Maillet Type Theorem and Gevrey Regularity in Time of Solutions to Nonlinear Partial Differential Equations

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I am a researcher of partial differential equations in the complex domain. Recently, I am very much interested in applying complex method to problems in the real domain.
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My intension is illustrated as follows.

- There are many good arguments in PDEs in the complex domain
- Apply
- Problems in PDEs in the real domain
In this talk, I will consider the equation

\[ t^\gamma \partial_t^m u = F(t, x, \{ \partial_t^j \partial_x^\alpha u \}_{j+|\alpha| \leq L} ) \]

where \( \gamma \geq 0 \) and \( L \geq m \geq 1 \)
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and I will present two results:

Part I: Maillet type theorem
   - this is a model in the complex domain -

Part II: Gevrey regularity in time of solutions of (E)
   - this is a result in the real domain -
Part I
Maillet type theorem in the complex PDEs
0.1. Notations

\( t \) the time variable in \( \mathbb{C}_t \),
\[ x = (x_1, \ldots, x_n) \] the space variables in \( \mathbb{C}^n_x \),
\( D_R = \{ x \in \mathbb{C}^n ; |x_i| \leq R \ (i = 1, \ldots, n) \} \).
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\( D_R = \{ x \in \mathbb{C}^n ; |x_i| \leq R \ (i = 1, \ldots, n) \} \).

We will use the following notations:

\( \mathcal{O}_R : \) the set of all holomorphic functions in \( x \) on \( D_R \),

\( \mathcal{O}_R[[t]] : \) the ring of formal power series in \( t \) with coefficients in \( \mathcal{O}_R \),

\( \mathcal{M}_R[[t]] : \) the subset of all \( f(t, x) \in \mathcal{O}_R[[t]] \) satisfying \( f(0, x) \equiv 0 \).
0.2. Some definitions

Definition 0.1. For $s \geq 1$ we denote by $\mathcal{O}\{t\}_s$ (or $\mathcal{E}^{(s)}$) the set of all formal power series $\sum_{k \geq 0} a_k(x)t^k \in \mathcal{O}_R[[t]]$ satisfying the following: there are $C > 0$ and $h > 0$ such that

$$\max_{x \in D_R} |a_k(x)| \leq Ch^k k!^{s-1}, \quad \forall k \in \mathbb{N}.$$
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$$\max_{x \in D_R} |a_k(x)| \leq C h^k k!^{s-1}, \quad \forall k \in \mathbb{N}.$$ 

If $f(t, x) \in \mathcal{O}\{t\}_s$ (or $\in \mathcal{E}\{s\}$), we say that $f(t, x)$ is a formal power series in the formal Gevrey class of order $s$. 
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Definition 0.2. Let $f(t, x) = \sum_{k \geq 0} a_k(x) t^k \in \mathcal{O}_R[[t]]$. We define the valuation of $f(t, x)$ with respect to $t$ by

$$\text{val}_t(f) = \min\{ k \in \mathbb{N} ; a_k(x) \not\equiv 0 \}$$

(if $a_k(x) \equiv 0$ for all $k \in \mathbb{N}$, we set $\text{val}_t(f) = \infty$).
0.3. Equation and assumption

Let $\gamma \geq 0$ and $1 \leq m \leq L$ be integers, and let us consider

(E) $\quad t^{\gamma} \partial_t^m u = F\left(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\leq L}\right)$
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$$(E) \quad t^{\gamma} \partial_t^m u = F(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\leq L})$$

under the following assumptions:

$c_1)$ $F(t, x, z)$ is a holomorphic function on $\Omega$,

c_2) $\hat{u}(t, x) \in \mathcal{M}_R[[t]]$ is a formal solution of (E)

where $\Omega$ is a neighborhood of the origin.
0.3. Equation and assumption

Let $\gamma \geq 0$ and $1 \leq m \leq L$ be integers, and let us consider

\[(E) \quad t^\gamma \partial_t^m u = F(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\leq L})\]

under the following assumptions:

1) $F(t, x, z)$ is a holomorphic function on $\Omega$,

2) $\hat{u}(t, x) \in M_R[[t]]$ is a formal solution of (E) where $\Omega$ is a neighborhood of the origin. We set:

$$k_{j,\alpha} = v_1 t \left( \frac{\partial F}{\partial z_{j,\alpha}}(t, x, D\hat{u}(t, x)) \right), \quad D\hat{u} = \{\partial_t^j \partial_x^\alpha \hat{u}\}_{j+|\alpha|\leq L},$$

and suppose

3) $k_{j,\alpha} \begin{cases} \geq \gamma - m + j, & \text{if } |\alpha| = 0, \\ \geq \gamma - m + j + 1, & \text{if } |\alpha| > 0. \end{cases}$
0.4. Maillet type theorem

Then, we have the following result:

Theorem 0.3 (Gérard-Tahara). Suppose the conditions $c_1$, $c_2$ and $c_3$: then, the formal solution $\hat{u}(t, x)$ in $c_2$ satisfies

$$\hat{u}(t, x) \in \mathcal{O}\{t\}_s \quad (\text{or } \in \mathcal{E}\{s\}) \quad \text{for any } \ s \geq s_0$$

where

$$s_0 = 1 + \max \left[ 0, \max_{|\alpha|>0} \left( \frac{j + |\alpha| - m}{k_{j,\alpha} - \gamma + m - j} \right) \right].$$

(Essentially, the proof was given in the book of Gérard-Tahara.)
Part II
Gevrey regularity in time in the real domain
In this PART II, I will consider the equation

\[
 t^\gamma \partial_t^m u = F\left( t, x, \left\{ \partial_t^j \partial_x^\alpha u \right\}_{j+|\alpha| \leq L, j < m} \right)
\]

where \( \gamma \geq 0 \) and \( L \geq m \geq 1 \)

in Gevrey classes,
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\[(E) \quad t^\gamma \partial_t^m u = F\left(t, x, \left\{ \partial_t^j \partial_x^\alpha u \right\}_{j+|\alpha| \leq L, j < m}\right)\]

where \(\gamma \geq 0\) and \(L \geq m \geq 1\)

in Gevrey classes, and give an answer to the following problem on time regularity:

**Problem.** Let \(u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))\) (with \(\sigma \geq 1\)) be a solution of (E); then can we have the property:

\[u(t, x) \in \mathcal{E}^{\{s, \sigma\}}([0, T] \times V)\]

for some \(s \geq 1\)?
The plan of part II is as follows:

- 1. Notations, Definitions of Gevrey classes, etc
- 2. Problem and examples
- 3. Main theorems
  - sufficient condition for time regularity -
- 4. Necessity of the condition
§1. Notations, definitions, etc
1.1. Notations

\[ t \text{ the time variable in } \mathbb{R}_t, \]
\[ x = (x_1, \ldots, x_n) \text{ the space variables in } \mathbb{R}_x^n, \]
\[ \mathbb{N} = \{0, 1, 2, \ldots \}, \quad \mathbb{N}^* = \{1, 2, \ldots \} \]
\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \]
\[ \partial_x = (\partial x_1, \ldots, \partial x_n) \text{ with } \partial x_i = \partial / \partial x_i \ (i = 1, \ldots, n), \]
\[ \partial^\alpha_x = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}. \]
1.2. Functions in Gevrey class (1)

Let $\sigma \geq 1$ and $V$ be an open subset of $\mathbb{R}^n_x$.

(1) We denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset $K$ of $V$ there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^\alpha f(x)| \leq Ch^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$
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A function in $\mathcal{E}\{\sigma\}(V)$ is called a function of Gevrey class of order $\sigma$. 
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$$\max_{x \in K} \left| \partial^\alpha x f(x) \right| \leq C h^{\left| \alpha \right|} \left| \alpha \right|! \sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

A function in $\mathcal{E}\{\sigma\}(V)$ is called a function of Gevrey class of order $\sigma$. If $1 < s_1 < s_2 < \infty$ we have

$$\mathcal{A}(V) = \mathcal{E}\{1\}(V) \subset \mathcal{E}\{s_1\}(V) \subset \mathcal{E}\{s_2\}(V) \subset C^\infty(V).$$

By this, we can say that functions in $\mathcal{E}\{s_1\}(V)$ is more regular than those in $\mathcal{E}\{s_2\}(V)$. 
1.3. Functions in Gevrey class (2)

(2) We denote by $C^\infty([0,T], \mathcal{E}\{\sigma\}(V))$ the set of all infinitely differentiable functions in $t \in [0,T]$ with values in $\mathcal{E}\{\sigma\}(V)$ equipped with the usual local convex topology.
1.3. Functions in Gevrey class (2)

We denote by $C^\infty([0, T], \mathcal{E}\{\sigma\}(V))$ the set of all infinitely differentiable functions in $t \in [0, T]$ with values in $\mathcal{E}\{\sigma\}(V)$ equipped with the usual local convex topology.

It is easy to see that $C^\infty([0, T], \mathcal{E}\{\sigma\}(V))$ is the set of all functions $u(t, x) \in C^\infty([0, T] \times V)$ satisfying the following: for any compact subset $K$ of $V$ and any $k \in \mathbb{N}$ there are $C_k > 0$ and $h_k > 0$ such that

$$\max_{[0,T] \times K} |\partial_t^k \partial_x^\alpha u(t, x)| \leq C_k h_k^{|\alpha|} |\alpha|!\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$
1.4. Functions in Gevrey class (3)

(3) Let \( s \geq 1 \): we denote by \( \mathcal{E}^{\{s, \sigma\}}([0, T] \times V) \) the set of all functions \( u(t, x) \in C^\infty([0, T] \times V) \) satisfying the following: for any compact subset \( K \) of \( V \) there are \( C > 0 \) and \( h > 0 \) such that

\[
\max_{[0,T] \times K} \left| \partial_t^k \partial_x^\alpha u(t, x) \right| \leq Ch^{k+|\alpha|}k!^s|\alpha|!^\sigma, \\
\forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n.
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\[ \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n. \]

This means that \( u(t, x) \) is a function of the Gevrey class of order \( s \) in \( t \) and of the Gevrey class of order \( \sigma \) in \( x \). We often write \( \mathcal{E}^{\{\sigma\}}([0, T] \times V) = \mathcal{E}^{\{\sigma, \sigma\}}([0, T] \times V) \).
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$$

\[ \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n. \]

This means that $u(t, x)$ is a function of the Gevrey class of order $s$ in $t$ and of the Gevrey class of order $\sigma$ in $x$. We often write $E^{\{\sigma\}}([0, T] \times V) = E^{\{\sigma,\sigma\}}([0, T] \times V)$.

The following is clear:

$$
E^{\{s,\sigma\}}([0, T] \times V) \subset C^\infty([0, T], E^{\{\sigma\}}(V)).
$$
§2. Problem and examples
2.1. Equation and problem

From now, I will consider the following nonlinear partial differential equation

\[(E) \quad t^\gamma \partial_t^m u = F(t, x, \{ \partial_t^j \partial_x^\alpha u \}_{j+|\alpha| \leq L, j < m})\]

where \(\gamma \geq 0\) and \(L \geq m \geq 1\) are integers, and \(F(t, x, \{ z_{j,\alpha} \}_{j+|\alpha| \leq L, j < m})\) is a suitable function in a Gevrey class.
2.1. Equation and problem

From now, I will consider the following nonlinear partial differential equation

\[ (E) \quad t^\gamma \partial_t^m u = F\left(t, x, \{ \partial_t^j \partial_x^\alpha u \}_{j + |\alpha| \leq L, j < m} \right) \]

where \( \gamma \geq 0 \) and \( L \geq m \geq 1 \) are integers, and \( F(t, x, \{ z_{j,\alpha} \}_{j + |\alpha| \leq L, j < m}) \) is a suitable function in a Gevrey class. And, we will consider the following problem on Gevrey regularity in time:

**Problem 1.1.** Let \( u(t, x) \in C^\infty([0, T], E^{\{\sigma\}}(V)) \) be a solution of \((E)\); can we have the result

\[ u(t, x) \in E^{\{s,\sigma\}}([0, T] \times V) \]

for some \( s \geq 1 \)?
Let us give three examples.

Example 2.1. Let us consider the periodic KdV equation:

\[(2.1) \quad \partial_t u + \partial_x^3 u + 6u \partial_x u = 0, \quad u(0, x) = \varphi(x) \text{ on } \mathbb{T}\]

where \(\varphi(x)\) is an analytic function on the torus \(\mathbb{T}\).
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where \(\varphi(x)\) is an analytic function on the torus \(\mathbb{T}\).

The following results are known:

(1) This problem (2.1) is well-posed in \(H^s(\mathbb{T})\) for \(s \gg 1\).
(2) Gorsky-Himonas showed:

\[u(t, x) \in C^\infty((−\delta, \delta), \mathcal{E}^{1}(\mathbb{T})).\]

(3) Hannah-Himonas-Petronilho showed:

\[u(t, x) \in \mathcal{E}^{3,1}(((−\delta, \delta) \times \mathbb{T}).\]

The proof of (3) just gives an answer to our problem in the KdV case.
2.3. Example (2)

Example 2.2. Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider

(2.2) \quad (t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x).
2.3. Example (2)

Example 2.2. Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider

\[(2.2) \quad (t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x).\]

The following results are known:

(1) (2.2) is uniquely solvable in $C^\infty([0, T], \mathcal{E}\{\sigma\}(\mathbb{R}))$ for any $\sigma \geq 1$.

(2) In the case $f(t, x) \in \mathcal{E}\{\sigma\}([0, T] \times \mathbb{R})$, we have the time regularity:

\[(2.3) \quad \begin{cases} 
    u(t, x) \in \mathcal{E}\{\sigma, \sigma\}([0, T] \times \mathbb{R}), & \text{if } k \geq 2, \\
    u(t, x) \in \mathcal{E}\{2\sigma-1, \sigma\}([0, T] \times \mathbb{R}), & \text{if } k = 1.
\end{cases}\]

The result (2.3) gives an answer to our problem in the case (2.2).
2.4. Example (3)

Example 2.3. Recently, Kinoshita-Taglialatela (Arkiv för Matematik, 49 (2011), 109-127) discussed time regularity problem for the following hyperbolic Cauchy problem:

\[
\begin{aligned}
\partial_t^2 u - a(t)\partial_x^2 u &= b(t)\partial_t u + c(t)\partial_x u, \\
u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x).
\end{aligned}
\]
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\begin{align*}
\partial_t^2 u - a(t) \partial_x^2 u &= b(t) \partial_t u + c(t) \partial_x u, \\
u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x).
\end{align*}
\]

And they showed under a suitable assumption that the problem (2.4) is well-posed in $\mathcal{E}^{s, \sigma}([0, T] \times \mathbb{R})$ for $0 \leq \sigma - 1 \leq (s - 1)/s$. 
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\partial_t^2 u - a(t) \partial_x^2 u = b(t) \partial_t u + c(t) \partial_x u, \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x).
\end{cases}
\]

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And they showed under a suitable assumption that the problem (2.4) is well-posed in \(E^{\{s, \sigma\}}([0, T] \times \mathbb{R})\) for \(0 \leq \sigma - 1 \leq (s - 1)/s\).

This result can be improved to \(1 \leq \sigma \leq s\) by solving our problem on time regularity.
2.5. Motivation

By looking at these examples, I have come to think that the mechanism of Gevrey regularity in time is very close to that of Maillet type theorem in the book of Gérard-Tahara. And so, by applying the argument in the proof of Maillet type theorem we will be able to solve time regularity problem.
§3. Main theorems
- sufficient condition for time regularity -
3.1. Equation and assumptions

We will consider

\[(E) \quad t^\gamma \partial_t^m u = F(t, x, Du), \quad Du = \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha| \leq L, j < m}\]
3.1. Equation and assumptions

We will consider

\[(E) \quad t^\gamma \partial_t^m u = F(t, x, Du), \quad Du = \{ \partial_t^j \partial_x^\alpha u \}_{j+|\alpha| \leq L}^{j < m} \]

Let \( \Omega \) be an open subset of \( \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^d_z \), and let \( F(t, x, z) \) be a \( C^\infty \) function on \( \Omega \). Let \( s_1 \geq 1, \sigma \geq 1 \) and \( s_2 \geq 1 \), let \( T > 0 \), and let \( V \) be an open subset of \( \mathbb{R}^n \).
3.1. Equation and assumptions

We will consider

\[(E) \quad t^\gamma \partial_t^m u = F\left(t, x, Du\right), \quad Du = \left\{ \partial_t^j \partial_x^\alpha u \right\}_{j+|\alpha| \leq L, j < m} \]

Let \( \Omega \) be an open subset of \( \mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d \), and let \( F(t, x, z) \) be a \( C^\infty \) function on \( \Omega \). Let \( s_1 \geq 1, \sigma \geq 1 \) and \( s_2 \geq 1 \), let \( T > 0 \), and let \( V \) be an open subset of \( \mathbb{R}^n \).

The main assumptions are as follows.

\( a_1 \) \( \gamma \geq 0 \) and \( L \geq m \geq 1 \) are integers.

\( a_2 \) \( s_1 \geq 1 \) and \( \sigma \geq s_2 \geq 1 \) are real numbers.

\( a_3 \) \( F(t, x, z) \in \mathcal{E}\{s_1, \sigma, s_2\}(\Omega) \).

\( a_4 \) \( u(t, x) \in C^\infty([0, T], \mathcal{E}\{\sigma\}(V)) \) is a solution of (E).
3.2. Additional assumption

Definition 3.1. For \( f(t, x) \in C^\infty([0, T] \times V) \) we define the order of the zero of \( f(t, x) \) at \( t = 0 \) by

\[
ord_t(f, V) = \min \{ k \in \mathbb{N} ; (\partial_t^k f)(0, x) \not\equiv 0 \text{ on } V \}.
\]
3.2. Additional assumption

Definition 3.1. For $f(t, x) \in C^\infty([0, T] \times V)$ we define the order of the zero of $f(t, x)$ at $t = 0$ by

$$ord_t(f, V) = \min\{k \in \mathbb{N}; (\partial^k_t f)(0, x) \not\equiv 0 \text{ on } V\}.$$

Under the conditions $a_1 \sim a_4$ we set

$$k_{j, \alpha} = ord_t\left(\frac{\partial F}{\partial z_{j, \alpha}}(t, x, Du(t, x)), V\right).$$

And we suppose

$a_5$) \[ \begin{cases} 
    k_{j, \alpha} \geq \gamma - m + j, & \text{if } |\alpha| = 0, \\
    k_{j, \alpha} \geq \gamma - m + j + 1, & \text{if } |\alpha| > 0.
\end{cases} \]
The sufficient condition for the time regularity is as follows:

**Theorem 3.2 (Gevrey regularity in time).** Suppose the conditions $a_1 \sim a_5$: then, we have

$$u(t, x) \in \mathcal{E}^{s, \sigma}([0, T] \times V)$$

for any $s \geq \max\{s_0, s_1, s_2\}$ where

$$s_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha} - \gamma + m - j, m - j\}} \right) \right].$$
3.4. In the case of examples

Example 1. In the case of KdV equation:

\[ \partial_t u + \partial_x^3 u + 6u \partial_x u = 0, \quad u(0, x) = \varphi(x) \text{ on } \mathbb{T} \]

we have \( \gamma = 0, \quad m = 1, \quad L = 3 \quad \text{and} \quad s_0 = 3\sigma. \)
3.4. In the case of examples

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\[ \partial_t u + \partial_x^3 u + 6u \partial_x u = 0, \quad u(0, x) = \varphi(x) \quad \text{on } \mathbb{T} \]
we have \( \gamma = 0, \; m = 1, \; L = 3 \) and \( s_0 = 3\sigma \).

Example 2. In the case
\[ (t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x) \]
we have \( \gamma = 2, \; m = 2, \; L = 2 \) and
\[ s_0 = 1 + \frac{2\sigma - 2}{\min\{k, 2\}} = \begin{cases} \sigma, & \text{if } k \geq 2, \\ 2\sigma - 1, & \text{if } k = 1. \end{cases} \]
3.4. In the case of examples

**Example 1.** In the case of KdV equation:
\[ \partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0, x) = \varphi(x) \text{ on } \mathbb{T} \]
we have \( \gamma = 0, \ m = 1, \ L = 3 \) and \( s_0 = 3\sigma. \)

**Example 2.** In the case
\[ (t\partial_t + a)^2u - t^k\partial_x^2 u = f(t, x) \]
we have \( \gamma = 2, \ m = 2, \ L = 2 \) and
\[ s_0 = 1 + \frac{2\sigma - 2}{\min\{k, 2\}} = \begin{cases} \sigma, & \text{if } k \geq 2, \\ 2\sigma - 1, & \text{if } k = 1. \end{cases} \]

**Example 3.** In the case
\[ \partial_t^2 u - t^k\partial_x^2 u = b(t)\partial_t u + c(t)\partial_x u \]
we have \( \gamma = 0, \ m = 2, \ L = 2 \) and \( s_0 = \sigma. \)
3.5. Formal solution

If we have a solution $u(t, x) \in C^\infty([0, T], \mathcal{E}\{\sigma\}(V))$, by the formal Taylor expansion at $t = 0$ we have a formal solution

$$\hat{u}(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k \in \mathcal{E}\{\sigma\}(V)[[t]].$$
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\]

We will write \( \hat{u}(t, x) \in E^{\{s, \sigma\}}(\{t\}; V) \) if the following property holds: for any compact subset \( K \) of \( V \) there are \( C > 0 \) and \( h > 0 \) such that

\[
\max_{x \in K} |\partial_\alpha^\alpha u_k(x)| \leq C h^{k+|\alpha|} k!^{s-1} |\alpha|^{!\sigma}, \quad \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n.
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We will write $\hat{u}(t, x) \in \mathcal{E}\{s, \sigma\}({\{t}\}; V)$ if the following property holds: for any compact subset $K$ of $V$ there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^\alpha u_k(x)| \leq Ch^{k+|\alpha|} k!^{s-1} |\alpha|!\sigma, \quad \forall (k, \alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

By Theorem 3.2 we have the result:

$$\hat{u}(t, x) \in \mathcal{E}\{s, \sigma\}({\{t}\}; V) \text{ for any } s \geq \max\{s_0, s_1, s_2\}.$$
3.6. Main theorem (2)

But, in the case of formal solutions we have more:

**Theorem 3.3 (Maillet type theorem).** Suppose the conditions \( a_1 \sim a_5 \): then, the formal Taylor expansion \( \hat{u}(t, x) \) satisfies \( \hat{u}(t, x) \in \mathcal{E}^{s, \sigma} (\{t\}; V) \) for any \( s \geq \max\{s^*_0, s_1, s_2\} \) with

\[
s^*_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma |\alpha| - m}{k_{j,\alpha} - \gamma + m - j} \right) \right].
\]
3.6. Main theorem (2)

But, in the case of formal solutions we have more:

Theorem 3.3 (Maillet type theorem). Suppose the conditions $a_1 \sim a_5$: then, the formal Taylor expansion $\hat{u}(t, x)$ satisfies $\hat{u}(t, x) \in \mathcal{E}^{s, \sigma}\{t\}; V)$ for any $s \geq \max\{s^*_0, s_1, s_2\}$ with

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We note that $s^*_0 \leq s_0$ holds: in general, the time regularity in the case of formal solutions is better than the case of actual solutions.
3.7. Example

Example. Let $\sigma > 1$. We consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x) \in \mathcal{E}^{1,\sigma}([0, T] \times \mathbb{R}) :$$

let $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\sigma}([0, T] \times \mathbb{R}))$ be the unique solution and let $\hat{u}(t, x)$ be the formal Taylor expansion of $u(t, x)$ at $t = 0$. Then we have:
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(1) $u(t, x) \in \begin{cases} \mathcal{E}^{\sigma,\sigma}([0, T] \times \mathbb{R}), & \text{if } k \geq 2, \\ \mathcal{E}^{2\sigma-1,\sigma}([0, T] \times \mathbb{R}), & \text{if } k = 1. \end{cases}$
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(2) $\hat{u}(t, x) \in \mathcal{E}\{s,\sigma\}([t], \mathbb{R})$ for

$$s = 1 + \frac{2\sigma - 2}{k} \begin{cases} < \sigma, & \text{if } k \geq 3, \\ = \sigma, & \text{if } k = 2, \\ = 2\sigma - 1, & \text{if } k = 1. \end{cases}$$
§4. Necessity of the condition
4.1. Fuchsian case (1)

Let us consider the Fuchsian partial differential equation:

\[(4.1)\quad C(t\partial_t)u = F(t, x, \Theta u)\]

where \(\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\leq L, j<m}\) and \(C(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0\).
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where \(\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha| \leq L, j < m}\) and \(C(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0\). We suppose:

\[\text{b}_1) \quad c_i \geq 0 \quad (i = 0, 1, \ldots, m - 1);\]
\[\text{b}_2) \quad F(t, x, z) \gg 0 \quad \text{(at } (t, x, z) = (0, 0, 0)\text{)}, \quad \text{and} \]
\[\liminf_{|\beta| \to \infty} \left( \frac{F^{(1, \beta, 0)}(0, 0, 0)}{|\beta|!\sigma} \right)^{1/|\beta|} > 0;\]
\[\text{b}_3) \quad u(t, x) \text{ is a solution satisfying } u(0, x) = 0, \quad \text{and} \]
\[\frac{\partial F}{\partial z_j, \alpha}(t, x, \Theta u) \bigg|_{t=0} \equiv 0 \quad \text{on } V \text{ for any } (j, \alpha).\]
4.2. Fuchsian case (2)

In this case, our indices $s_0$ and $s^*_0$ are written as

$$s_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{\min\{q_{j,\alpha}, m-j\}} \right) \right],$$

$$s^*_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{q_{j,\alpha}} \right) \right]$$

with $q_{j,\alpha} = \text{ord}_t((\partial F/\partial z_{j,\alpha})(t, x, \Theta u(t, x)), V)$. 
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$$s^*_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{q_j,\alpha} \right) \right]$$

with $q_j,\alpha = \text{ord}_t((\partial F/\partial z_j,\alpha)(t, x, \Theta u(t, x)), V)$. Then we have the expression

$$\frac{\partial F}{\partial z_j,\alpha}(t, x, \Theta u(t, x)) = a_{j,\alpha}(x)t^{q_j,\alpha} + O(t^{q_j,\alpha+1})$$

for some $a_{j,\alpha}(x) \gg 0$ (at $x = 0$). We set

$$\Lambda(+) = \{(j, \alpha) \in \Lambda; a_{j,\alpha}(0) > 0, |\alpha| > 0\}.$$
4.3. Necessity of the condition in Fuchsian case

Then, we have the necessity of the condition:

Theorem 4.1 (Fuchsian case). If \( u(t, x) \in \mathcal{E}^{s, \sigma}([0, T] \times V) \) or \( \hat{u}(t, x) \in \mathcal{E}^{s, \sigma}(\{t\}; V) \) holds for some \( s \geq 1 \), we have

\[
s \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda(+)} \left( \frac{j + \sigma |\alpha| - m}{q_j, \alpha} \right) \right].
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\[
s \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda(\mathbb{R})} \left( \frac{j + \sigma |\alpha| - m}{q_j, \alpha} \right) \right].
\]

Recall that the sufficient condition is \( s \geq s_0 \) or \( s \geq s_0^* \) with

\[
s_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma |\alpha| - m}{\min \{q_j, \alpha, m-j\}} \right) \right],
\]

\[
s_0^* = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma |\alpha| - m}{q_j, \alpha} \right) \right].
\]
4.4. Non-singular case (1)

Let us consider the initial value problem

\[
\begin{align*}
\partial_t^m u &= F(t, x, Du) \text{ on } [0, T] \times V, \\
\partial_t^i u \big|_{t=0} &= \varphi_i(x) \text{ on } V, \quad i = 0, 1, \ldots, m - 1,
\end{align*}
\]

(4.2) where \( Du = \{ \partial_i^j \partial_x^\alpha u \}_{j+|\alpha|\leq L, j< m} \).
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In this case, our indices \( s_0 \) and \( s_0^* \) are written as

\[
s_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{m - j} \right) \right],
\]

\[
s_0^* = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{k_{j,\alpha} + m - j} \right) \right].
\]

where \( k_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t, x, Du(t, x)), V) \).
4.4. Non-singular case (2)

We set \( \Lambda = \{(j, \alpha) ; j + |\alpha| \leq L, j < m\} \),

\[
\varphi_m(x) = F(0, x, \{\varphi_j^{(\alpha)}(x)\}_{(j, \alpha) \in \Lambda}), \quad \text{and}
\]

\[
a(x) = \frac{\partial F}{\partial t}(0, x, \{\varphi_j^{(\alpha)}(x)\}_{(j, \alpha) \in \Lambda})
\]

\[
+ \sum_{(j, \alpha) \in \Lambda} \frac{\partial F}{\partial z_{j, \alpha}}(0, x, \{\varphi_j^{(\alpha)}(x)\}_{(j, \alpha) \in \Lambda}) \varphi_{j+1}^{(\alpha)}(x).
\]

We assume:

1. \( c_1 \) \( F(t, x, z) \gg 0 \) (at \( (t, x, z) = (0, 0, p) \));
2. \( c_2 \) \( \varphi_i(x) \gg 0 \) (at \( x = 0 \)), \( i = 0, 1, \ldots, m - 1 \);
3. \( c_3 \) \( \liminf_{|\beta| \to \infty} \left(a^{(\beta)}(0)/|\beta|!\sigma\right)^{1/|\beta|} > 0. \)
4.6. Necessity of the condition in non-singular case

Let \( u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V)) \) be a solution of (4.2). We set \( k_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t, x, Du(t, x), V) \): then

\[
\frac{\partial F}{\partial z_{j,\alpha}}(t, x, Du(t, x)) = a_{j,\alpha}(x)t^{k_{j,\alpha}} + O(t^{k_{j,\alpha}+1})
\]

for some \( a_{j,\alpha}(x) \gg 0 \) (at \( x = 0 \)). We set

\[
\Lambda(+) = \{(j, \alpha) \in \Lambda ; a_{j,\alpha}(0) > 0, |\alpha| > 0\}.
\]
Theorem 4.2 (Non-singular case). If
\( u(t, x) \in \mathcal{E}^{s, \sigma}([0, T] \times V) \) or \( \hat{u}(t, x) \in \mathcal{E}^{s, \sigma}({\{t}\}; V) \)
holds for some \( s \geq 1 \), we have

\[
s \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda(+)} \left( \frac{j + \sigma|\alpha| - m}{k_{j, \alpha} + m - j} \right) \right].
\]
Theorem 4.2 (Non-singular case). If $u(t, x) \in \mathcal{E}^{s, \sigma}([0, T] \times V)$ or $\hat{u}(t, x) \in \mathcal{E}^{s, \sigma}({t}; V)$ holds for some $s \geq 1$, we have

$$s \geq 1 + \max \left[ 0, \max_{(j, \alpha) \in \Lambda^+} \left( \frac{j + \sigma|\alpha| - m}{k_{j, \alpha} + m - j} \right) \right].$$

Recall that the sufficient condition is $s \geq s_0$ or $s \geq s^*_0$ with

$$s_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{m - j} \right) \right],$$

$$s^*_0 = 1 + \max \left[ 0, \max_{|\alpha| > 0} \left( \frac{j + \sigma|\alpha| - m}{k_{j, \alpha} + m - j} \right) \right].$$

where $k_{j, \alpha} = \text{ord}_t((\partial F/\partial z_{j, \alpha})(t, x, Du(t, x)), V)$. 