a function such that* $g; f_2 = f_1; \mathbf{Symb}(\sigma)$ *and* $f_i\colon A_i \to \mathbf{Symb}(\Sigma^i_<)$*, for* $i \in \{1, 2\}$*, are two* $(\mathbf{Set} \downarrow \mathbf{Symb})$*-objects, a* $\mathbf{AlgSigDep}$*-morphism between "reconstructed" signatures is defined as*
$$\mathbf{Rec}(\langle g, \sigma\rangle) = \langle \sigma'_S, \sigma'_{\Omega_{S^{A_1}}}\rangle\colon \mathbf{Rec}(f_1) \to \mathbf{Rec}(f_2)$$
where $\sigma'_S = g|_{S^{A_1}}$ *and for every* $e \in S^{A_1+}$, $\sigma'_{\Omega_e} = g|_{\Omega^{A_1}_e}$.

Note that not all symbols from a given set stays in the "reconstructed" signature – only these that meet the dependency compatibility conditions. The following definition unveils exactly what kind of inverse is **Rec** to **Symb**. The subsequent lemma proves that they are locally adjoint.

Definition 37 *For every signature with dependent symbols,* $\Sigma_< = \langle S, \Omega_{S^+}\rangle \in \mathbf{AlgSigDep}$*, there exist the following two functors. The* under-$\Sigma_<$ "symbol" functor

$$\mathbf{Symb}_{\Sigma_<}\colon (\mathbf{AlgSigDep} \downarrow \Sigma_<) \to (\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<))$$

defined

- *for an* $(\mathbf{AlgSigDep} \downarrow \Sigma_<)$*-object* $\sigma\colon \Sigma'_< \to \Sigma_<$ *as*
$$\mathbf{Symb}_{\Sigma_<}(\sigma) = \mathbf{Symb}(\sigma)\colon \mathbf{Symb}(\Sigma'_<) \to \mathbf{Symb}(\Sigma_<)$$

– *for an* $(\mathbf{AlgSigDep} \downarrow \Sigma_<)$*-morphism* $\varphi\colon \Sigma^1_< \to \Sigma^2_<$ *as*

$$\mathbf{Symb}_{\Sigma_<}(\varphi) = \mathbf{Symb}(\varphi)\colon \mathbf{Symb}(\Sigma^1_<) \to \mathbf{Symb}(\Sigma^2_<)$$

and the under-$\Sigma_<$ "reconstruction" functor

$$\mathbf{Rec}_{\Sigma_<}\colon (\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<)) \to (\mathbf{AlgSigDep} \downarrow \Sigma_<)$$

defined

– *for an* $(\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<))$*-object* $f\colon A \to \mathbf{Symb}(\Sigma_<)$ *as*

$$\mathbf{Rec}_{\Sigma_<}(f) = \langle \varphi_S, \varphi_{\Omega_S} \rangle\colon \mathbf{Rec}(f) \to \Sigma_<$$

where $\mathbf{Rec}(f) = \langle\langle S', \Omega'_{S'^+}\rangle, <_{\mathbf{Rec}(f)}\rangle$, $\varphi_S = f|_{S'}$ *and for every* $e \in S'^+$, $\varphi_{\Omega_e}\colon \Omega'_e \to \Omega_{\varphi^+_S(e)}$ *is defined as* $\varphi_{\Omega_e}(o) = f(o)$*;*

– *for an* $(\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<))$*-morphism* $g\colon A^1_< \to A^2_<$*, with* $f_1\colon A^1_< \to \mathbf{Symb}(\Sigma_<)$ *and* $f_2\colon A^2_< \to \mathbf{Symb}(\Sigma_<)$*, as*

$$\mathbf{Rec}_{\Sigma_<}(g) = \langle \gamma_S, \gamma_{\Omega_S} \rangle\colon \mathbf{Rec}(f_1) \to \mathbf{Rec}(f_2)$$

where $\mathbf{Rec}(f_1) = \langle\langle S_1, \Omega^1_{S_1{}^+}\rangle, <_{\mathbf{Rec}(f_1)}\rangle$, $\gamma_S = g|_{S_1}$ *and for every* $e \in S^+_1$, $\gamma_{\Omega_e}\colon {\Omega^1}_e \to \Omega_{\gamma^+_S(e)}$ *is defined as* $\gamma_{\Omega_e}(o) = g(o)$

Lemma 38 *For any* $\Sigma_< \in \mathbf{AlgSigDep}$*, the two functors from Def. 37 are adjoint in the following way*

$$\mathbf{Rec}_{\Sigma_<} \dashv \mathbf{Symb}_{\Sigma_<}$$

Proof: Given a $(\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<))$-object $f\colon A \to \mathbf{Symb}(\Sigma_<)$ and a $(\mathbf{Set} \downarrow \mathbf{Symb}(\Sigma_<))$-morphism $g\colon \mathbf{Symb}(\Sigma'_<) \to A$, for some $(\mathbf{AlgSigDep} \downarrow \Sigma_<)$-object $h\colon \Sigma'_< \to \Sigma_<$, there exists exactly one $(\mathbf{AlgSigDep} \downarrow \Sigma_<)$-morphism $g^\#\colon \Sigma'_< \to \mathbf{Rec}(f)$ such that $\mathbf{Symb}(g^\#); \epsilon_f = g$, where ϵ_f is the counit function $\epsilon_f\colon \mathbf{Symb}(\mathbf{Rec}(f)) \to A$ naturally defined as $\epsilon_f(s) = s$ for $s \in S^A$ and $\epsilon_f(o : e) = o$, for $o \in \Omega^A_e$, $e \in {S^A}^+$. To prove the existence of $g^\#$, let $\Sigma = \langle S, \Omega_{S^+}\rangle$ and $\Sigma' = \langle S', \Omega'_{S'^+}\rangle$. The morphism $g^\#\colon \Sigma'_< \to \mathbf{Rec}(f)$ is defined as $g^\# = \langle \sigma_s, \sigma_{\Omega_S}\rangle$, where $\sigma_s\colon S' \to S$ and $\sigma_{\Omega'_{S'}}\colon \Omega_{S'^+} \to \Omega_{S^+}$ are given as: $\sigma_s(s) = g(s)$, for $s \in S'$, $\sigma_{\Omega_e}(o) = g(o : e)$, for $e \in {S'}^+$ and $o \in \Omega'_e$. By construction, the morphism $g^\#$ is the unique such that $\mathbf{Symb}(g^\#); \epsilon_f = g$. □

Corollary 39 *Given a function* $f\colon A \to \mathbf{Symb}(\Sigma_<)$*, it holds that*

$$\mathbf{Symb}(\mathbf{Rec}_{\Sigma_<}(f)) = \epsilon_f; f$$

6.2 Limits and Colimits in AlgSigDep

The rich technical background defined in the previous sections, regarding the signature symbols and their reconstructions, is to be used here to prove the existence of (co)limits in **AlgSigDep**.

Before we begin we would like to remind the reader that this is a standard result that the category **AlgSig** is both complete and cocomplete.

As in Sect. 4 we start our review of limits from the final object. Unfortunately it doesn't exist in **AlgSigDep**.

Theorem 40 *The category* **AlgSigDep** *does not have the final object.*

This is the straight consequence of the lack of the final object in $\mathbf{Soset_{wf}}{\downarrow}$. Fortunately all other (co)limits are present in this category

Theorem 41 *The category* **AlgSigDep** *has all equalizers.*

We use the Theorem 16 and the "reconstructing" functor from Def. 37. For the complete proof see App. A of [Mar10].

The construction of products in **AlgSigDep** follows these in $\mathbf{Soset_{wf}}{\downarrow}$. The result is then "reconstructed" to the form of a signature with dependency structure.

Theorem 42 *The category* **AlgSigDep** *has all binary products.*

Proof: The construction of the binary product in **AlgSigDep** follows the construction of a product candidate in $\mathbf{Rset}{\downarrow}$ (cf. Def. 17).

Let $\Sigma^A_<$ and $\Sigma^B_<$ be two **AlgSigDep**-objects and let $\Sigma = \Sigma^A \times \Sigma^B$ and two projection morphism $\pi^\Sigma_A \colon \Sigma \to \Sigma^A$ and $\pi^\Sigma_B \colon \Sigma \to \Sigma^B$ be the product in **AlgSig**. Now, let $P = \mathbf{SetSymb}(\Sigma)$, $\pi_A = \mathbf{SetSymb}(\pi^\Sigma_A)$, $\pi_B = \mathbf{SetSymb}(\pi^\Sigma_B)$ and $<_P = \{\langle p_1, p_2\rangle \in P^2 \mid \pi_A(p_1) <_{\Sigma_A} \pi_A(p_2) \text{ and } \pi_B(p_1) <_{\Sigma_B} \pi_B(p_2)\}$.

Let a set[4] Q be defined as:

$$Q = \bigcup_{p\in P} \{\langle [X_<, P_<, \mu], p\rangle \mid$$

$[X_<, P_<, \mu]$ is a well founded strictly ordered R-mset such that:

- $K([X_<, P_<, \mu]) = id_X,$
- $\mu; \pi_A$ and $\mu; \pi_B$ are morphisms in $\mathbf{Rset}{\downarrow}$,
- there exists $x \in X,$ such that $\mu(x) = p$ and for all $x' \in X,$ if $x' \neq x$ then $x' <_X x$

$\}$

The well-founded strict order $<_Q$ is defined as follows:

$$\begin{aligned} <_Q = \{\langle\langle[X'_<, P_<, \mu'], p'\rangle, \langle[X_<, P_<, \mu], p\rangle\rangle \in Q^2 \mid \\ \text{there exists } x' \in X, \mu(x') = p', p' < p \\ \text{and } [X'_<, P_<, \mu'] = [X_<, P_<, \mu]^{x'}{\downarrow}\} \end{aligned}$$

[4] Cf. Theorem 20 for the proof that a product candidate in $\mathbf{Soset_{wf}}{\downarrow}$ is a set

Let the top-element labeling function $l\colon Q \to P$ be defined as $l(\langle [X^p_<, P_<, \mu_p], p\rangle) = p$. Let the projections be defined as $\rho_A = l; \pi_A\colon Q_< \to \mathbf{DepSymb}(\Sigma^A_<)$ and $\rho_B = l; \pi_B\colon Q_< \to \mathbf{DepSymb}(\Sigma^B_<)$.

Following the line of the proof of Lemma 18, the relation $<_Q$ is indeed a well-founded strict order and ρ_A and ρ_B are indeed $\mathbf{Soset_{wf}}{\downarrow}$-morphisms.

The product of $\Sigma^A_\leq$ and $\Sigma^B_\leq$ in **AlgSigDep** is an **AlgSigDep**-object

$$\Sigma^Q_< = \langle \mathbf{Rec}(l), <_Q|_{\epsilon_l}\rangle$$

and two projection **AlgSigDep**-morphisms

$$\rho^\Sigma_A = \mathbf{Rec}_{\mathbf{Dep}(\Sigma)}(l); \pi^\Sigma_A\colon \Sigma^Q_< \to \Sigma^A_<$$
$$\rho^\Sigma_B = \mathbf{Rec}_{\mathbf{Dep}(\Sigma)}(l); \pi^\Sigma_B\colon \Sigma^Q_< \to \Sigma^B_<$$

where $\epsilon_l\colon \mathbf{Symb}(\mathbf{Rec}(l)) \to Q$ is the counit function from Lemma 38. Above morphisms are well defined because, by Lemma 35, $P = \mathbf{Symb}(\mathbf{Dep}(\Sigma))$, thus $l\colon Q \to \mathbf{Symb}(\mathbf{Dep}(\Sigma))$. Note that, in general, it may happen that $Q \neq \mathbf{Symb}(\Sigma^Q)$ when l is not injective on sort symbols.

Now, let $\Sigma^T_<$ be an algebraic signature with dependent symbols and $\theta_A\colon \Sigma^T_< \to \Sigma^A_<$ and $\theta_B\colon \Sigma^T_< \to \Sigma^B_<$ be two **AlgSigDep**-morphisms. Let $T_< = \mathbf{DepSymb}(\Sigma^T_<)$. Like in the proof of Lemma 18 we define the monotone function $\theta\colon T \to P$ as $\theta(t) = \langle \mathbf{DepSymb}(\theta_A)(t), \mathbf{DepSymb}(\theta_B)(t)\rangle$ and the $\mathbf{Soset_{wf}}{\downarrow}$-morphism $u\colon T_< \to Q_<$ as $u(t) = \langle [T_<, P_<, \theta]_{/K}{}^{[t]_K}{\downarrow}, \theta(t)\rangle$ for any $t \in T$. Mimicking the proof of Lemma 18, we learn that $u(t) \in Q$, the morphism u is indeed a $\mathbf{Soset_{wf}}{\downarrow}$-morphism and that it is unique such, that $u; \rho_A = \mathbf{DepSymb}(\theta_A)$ and $u; \rho_B = \mathbf{DepSymb}(\theta_B)$. By Lemma 38, there exists the unique signature morphism $u^\#\colon \Sigma^T \to \Sigma^Q$ such that $\mathbf{Symb}(u^\#); \epsilon_l = u$. Using the above, since **Symb** is a functor and by Corollary 39, we prove

$$\begin{aligned}
\mathbf{Symb}(u^\#; \rho^\Sigma_A) &= \mathbf{Symb}(u^\#); \mathbf{Symb}(\rho^\Sigma_A)\\
&= \mathbf{Symb}(u^\#); \mathbf{Symb}(\mathbf{Rec}_{\mathbf{Dep}(\Sigma)}(l); \pi^\Sigma_A)\\
&= \mathbf{Symb}(u^\#); \mathbf{Symb}(\mathbf{Rec}_{\mathbf{Dep}(\Sigma)}(l)); \mathbf{Symb}(\pi^\Sigma_A)\\
&= \mathbf{Symb}(u^\#); \epsilon_l; l; \pi_A\\
&= \mathbf{Symb}(u^\#); \epsilon_l; \rho_A\\
&= u; \rho_A\\
&= \mathbf{Symb}(\theta_A)
\end{aligned}$$

Therefore, since the functor **Symb** is faithful (cf. Lemma 33), we get $u^\#; \rho^\Sigma_A = \theta_A$. Similarly we get $u^\#; \rho^\Sigma_B = \theta_B$.

□

The colimits are inherited from **AlgSig**.

Theorem 43 *The category* **AlgSigDep** *has all colimits.*

At this point we proved all that was needed to say that the category **AlgSigDep** is the natural extension of the category of algebraic signatures **AlgSig** that simply adds dependencies on signatures' symbols. The only cost we pay while shifting from one category to another is the loss of the final object. The other vital properties are preserved. Moreover, the limitation of dependency structure to well-founded strict orders seems to be reasonable and reflects the intuition one may have regarding the matter described in Sect. 2.

Here, this is also the right place to mention, that we imagine models of new signatures being exactly the same as these of the original ones. The whole "dependency thing" is about syntactical analysis, disregarded in models.

7 Conclusion

In the paper we proposed the category of algebraic signatures with symbol dependencies **AlgSigDep**. The proposal was proceeded by the long analysis of several possible orderings and proofs of the existence of the (co)limits in the respective categories. Then we proved the existence of all pushouts and pullbacks in **AlgSigDep**. Unfortunately the category lacks the final object. On the way we defined a product candidate in the category R-sets and p-morphisms, aka the category of transitive Kripke frames.

In the paper we used **AlgSig** and added the dependency structure. However, the construction is more generic than that and should work with most signature categories in the similar way.

Future work concerns the use of algebraic signatures enriched by dependency structure in covariant definition of signatures of generic modules in the architectural specifications framework.

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A Proofs of Lemmas and Theorems

Here we give proofs of Lemmas and Theorems for whom there was no place in the main sections. Before we get to the proofs let us introduce a bit more of technicalities.

Definition 44 (Quotient of R-mset wrt. $\equiv$) *Given an R-mset $[A_R, P_R, \mu]$ and an equivalence relation $\equiv \subseteq A^2$, a quotient R-mset is defined as an R-mset:*

$$[A_R, P_R, \mu]_{/\equiv} = [A'_R, P_R, \mu']$$

where $A' = A_{/\equiv}$, $R_{A'}$ is defined by

$$[a_1]_\equiv R_{A'} [a_2]_\equiv \text{ iff for any } a'_1 \in [a_1]_\equiv \text{ and } a'_2 \in [a_2]_\equiv,\ a'_1 R_A a'_2$$

and the labeling

$$\mu'([a]_\equiv) = \mu(a)$$

where $[a]_\equiv$ is the equivalence class of a wrt. $\equiv$.

Definition 45 (Kernel of an R-mset) *Given an R-mset $[A_R, P_R, \mu]$, its* kernel *is defined as*

$$[A_R, P_R, \mu]_{/K} = [A_R, P_R, \mu]_{/K([A_R, P_R, \mu])}$$

Elements of the kernel are equivalence classes, for $a \in A$, $[a]_K = [a]_{K([A_R, P_R, \mu])}$.

Lemma 46 *Given an R-mset $[A_R, P_R, \mu]$*

$$K([A_R, P_R, \mu]_{/K}) = id_{A_{/K([A_R, P_R, \mu])}}$$

Proof: Let $\sim' \subseteq (A_{/K([A_R, P_R, \mu])})^2$ be a dependency bisimulation on $[A_R, P_R, \mu]_{/K}$, then $\sim \subseteq A^2$, defined as $a_1 \sim a_2$ iff $[a_1]_K \sim' [a_2]_K$, for $a_1, a_2 \in A$, is a dependency bisimulation on $[A_R, P_R, \mu]$. Therefore, $\sim \subseteq K([A_R, P_R, \mu])$ and for any $a_1, a_2 \in A$, if $[a_1]_K \sim' [a_2]_K$ then $a_1 \sim a_2$ and $\langle a_1, a_2 \rangle \in K([A_R, P_R, \mu])$, thus $[a_1]_K = [a_2]_K$ and $\sim' = id_{A_{/K([A_R, P_R, \mu])}}$. $\square$

Lemma 47 *Given an R-mset $[A_R, P_R, \mu]$ and $a \in A$,*

$$K([A_R, P_R, \mu]^a\!\downarrow) = K([A_R, P_R, \mu]) \cap ([A_R, P_R, \mu]^a\!\downarrow)^2$$

Proof: The proof is straightforward. Let us just notice that the kernel relation is based solely on the structure down from the given element wrt. R in the R-mset. $\square$

Corollary 48 *Given an R-mset $[A_R, P_R, \mu]$, and $a \in A$, it holds that*

$$([A_R, P_R, \mu]^a\!\downarrow)_{/K} = ([A_R, P_R, \mu]_{/K})^{[a]_K}\!\downarrow$$

Lemma 49 *Given an* $\mathbf{Rset}{\downarrow}$*-morphism* $f\colon A_R \to B_R$*, and an element* $a \in A$*, the reduct* $f|_{A_R{}^a{\downarrow}}\colon A_R{}^a{\downarrow} \to B_R$ *is also an* $\mathbf{Rset}{\downarrow}$*-morphism.*

Proof: The reduct $f|_{A_R{}^a{\downarrow}}$ is monotone, because f is so. Since R is transitive, for any $a' \in A_R{}^a{\downarrow}$, all $a'' R_A a'$ are in $A_R{}^a{\downarrow}$ as well and so $f|_{A_R{}^a{\downarrow}}$ meets the requirement (2) of Def. 2, as f does. $\square$

Lemma 50 (Rset↓-morphisms Preserve Dependency Bisimulations) *Consider two R-msets* $[A_R, P_R, \mu]$ *and* $[A'_R, P_R, \mu']$ *and a* $\mathbf{Rset}{\downarrow}$*-morphism* $f\colon A_R \to A'_R$*, such that* $\mu = f;\mu'$*. If* $\sim\, \subseteq A^2$ *is a dependency bisimulation on* $[A_R, P_R, \mu]$*, then* $f(\sim) \subseteq f(A)^2$ *is a dependency bisimulation on* $[f(A)_R, P_R, \mu'|_{f(A)}]$*, where* $R_{f(A)} = R_{A'}|_{f(A)}$*.*

Proof: Let $\sim' = f(\sim)$. Then, given $a_1, a_2 \in A$, $a_1 \sim a_2$, we have $f(a_1) \sim' f(a_2)$. By $\mu = f;\mu'$, we have $\mu'(f(a_1)) = \mu'(f(a_2))$ and for any $p' \in P$, $p' \, R \, \mu(f(a_1))$ and $b'_1 \in A'$, if $\mu'(b'_1) = p'$ and $b'_1 \, R \, f(a_1)$, by requirement (2) of Def. 2, there exists $a'_1 \in A$, $a'_1 \, R \, a_1$, $f(a'_1) = b_1$ and, since $\sim$ is a dependency bisimulation, there exists $a'_2 \, R \, a_2$, $a'_1 \sim a'_2$, thus $f(a'_2) \, R \, f(a_2)$ and $f(a'_2) \sim' f(a'_1) = b'_1$, hence $b'_1 \in f(A)$. By symmetry, this proves that $f(\sim)$ is a dependency bisimulation on $[f(A)_R, P_R, \mu']$. $\square$

Corollary 51 *Consider two R-msets* $[A_R, P_R, \mu]$ *and* $[A'_R, P_R, \mu']$ *and a surjective* $\mathbf{Rset}{\downarrow}$*-morphism* $f\colon A_R \to A'_R$*, such that* $\mu = f;\mu'$*. If* $\sim\, \subseteq A^2$ *is a dependency bisimulation on* $[A_R, P_R, \mu]$*, then* $f(\sim) \subseteq {A'}^2$ *is a dependency bisimulation on* $[A'_R, P_R, \mu']$*.*

Lemma 52 *Given two R-msets* $[A_R, P_R, \mu]$ *and* $[A'_R, P_R, \mu']$ *and a* $\mathbf{Rset}{\downarrow}$*-morphism* $f\colon A_R \to A'_R$*, such that* $\mu = f;\mu'$*, it holds that*

$$K(f) \subseteq K([A_R, P_R, \mu])$$

where $K(f) \subseteq A^2$ *is the kernel relation for the function* f*. Moreover, if* $K([A'_R, P_R, \mu]) = id_{A'}$*, then*

$$K(f) = K([A_R, P_R, \mu])$$

Proof: A $\mathbf{Rset}{\downarrow}$-morphism f weakly reflects R-set down-closures. Therefore, if $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$, then $\mu(a_1) = \mu(a_2)$ and for any $p' R \mu(a_1)$ and any $a_3 \, R \, a_1$ such that $\mu(a_3) = p'$, since $f(a_3) \, R \, f(a_1) = f(a_2)$ there exists $a_4 \, R \, a_2$, $f(a_4) = f(a_3)$. By symmetry, this implies that $K(f)$ is a dependency bisimulation on $[A_R, P_R, \mu]$ and since $K([A_R, P_R, \mu])$ is the greatest such relation, $K(f) \subseteq K([A_R, P_R, \mu])$.

When additionally $K([A'_R, P_R, \mu]) = id_{A'}$, let $\sim\, \subseteq A^2$ be a dependency bisimulation on $[A_R, P_R, \mu]$ and $a_1, a_2 \in A$. If $a_1 \sim a_2$ then, by definition of f, $f(a_1) f(\sim) f(a_2)$ and, since $f(\sim)$ is a dependency bisimulation on $[f(A)_R, P_R, \mu|_{f(A)}]$ (cf. Lemma 50), $f(\sim) \subseteq K([f(A)_R, P_R, \mu|_{f(A)}]) \subseteq K([A'_R, P_R, \mu']) = id_{A'}$, thus

$f(a_1) = f(a_2)$, therefore $K([A_R, P_R, \mu]) \subseteq K(f)$. Together with the previous result this gives $K(f) = K([A_R, P_R, \mu])$. □

Proof of Theorem 13:

The relations are reflexive, therefore there exist unique morphisms from any of their objects to the singleton ordered by identity. □

Proof of Theorem 14: Since the relations in objects of **Soset**$\downarrow$ (and $\mathbf{Soset_{wf}}\downarrow$) are irreflexive, their morphisms must not glue together any elements being in relation. Hence, if the final object existed, there would be an injective map from any ordinal (represented as an R-set with natural strict well-founded "ordering") into it. Hence, such a final object can not be a proper set. □

Proof of Theorem 16: Given two morphisms $f, g \colon A_R \to B_R$ in **Rset**$\downarrow$, their equalizer is an inclusion $e \colon C_R \to A_R$, where

$$C = \{a \in A \mid \text{ for all } a' \in {A_R}^a\!\downarrow,\ f(a') = g(a')\}$$

See Def. 3 for the definition of closed down sub R-set ${A_R}^a\!\downarrow$ induced by a. The relation is defined as $R_C = R_A|_C$. Trivially, e is an **Rset**$\downarrow$-morphism. Let us check that it is universal. Let $h \colon D_R \to A_R$ be such **Rset**$\downarrow$-morphism that $h; f = h; g$. We need to find the unique $u \colon D_R \to C_R$, such that $u; e = h$. Since e is an inclusion and because $h(D) \subseteq C$, putting $u(d) = h(d)$ for $d \in D$ yields the only such morphism. The inclusion $h(D) \subseteq C$ is the consequence of the fact that h, as an **Rset**$\downarrow$-morphism, weakly reflects R-set down-closures. To complete the proof, it is enough to notice that if A_R and B_R are in **Preord**$\downarrow$, **Soset**$\downarrow$ or $\mathbf{Soset_{wf}}\downarrow$, respectively, then so is C_R. □

Lemma 53 *Every R-mset component of $\langle [X_R, P_R, \mu], p\rangle \in (A \prod B)_R$ has a distinguished top-element x, i.e. there exists exactly one $x \in X$, such that $\mu(x) = p$ and for all $x' \neq x \in X$, $x' \, R \, x$.*

Proof: By contradiction, let x_1 be another such element, i.e. $x_1 \neq x$ and for all $x' \neq x_1 \in X$, $x' \, R \, x_1$. As a consequence $x \, R \, x_1$ and $x_1 \, R \, x$, thus, since R is transitive, $\langle x, x_1\rangle \in K([X_R, P_R, \mu]) = id_X$. Contradiction. □

Lemma 54 (Product Candidate is Self-adequate) *The class $A \prod B$ is* self-adequate*, meaning that for all $\langle [X_R, P_R, \mu], p\rangle \in A \prod B$ and for all $x \in X$*

$$\langle [X_R, P_R, \mu]^x\!\downarrow, \mu(x)\rangle \in A \prod B$$

Proof of Lemma 18: Let us assume that $A \prod B$ is a set. For the notation convenience we name

$$\langle Q, R_Q\rangle = \langle A \prod B, R_{A \prod B}\rangle$$

We notice that, by definition, R_Q is transitive, which makes Q_R indeed an **Rset**$\downarrow$-object. Moreover, both ρ_A and ρ_B are **Rset**$\downarrow$ morphisms. They obviously preserve the relation. Let $q_1, q_2 \in Q$ be $q_1 = \langle [X^1_R, P_R, \mu_1], p_1\rangle$ and $q_2 = \langle [X^2_R, P_R, \mu_2], p_2\rangle$, such that $q_1 \, R_Q \, q_2$, then, by definition, $p_1 \, R_P \, p_2$, thus $(\rho_A(q_1)) \, R_A (\rho_A(q_2))$ (and similarly for ρ_B). They also meet the requirement (2) of Def. 2, which makes them **Rset**$\downarrow$-morphisms. Namely, given $q = \langle [X_R, P_R, \mu], p\rangle \in Q$ and $a' \in A$, $a' \, R_A \, \rho_A(q)$, by definition of Q there exists $x \in X$, $\mu(x) = p$, $\mu; \pi_A(x) = a$ and, since $\mu; \pi_A$ is a **Rset**$\downarrow$-morphism, there exists $x' \, R \, x$, $\mu; \pi_A(x') = a'$. By self-adequacy of Q (cf. Lemma 54), there is $q' = \langle [X_R, P_R, \mu]^{x'}\downarrow, \mu(x')\rangle$, and of course $q' \, R \, q$ and $\rho_A(q') = \pi_A(\mu(x')) = a'$. For ρ_B the proof goes likewise.

Now we show that for each object $T_R \in$ **Rset**$\downarrow$ and two morphisms $\theta_A : T_R \to A_R$ and $\theta_B : T_R \to B_R$ in **Rset**$\downarrow$, there exists a unique **Rset**$\downarrow$-morphism $u : T_R \to Q_R$ such that $u; \rho_A = \theta_A$ and $u; \rho_B = \theta_B$.

Let an **Rset**$\downarrow$-object T_R and morphisms θ_A and θ_B be as described above. We define a function $\theta : T \to P$ as

$$\theta(t) = \langle \theta_A(t), \theta_B(t)\rangle$$

The monotonicity of θ follows the monotonicity of θ_A and θ_B and the definition of R_P. By definition, it holds that $\theta; \pi_A = \theta_A$ and $\theta; \pi_B = \theta_B$. We notice that $[T_R, P_R, \theta]$ is an R-mset. Let

$$[T'_R, P_R, \theta'] = [T_R, P_R, \theta]_{/K}$$

It is easy to prove that $\theta'; \pi_A$ and $\theta'; \pi_B$ are **Rset**$\downarrow$-morphisms. Let the morphism $u : T_R \to Q_R$ be defined as

$$u(t) = \langle [T_R, P_R, \theta]_{/K}{}^{[t]_K}\downarrow, \theta(t)\rangle$$

for any $t \in T$, where $[t]_K$ is an equivalence class of t wrt. $K([T_R, P_R, \theta])$ (cf. Def. 12 and Def. 45). Before we proceed with the proof let us simplify the notation by naming

$$[T^t_R, P_R, \theta_t] = [T_R, P_R, \theta]_{/K}{}^{[t]_K}\downarrow$$

for $t \in T$.

Let us show that for every $t \in T$, $u(t) \in Q$. The R-mset $[T^t_R, P_R, \theta_t]$ indeed meets all requirements from the definition of Q. By Def. 10, $\theta_t = \theta'|_{T^t}$ and, since $\theta'; \pi_A$ and $\theta'; \pi_B$ are **Rset**$\downarrow$-morphisms, by Lemma 49 $\theta_t; \pi_A$ and $\theta_t; \pi_B$ are also **Rset**$\downarrow$-morphisms. Moreover, the element $[t]_K$ is such that for all $x \in T^t$, $(x) \, R([t]_K)$ and by Lemma 46 and Corollary 48, $K([T^t_R, P_R, \theta_t]) = id_{[T^t_R, P_R, \theta_t]}$. Now, let us check that u is monotone. Let $t', t \in T$ and $t' \, R_T \, t$. To prove that $u(t') \, R_Q \, u(t)$ we need to show that there exists $x' \in T^t$, $\theta_t(x') = \theta(t')$ and that $[T^{t'}_R, P_R, \theta_{t'}] = [T^t_R, P_R, \theta_t]^{x'}\downarrow$. Let us take $x' = [t']_K$. Obviously $\theta_t([t']_K) = \theta(t')$. The second requirement, $[T^{t'}_R, P_R, \theta_{t'}] = [T^t_R, P_R, \theta_t]^{[t']_K}\downarrow$ also holds because $([t']_K) R_{T^t}([t]_K)$ and by Lemma 11. It is trivial to show that $u; \rho_A = \theta_A$ and $u; \rho_B = \theta_B$. To finish the proof we need to show that a function u is a **Rset**$\downarrow$-morphism and that the choice of u is unique.

We prove that the function u is a **Rset**$\downarrow$-morphism. We already have shown that it is monotone. The requirement (2) of Def. 2 says that for any $t \in T$ and $q' = \langle [X'_R, P_R, \mu'], p' \rangle \in Q$ such that $(q')\, R_Q(u(t))$, there must exist $t' \in T$, $t'\, R\, t$ and $u(t') = q'$. Since $(q')\, R_Q(u(t))$, there exists $x' \in T^t$ that $\theta_t(x') = p'$ and, by definition of u, there exists $t' \in T$ that $[t']_K = x'$ and $t'\, R\, t$. Of course $\theta(t') = p'$. Moreover, by definition of R_Q, $[X'_R, P_R, \mu'] = [T^t_R, P_R, \theta_t]^{[t']_K}\downarrow$ and by Lemma 11 we have $[T^t_R, P_R, \theta_t]^{[t']_K}\downarrow = [T^{t'}_R, P_R, \theta_{t'}]$. This proves that $q' = u(t')$.

To show the uniqueness of u such that $u; \theta_A = \rho_A$ and $u; \theta_B = \rho_B$ let us have some **Rset**$\downarrow$-morphism $u' \colon T_R \to Q_R$ that $u; \theta_A = \rho_A$ and $u; \theta_B = \rho_B$. For any $t \in T$, $u'(t) = \langle [X_R, P_R, \mu], \theta(t) \rangle$, for some R-set X_R and a monotone function $\mu \colon X_R \to P_R$. Let us define a surjective **Rset**$\downarrow$ morphism $u'_t \colon {T_R}^t\!\downarrow \to X_R$ as follows. For any $t' \in {T_R}^t\!\downarrow$ we have $(u'(t'))\, R_Q(u(t))$, thus $u'(t') = \langle [X, P_R, \mu]^{x'}\downarrow, \theta(t') \rangle$ for exactly one $x' \in X$ (cf. Lemma 53), let

$$u'_t(t') = x'$$

The morphism u' meets both requirements of Def. 2 – it is monotone and, since Q is self adequate (cf. Lemma 54), for each $x'' \in X$, if $x''\, R_X\, x'$ then there exists $q'' \in Q$ such that $(q'')\, R_Q(u'(t'))$ and, since u' is a **Rset**$\downarrow$-morphism, there must exist $t'' \in T$ such that $t''\, R\, t'$ and $u'(t'') = q'' = \langle [X_R, P_R, \mu]^{x''}\downarrow, \theta(t'') \rangle$, thus $u'_t(t'') = x''$. This also proves that u'_t is surjective. Hence, there is a bijection between X and $({T_R}^t\!\downarrow)_{/K(u'_t)}$, where $K(u'_t)$ is the kernel of the function u'_t. By Lemma 52, since u'_t is surjective and $K([X_R, P_R, \mu]) = id_X$,

$$K(u'_t) = K([{T_R}^t\!\downarrow, P_R, \theta|_{{T_R}^t\downarrow}]) = K([T_R, P_R, \theta]^t\!\downarrow)$$

This means that $[X, P_R, \mu] = ([T_R, P_R, \theta]^t\!\downarrow)_{/K}$. By Corollary 48, we get $[X, P_R, \mu] = ([T_R, P_R, \theta]_{/K})^{[t]_K}\downarrow$; therefore, $u' = u$.

This shows the uniqueness of u and completes the proof that $\langle Q, R_Q \rangle$, together with projections ρ_A and ρ_B, is the product of A_R and B_R in **Rset**$\downarrow$.

□

Proof of Lemma 19: (reflexive) If R_A and R_B are reflexive, then clearly R_P is also reflexive. Let $q = \langle [X_R, P_R, \mu], p \rangle \in A \prod B$. By Lemma 53 there exists a top-element $x \in X$ and we notice that $[X_R, P_R, \mu] = [X_R, P_R, \mu]^x\downarrow$. By reflexivity of R_P we have $p\, R\, p$. All together this makes $q\, R\, q$ (cf. Def.17).

(irreflexive) Let R_A and R_B be irreflexive. By contradiction. Let $q = \langle [X_R, P_R, \mu], p \rangle \in A \prod B$ and $q\, R\, q$. Thus, $p\, R\, p$; therefore, $(\pi_A(p)\, R_A(\pi_A(p))$, which contradicts the irreflexivity of R_A.

(asymmetric) Let R_A and R_B, thus also R_P, be asymmetric. Let $q = \langle [X_R, P_R, \mu], p \rangle, q' = \langle [X'_R, P_R, \mu'], p' \rangle \in A \prod B$, and let $q\, R\, q'$. This means that $p\, R\, p'$ and, by asymmetry, it doesn't hold that $p'\, R\, p$. Therefore, by definition of $R_{A \prod B}$ (cf. Def. 53), it doesn't hold that $q'\, R\, q$.

(strict well-founded) Let R_A and R_B be strict well-founded relations. In this case R_P is also strict well founded. Let $C \subseteq A \prod B$ be a descending chain wrt. $R_{A \prod B}$. If it is empty, then it is finite. Otherwise, let $q = \langle [X_R, P_R, \mu], p \rangle \in C$.

Let us notice that due to Def. 17, for any $q' \in C$, $q' \, R \, q$, there exists $x' \in X$ such that $q' = \langle [X_R, P_R, \mu]^{x'}\!\downarrow, \mu(x') \rangle$. It means that if $C^q\!\downarrow$ was infinite, then $X_R{}^{x'}\!\downarrow$ would also be so. However, this is not the case, because $\mu(X_R)$ is finite (as a chain wrt. well-founded R_P) and μ is monotone. Therefore, $C^q\!\downarrow$ is finite and $R_{A \prod B}$ is strict well-founded. □

Proof of Theorem 20: By Lemma 19 we know that $<_{A \prod B}$ is a well-founded strict order, thus in fact $A \prod B$, if proved to be a set, is the $\mathbf{Soset_{wf}}\!\downarrow$-object. To show that it is indeed a product we need to show that $A \prod B$ is a set to be able to use Lemma 18. To do so, it is enough to bound the number of $A \prod B$ elements that share the given label $p \in P$ (using notation from Def. 17). Given $q = \langle [X_<, P_<, \mu], p \rangle \in A \prod B$ and a label $p' \in P_<{}^p\!\downarrow$, let us bound the cardinality of $\mu^{-1}(p')$ by cases:

- (base case) if $P_<{}^p\!\downarrow = \{p\}$, then $p' = p$ and $|\mu^{-1}(p')| <= 1$, i.e. $l_{p'} = 1$, because there may be only one element labeled by p' distinct wrt. the kernel relation (cf. Def. 12);
- (induction step) otherwise, let $\mathcal{L} = \sum_{p'' < p'} l_{p''}$, where $l_{p''}$ is the bound of $\mu^{-1}(p'')$ for $p'' < p'$, then $|\mu^{-1}(p')| <= 2^{\mathcal{L}}$, i.e. $l_{p'} = 2^{\mathcal{L}}$; it is impossible to have more elements labeled by p' distinct wrt. the kernel relation, than all combinations of elements lower wrt. $<_X$.

The cardinal number $l_{p'}$ is well defined for every $p' \in P_<{}^p\!\downarrow$, because $<_X$ is a well-founded strict order. Finally, by definition of $<_{A \prod B}$, we conclude that the cardinality of the set of elements that share the label p is bounded by l_p, therefore, $A \prod B$ is a set. By Lemma 18 a $\mathbf{Soset_{wf}}\!\downarrow$-object $\langle A \prod B, <_{A \prod B} \rangle$ together with morphisms ρ_A and ρ_B, as defined in Def. 17, is a product of $A_<$, $B_<$ in $\mathbf{Soset_{wf}}\!\downarrow$. □

Lemma 55 *Given two* $\mathbf{Rset}\!\downarrow$*-morphisms* $f, g \colon A_R \to B_R$ *and a relation* $\sim \subseteq B^2$ *defined as* $b_1 \sim b_2$ *iff there exists* $a \in A$ *such that* $b_1 = f(a)$ *and* $b_2 = g(a)$ *and its reflexive, symmetric and transitive closure* $\equiv = Trans(Sym(Ref(\sim)))$*, it holds that: for any* $b_1, b_2 \in B$*, if* $b_1 \, R_B \, b_2$ *then for any* $b_2' \equiv b_2$ *there exists* $b_1' \equiv b_1$ *such that* $b_1' \, R_B \, b_2'$*.*

Proof: For any $b_2 \equiv b_2'$ there exists a path from b_2 to b_2' in the undirected graph $Graph(f) \cup Graph(g)$. Let $a_2 \in A$ be such that $f(a_2) = b_2$ and let $b_1 \, R_B \, b_2$. Since f is $\mathbf{Rset}\!\downarrow$-morphism, by requirement (2) of Def. 2, there exists $a_1 \in A$ such that $f(a_1) = b_1$ and $a_1 \, R_A \, a_2$. Since g is also an $\mathbf{Rset}\!\downarrow$-morphism, by requirement (1) of the same definition, it is monotone, i.e., $g(a_1) \, R_A \, g(a_2)$ and $g(a_1) \equiv b_1$ and $g(a_2) \equiv b_2$. The above procedure executed along the path between b_2 and b_2' (the same that served the transitive closure in definition of $\equiv$) results in existence of the required $b_1' \equiv b_1$ such that $b_1' \, R_B \, b_2'$. □

Proof of Lemma 22: Let $f, g \colon A_R \to B_R$ be two $\mathbf{Rset}\!\downarrow$-morphisms and $e \colon B \to C$ be their coequalizer in $\mathbf{Set}$. Let us notice that $C = B_{/\equiv}$, where

$\equiv$ is an equivalence defined as in Lemma 55. We need to show that R_C is transitive and that e meets both conditions of Def. 2. In fact R_C is transitive. Let $c_1\, R_C\, c_2\, R_C\, c_3$. Function e as a coequalizer in **Set** is surjective, thus there exist $b_1, b_2, b_2', b_3 \in B$ such that $e(b_1) = c_1$, $e(b_2) = e(b_2') = c_2$, $e(b_3) = c_3$ and $b_1\, r_B\, b_2$ and $b_2'\, R_B\, b_3$. By Lemma 55, since $b_2 \equiv b_2'$, there exists $b_1' \equiv b_1$ such that $b_1'\, R_B\, b_2'$. Relation R_B is transitive thus $b_1'\, R_B\, b_3$. Of course, since e is the coequalizer of f and g, we get $e(b_1') = e(b_1) = c_1$. Therefore $c_1\, R_C\, c_3$. Function e trivially meets the first condition of Def. 2, because $R_C = e(R_B)$. To prove the second condition let $b_2 \in B$ and $c_1 \in C$ such that $c_1\, R_C\, e(b_2)$. From the surjectivity of e and the definition of R_C we get the existence of $b_2', b_1' \in B$ such that $e(b_1') = c_1$, $e(b_2') = e(b_2)$ and $b_1'\, R_B\, b_2'$. Therefore, $b_2' \equiv b_2$ and by Lemma 55, there exists $b_1 \in B$ such that $b_1 \equiv b_1'$ and $b_1\, R_B\, b_2$, and accordingly $e(b_1) = e(b_1') = c_1$. □

Proof of Theorem 23: The coequalizer of two morphisms $f, g\colon A_R \to B_R$ in **Rset↓** is $e\colon B_R \to C_R$, where e is the coequalizer of f and g in **Set** and a relation $R_C = e(R_B)$. The universal properties of e in **Rset↓** are inherited from **Set**. Namely, given a morphism $h\colon B_R \to D_R$ in **Rset↓** such that $f;h = g;h$, there exists a unique function $k\colon C \to D$ such that $e;k = h$. It is monotone. Given $c_1, c_2 \in C$, $c_1\, R_C\, c_2$ since e in surjective (as a coequalizer in **Set**) and monotone there exist $b_1, b_2 \in B$ such that $b_1\, R_B\, b_2$ and $e(b_1) = c_1$ and $e(b_2) = c_2$. We have $h(b_1)\, R_D\, h(b_2)$, because h is monotone, and finally

$$k(c_1) = k(e(b_1)) = h(b_1)\, R_D\, h(b_2) = k(e(b_1)) = k(c_2)$$

Function k also meets the requirement (2) of Def. 2. Let $c_1 \in C$ and $d_2 \in D$ be such that $d_2\, R_D\, k(c_1)$. Since h is a p-morphism, for any $b_1 \in B$ such that $h(b_1) = k(c_1)$ there exits $b_2 \in B$, $b_2\, R_B\, b_1$ and $h(b_2) = d_2$. Let us choose the b_1 such that $e(b_1) = c_1$. It exists because e is surjective. Of course $h(b_1) = k(e(b_1)) = k(c_1)$, so there is b_2 with properties given above. Function e is monotone, thus, $e(b_2)\, R_C\, e(b_1)$. Moreover, $k(e(b_2)) = h(b_2) = d_2$. The two, above shown, properties of function k make it an **Rset↓**-morphism.

Due to Lemma 22, e is an **Rset↓**-morphism and $e(R_B)$ is a transitive relation. Note that reflexivity of R_B guarantees reflexivity of $e(R_B)$. □

Proof of Theorem 24: Let us show a counterexample. A strictly ordered set $\langle \mathbb{N}, \mathit{prev}\rangle$, where $\mathit{prev} \subseteq \mathbb{N} \times \mathbb{N}$ is defined as $\mathit{prev} = \{\langle n+1, n\rangle \mid n \in \mathbb{N}\}$, is a **Soset↓**-object. Let $f, g\colon \langle \mathbb{N}, \mathit{prev}\rangle \to \langle \mathbb{N}, \mathit{prev}\rangle$ be two **Soset↓**-morphisms be defined as $f = \mathit{id}_{\mathbb{N}}$ and $g = \mathit{succ}$, where succ is a successor function. Their coequalizer in **Rset↓** is $e\colon \langle \mathbb{N}, \mathit{prev}\rangle \to \langle C, R_C\rangle$, where $C = \{*\}$ and $R_C = \{\langle *, *\rangle\}$, i.e. a singleton ordered by identity. However, $\langle C, R_C\rangle$ fails to be a **Soset↓**-object, because R_C is reflexive. Moreover, no other function may in the same time coequalize functions f and g and stay monotone. Therefore, they have no coequalizer in **Soset↓**. □

Proof of Theorem 25: The coequalizer of two morphisms $f, g \colon A_R \to B_R$ in $\mathbf{Soset_{wf}}{\downarrow}$ is $e \colon B_R \to C_R$, where e is the coequalizer of f and g in **Set** and a relation $R_C = e(R_B)$. If all descending chains in R_B are finite, for any $b_1, b_2 \in B$, if $b_1 < b_2$ then $b_1 \not\equiv b_2$, where $\equiv$ is the equivalence relation as in Lemma 55. To prove this fact, let us assume that there are $b_1, b_2 \in B$ such that $b_1 < b_2$, the proof goes by induction on the length of the descending chain lower to b_1 wrt. $<$. In the base case let for all $b \in B$, $b \not< b_1$. By contradiction let $b_1 \equiv b_2$. Due to Lemma 55, since $b_1 \equiv b_2$ and $b_1 < b_2$, there must exist $b_1' < b_1$. Contradiction. In the induction step let us assume that for all $b < b_1$, $b \not\equiv b_1$. Again by contradiction, let $b_1 \equiv b_2$. Using once more Lemma 55, since $b_1 \equiv b_2$ and $b_1 < b_2$, we get the existence of $b_1' < b_1$, that $b_1' \equiv b_1$. Contradiction. Therefore, irreflexivity of well-founded R_B guarantees irreflexivity of $e(R_B)$ in $\mathbf{Soset_{wf}}{\downarrow}$. □

Proof of Theorem 26: The empty set ordered by the empty relation is an initial object in all above-listed categories. A binary coproduct of $\langle A, R_A \rangle$ and $\langle B, R_B \rangle$ is $\langle A \uplus B, R_A {\uplus} R_B \rangle$. Other finite and infinite coproducts are defined in the same way. □

Proof of Theorem 40: The category $\mathbf{Soset_{wf}}{\downarrow}$ is embeddable into **AlgSigDep** as a full subcategory (of sort-symbols-only signatures); thus, by Theorem 14, the final object does not exist. □

Proof of Theorem 41: Given two **AlgSigDep**-morphisms $f, g \colon \Sigma^A_< \to \Sigma^B_<$, let $e \colon C \to \mathbf{Symb}(\Sigma^A_<)$ be the equalizer of

$$\mathbf{Symb}(f), \mathbf{Symb}(g) \colon \mathbf{Symb}(\Sigma^A_<) \to \mathbf{Symb}(\Sigma^B_<)$$

in **Set**. The equalizer of f and g in **AlgSigDep** is

$$\mathbf{Rec}_{\Sigma^A_<}(e) \colon \mathbf{Rec}(e) \to \Sigma^A_<$$

See Def. 37 for a definition of signature morphism reconstruction. See also a proof of Theorem 16 showing that the equalizer from **Set** is also the equalizer in $\mathbf{Soset_{wf}}{\downarrow}$. □

Proof of Theorem 43: A coequalizer of two morphisms $f, g \colon \langle \Sigma_A, <_A \rangle \to \langle \Sigma_B, <_B \rangle$ in **AlgSigDep** is $e \colon \langle \Sigma_B, <_B \rangle \to \langle \Sigma_C, <_C \rangle$, where e is the coequalizer of f and g in **AlgSig** and the strict order $<_C = \mathbf{Symb}(e)(<_B)$ (cf. Theorem 16). The initial object in **AlgSigDep** is the empty signature with the empty relation. Binary coproducts in **AlgSigDep** are binary coproducts in **AlgSig** ordered by the union of the component orders. Other finite and infinite coproducts are defined in the same way. □