The Newton-Puisuex Polygon for $q$–difference equations. Applications.

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First things first

1.-Thanks to the Scientific Committee

2.-Thanks to the Organizing Committee (Prof. Walser, Greg, Galina, Sławomir)

3.-Enjoy
Prior Art

■ Maillet-Malgrange (1903-1989): Any power series solution of an analytic ODE is of Gevrey type.

■ J. Cano (1993): Any power series solution of a Gevrey ODE is Gevrey, explicit computation (Newton Polygon).


This work is the natural next step. Techniques like Cano’s.
Aims: solve & \((q\text{--Gevrey})\) bound

Given a general \(q\text{--difference equation like}\) (no independent term)

\[
P(x, y, \sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0
\]
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- Compute solutions
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  - Convergence, \(q\text{—Gevrey asymptotics}\)
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- Compute solutions
- Behaviour of solutions
  - Convergence, \(q–\text{Gevrey asymptotics}\)
  - Rational rank, “size” of the exponent's semigroup
Framework

Equation: A q−difference equation is $|q| > 1$

Solution: A solution of $P(x)$ is a generalized power series $f(x) = \sum_{\gamma \in \Gamma} f_{\gamma} x^\gamma$, where $\Gamma$ is well ordered, such that $P(x, f(x), f(qx), \ldots, f(q^n x)) = 0$ (a determination of the logarithm is fixed).
Framework

Equation: A $q$–difference equation is [we use $|q| > 1$]

$$P(x, y_0, \ldots, y_n) \in \mathbb{C}[[x^{\mathbb{R} > 0}][[y_0, \ldots, y_n]]$$

(too general, usually $\mathbb{C}[[x, y_0, \ldots, y_n]]$), $P(0) = 0.$
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$$f(x) = \sum_{\gamma \in \Gamma} f_{\gamma}x^{\gamma}, \text{ with } \Gamma \subset \mathbb{R}^>0$$

where $\Gamma$ is well ordered, such that

$$P(x, f(x), f(qx), \ldots, f(q^n x)) = 0$$

(a determination of the logarithm is fixed).
Solution of [implicit equation]

\[ y + xy^3 + y^5 + x^3 = 0 \]
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Start with an “obvious” term (least degree, \(y = -x^3\)).
Solution of \([\text{implicit equation}]\)

\[ y + xy^3 + y^5 + x^3 = 0 \]

Start with an “obvious” term (least degree, \( y = -x^3 \)).
Substitute: \( y = y - x^3 \) to get

\[ y - x^{10} - x^{15} + [... ] + 10x^9y^2 + [... ] + y^5 = 0 \]
Solution of [implicit equation]

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Algebraic Curves (Newton)

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Following “obvious” term is \( y = x^{10} \).
Go on. This one was easy.
Algebraic Curves (Puiseux)

What happens with

\[ y^2 - x^3 = 0? \]

(no formal power series solution).

Obviously \( y = x^3/2 \) is a "solution".

What with . . .

\[ y^6 + x^5 y^5 + xy^4 + x^3 y^2 + x^3 = 0? \]
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\[ y^6 + x^5y^5 + xy^4 + x^3y^2 + x^{10} = 0 \]
The Polygon

For $y^2 - x^3$, the number $\frac{3}{2}$ corresponds to the side *inclination*.
The Polygon

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The coefficient $y = 1 \cdot x^{3/2}$ comes from substituting $y = y + cx^{3/2}$, which gives at $y = 0$:

$$x^3 + c^2 x^3 = 0 \Rightarrow c = \pm 1.$$
The Polygon: positive convex hull of cloud

For $y^6 + x^5y^5 + xy^4 + x^3y^2 + x^{10} = 0$:

The inner point (5, 5) is irrelevant for starting a solution. Solutions $y = cx^\alpha + \ldots$ admit $\alpha = \frac{1}{2}, 1, \frac{7}{2}$, the inclinations.
The Polygon: coefficients

If a solution starts with \( y = cx^{\alpha} \), \( \alpha \) is the inclination, and \( c \)?
The Polygon: coefficients

If a solution starts with \( y = cx^\alpha \), \( \alpha \) is the inclination, and \( c \) is constant.

Substitute: \( y = y + cx^\alpha \). Example: \( y = y + cx \) \((\alpha = 1)\):

\[
\begin{align*}
xy^4 & = 4 \\
x^3y^2 & = 2 \\
y + x & = 5
\end{align*}
\]
The Polygon: coefficients

If a solution starts with $y = cx^\alpha$, $\alpha$ is the inclination, and $c$?
Substitute: $y = y + cx^\alpha$. Example: $y = y + cx$ ($\alpha = 1$):

Lowest $x-$term for that side must be 0:

$$P_\alpha(C) = x(y + cx)^4 + x^3(y + cx)^2 \bigg|_{y=0} = 0 \simeq c^4 + c^2 = 0.$$
The Polygon: coefficients

If a solution starts with \( y = cx^\alpha \), \( \alpha \) is the inclination, and \( c \)?

Substitute: \( y = y + cx^\alpha \). Example: \( y = y + cx \ (\alpha = 1) \):

```
0  2  4  6  8  10  
0  2  4  6           
   y + 2x = 6        
```

```
0  2  4  6  8  10  
0  2  4  6           
   y + x = 5         
```

```
0  2  4  6  8  10  
0  2  4  6           
   y + \frac{1}{2}x = 3
```

Lots of inner points

```
0  2  4  6  8  10  
0  2  4  6           
   y + x = 5         
```

Setting \( c = i \), a new Newton Polygon appears. The green side is new, and has inclination \( > \alpha = 1 \).

Recurse with new equation and Polygon.
Lexicon

- **N-P Polygon**: convex envelope of “cloud of points \((i,j)\).”
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- Given a slope \(\mu\), the **valuation** \(\nu_\mu(P)\) is the minimal value for \(j + \mu i\) and the **initial form**

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\ln_\mu(P) = \sum_{i+\mu x = \nu_\mu(P)} p_{ij} x^i y^j
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Q_\mu(T) \equiv ln_\mu(P)(1, T) = 0
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But: in the algebraic case, \(\mu\) is always rational and any slope and root give rise to a solution.
Newton-Puisuex for q-difference equations

The polygon for $q$–diff equation $P = \sum p_{ij_0...j_n} x^i y_0^{j_0} \ldots y_n^{j_n}$. 
Newton-Puisseux for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\ldots j_n} x^i y_0^{j_0} \ldots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
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- So, one should place $x^i y_0^{j_0} \ldots y_n^{j_n}$ at $(i, j)$ with $j = j_0 + \cdots + j_n$. 
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For the equation $y_2^2 - q^3 y_1 y_0 + xy_1^2 y_0 + x^3 = 0$
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The polygon for \( q \)-diff equation \( P = \sum p_{ij_0 \ldots j_n} x^i y_{j_0}^j \ldots y_{j_n}^j \).

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- So, one should place \( x^i y_{j_0}^j \ldots y_{j_n}^j \) at \((i, j)\) with \( j = j_0 + \cdots + j_n \).

For the equation \( y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \)
Pathologies

Sample equation: \( y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \)
Pathologies

Sample equation: $y^2_2 - q^3 y_1 y_0 + x y^2_1 y_0 + x^3 = 0$

Null $Q$: For $\mu > \frac{2}{3}$ $\rightarrow \ln_{\mu} = y^2_2 - q^3 y_1 y_0$, so:

$$Q_{\mu}(T) = (q^2 T)^2 - q^3 (q T) T \equiv 0.$$  

Any $c$ will make $Q_{\mu}(c) = 0$: free slopes.
Pathologies

Sample equation: \( y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \)

**Null Q:** For \( \mu > \frac{2}{3} \rightarrow l n_\mu = y_2^2 - q^3 y_1 y_0 \), so:

\[
Q_\mu(T) = (q^2 T)^2 - q^3 (qT) T \equiv 0.
\]

Any \( c \) will make \( Q_\mu(c) = 0 \): free slopes.

**Constant Q:** For \( \mu = \frac{2}{3} \rightarrow l n_\mu = y_2^2 - q^3 y_1 y_0 + x^3 \), so

\[
Q_\mu(T) = (q^2 T)^2 - q^3 (qT) T + 1 \equiv 1
\]

No \( c \) will make \( Q_\mu(c) = 0 \). Useless slope.
Exceptional behaviour

Recall: \[ y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \]
Exceptional behaviour

Recall: \( y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \)

- \( Q_\mu(T) = 0 \) for \( \mu > 2/3 \): \( y = cx^{1/\mu} + \ldots \) possible (may have \( \mu \not\in \mathbb{Q} \): not so in algebraic curves).
Exceptional behaviour

Recall: \( y^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0 \)

- \( Q_\mu(T) = 0 \) for \( \mu > 2/3 \): \( y = c x^{1/\mu} + \ldots \) possible (may have \( \mu \notin \mathbb{Q} \): not so in algebraic curves).
- \( Q_{2/3}(T) = 1 \): no solution starting with \( c x^{3/2} \).
Substitution step

After substituting

\[ y = y + cx^\alpha \]

new Polygon, continue with greater inclination (to the right).

Say \( y = y + x \) in \( y_2^2 - q^3 y_1 y_0 + xy_1^2 y_0 + x^3 = 0, \)
## Results: Solution construction

### Lemma

*Generically*, the Newton-Puiseux algorithm gives rise to a solution. And solutions correspond to the algorithm.
Results: Solution construction

**Lemma**

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**Theorem**

*The semigroup generated by $\Gamma$ (exponents) is finitely generated* (like for ODE). [Hence $\sum x^{(2-\frac{1}{n})}$ is forbidden].
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**Lemma**

*Generically*, the Newton-Puiseux algorithm gives rise to a solution. And solutions correspond to the algorithm.

**Theorem**

The semigroup generated by \( \Gamma \) (exponents) is *finitely generated* (like for ODE). [Hence \( \sum x^{(2-\frac{1}{n})} \) is forbidden].

**Theorem**

If \( P = A(x, y_0) + B(x, y_0)y_1 \) ("order and degree 1"), then \( \text{rational rank}(\Gamma) \leq 2 \). ("Only one irrational exponent").
Pivot point

In a finite number of steps, the topmost vertex of the interesting side is fixed (usually at height 1).
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\[ y = [3 \text{ terms}] + c_2 x^2 + c_3 x^3 + c_8 x^8 + \ldots \]
Results: $q$—Gevrey bounds (intro)

$t + 1$— Gevrey $\iff \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n$ convergent

Assume Pivot Point $(a, 1)$ contributions up to $[r]$, $\text{ord}(P) = n$.
Worst point: $(e_k, 1)$, high order “bad”, far from pivot “good”.

\begin{itemize}
  \item $p_{\text{ar}y_{r}}$
  \item $y_k \simeq (e_k, 1)$
  \item $y_n \simeq (e_n, 1)$
\end{itemize}
Results: $q$–Gevrey bounds (intro)

$$t + 1 - \text{Gevrey } \Leftrightarrow \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n \text{ convergent}$$

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Compute $s = \max \left\{ \frac{k-r}{e_k-a} \right\}$ for $k > r$

$p_0y_0 + \cdots + [p_{ar}y_r]$ $y_k \simeq (e_k, 1)$ $y_n \simeq (e_n, 1)$
Results: $q$–Gevrey bounds (intro)

\[ t + 1 \text{– Gevrey} \iff \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n \text{ convergent} \]

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Compute \(s = \max \left\{ \frac{k-r}{e_k-a} \right\} \) for \(k > r\)

For pivot at \((a, b)\), analogous computation.
Formal power series solution \( f(x) \in \mathbb{C}[[x]] \) (can be generalized to rational exponents easily). Recall \( s = \frac{\Delta_{\text{ord}}}{\text{distance}} \).

**Theorem (q–Gevrey Malgrange-Maillet)**

If \( P \) has q–Gevrey order \( t + 1 \), then \( f(x) \) has q–Gevrey order \( \leq s + t + 1 \).
Results: $q$–Gevrey bounds (I)

Formal power series solution $f(x) \in \mathbb{C}[[x]]$ (can be generalized to rational exponents easily). Recall $s = \frac{\Delta_{\text{ord}}}{\text{distance}}$.

**Theorem ($q$–Gevrey Malgrange-Maillet)**

*If $P$ has $q$–Gevrey order $t + 1$, then $f(x)$ has $q$–Gevrey order $\leq s + t + 1$.***

**Corollary**

*If the pivot point of $f(x)$ contains order $n$, then $f(x)$ has the same $q$–Gevrey order as $P$.***
Results: closed-eyed bound

**Corollary**

If $P(x, y_0, \ldots, y_n)$ has $q$–Gevrey order $t + 1$, then any solution has at most $q$–Gevrey order $n + t + 1$.

No information required on the solution, whereas the results above assume the pivot point is known.
Thanks, etc

Questions?

And many thanks