

Lemma 3. Let $a = p_D/p_R$ and let $0 < p < 1/(3+a)$. For all $n \geq 1$ we have

$$\frac{\partial f_n}{\partial p}(p, x) \leq 0, \text{ if } 0 \leq x \leq 1,$$

and

$$\frac{\partial f_n}{\partial p}(p, x) \geq 0, \text{ if } -a^{-1} < x \leq 0 \text{ or } 1 \leq x < a^{-1}.$$

Proof. Consider the polynomial

$$\psi(p, x) = 1 - \varphi(p, 1 - x).$$

Clearly we have

$$\psi^n(p, x) = 1 - \varphi^n(p, 1 - x).$$

Thus

$$f_n(p, x) = 1 - \frac{\psi^n(p, 1 - x)}{\psi^n(p, 1)}.$$

Let

$$F_n(p, x) = \frac{\partial \psi^n}{\partial p}(p, x) \cdot \frac{1}{\psi^n(p, x)} \text{ for } x \neq 0.$$

We prove the following property, for all $n \geq 1$, $x \leq 1+a^{-1}$, and $0 < p < 1/(3+a)$,

$$\frac{\partial F_n}{\partial x}(p, x) \leq 0 \tag{27}$$

Let $\mu = 1 - p + ap$. We have $\psi(p, x) = \mu x - apx^2$. Thus $\psi^{n+1} = \mu\psi^n - ap[\psi^n]^2$. Hence we have

$$\frac{\partial \psi^{n+1}}{\partial p}(p, x) = [(\mu - 2ap\psi^n(p, x)) \cdot F_n(p, x) - 1 + a - a\psi^n(p, x)] \cdot \psi^n(p, x).$$

Hence

$$F_{n+1}(p, x) = \frac{\mu - 2ap \cdot \psi^n(p, x)}{\mu - ap \cdot \psi^n(p, x)} \cdot F_n(p, x) + \frac{-1 + a - a \cdot \psi^n(p, x)}{\mu - ap \cdot \psi^n(p, x)}.$$

We have $\psi(p, x) \leq \frac{\mu}{2ap}$ for $p > 0$ and all x . It is also easy to verify that $x \leq 1 + a^{-1} \leq \frac{\mu}{2ap}$ for $p \leq \frac{1}{a+3}$. Thus we have for all $n \geq 1$

$$\frac{\partial \psi^n}{\partial x}(p, x) = (\mu - 2ap\psi^{n-1}(p, x)) \dots (\mu - 2ap\psi(p, x))(\mu - 2apx) \geq 0.$$

Since for all $p > 0$,

$$\frac{\partial}{\partial x} \left(\frac{\mu - 2apx}{\mu - apx} \right) < 0$$

and since for all $n \geq 1$ the function $\psi^n(p, x)$ is monotone increasing with respect to x for $x < 1 + a^{-1}$ and $0 < p \leq 1/(3+a)$, we have for all $n \geq 1$

$$\frac{\partial}{\partial x} \left(\frac{\mu - 2ap \cdot \psi^n(p, x)}{\mu - ap \cdot \psi^n(p, x)} \right) \leq 0.$$

In a similar way we show that

$$\frac{\partial}{\partial x} \left(\frac{-1 + a - a \cdot \psi^n(p, x)}{\mu - ap \cdot \psi^n(p, x)} \right) \leq 0.$$

It is easy to show that

$$F_{n+1} = A_n \cdot A_{n-1} \cdots A_1 \cdot B_0 + A_n \cdot A_{n-1} \cdots A_2 \cdot B_1 + \cdots + A_n \cdot B_{n-1} + B_n,$$

where

$$A_i = \frac{\mu - 2ap \cdot \psi^i(p, x)}{\mu - ap \cdot \psi^i(p, x)}$$

and

$$B_i = \frac{-1 + a - a \cdot \psi^i(p, x)}{\mu - ap \cdot \psi^i(p, x)}.$$

Since each A_i and B_i is non-negative and both have non-positive derivatives with respect to x , it follows that for all $n \geq 1$, for all $x \leq 1 + a^{-1}$, and $0 < p < 1/(3+a)$,

$$\frac{\partial F_{n+1}}{\partial x}(p, x) \leq 0.$$

This proves (27).

Now we complete the proof of the lemma. We have

$$\begin{aligned} \frac{\partial f_n}{\partial p}(p, x) &= - \frac{\frac{\partial \psi^n}{\partial p}(p, 1-x) \cdot \psi^n(p, 1) - \frac{\partial \psi^n}{\partial p}(p, 1) \cdot \psi^n(p, 1-x)}{[\psi^n(p, 1)]^2} \\ &= - \frac{(F_n(p, 1-x) - F_n(p, 1))\psi^n(p, 1)\psi^n(p, 1-x)}{[\psi^n(p, 1)]^2}. \end{aligned}$$

It is easy to show that for all $n \geq 1$, $\psi^n(p, x) \geq 0$ for $0 \leq x \leq 1 + a^{-1}$ and $\psi^n(p, x) \leq 0$ for $x \leq 0$. It follows from (27) that for all $0 < p < 1/(3+a)$ we have $F_n(p, 1-x) \geq F_n(p, 1)$, whenever $0 \leq x < a^{-1}$. Thus $\frac{\partial f_n}{\partial p}(p, x) \leq 0$, if $0 < x < 1$ (since then $\psi^n(p, 1-x) \geq 0$) and $\frac{\partial f_n}{\partial p}(p, x) \geq 0$, if $1 < x < a^{-1}$ (since then $\psi^n(p, 1-x) \leq 0$). We also have $F_n(p, 1-x) \leq F_n(p, 1)$, whenever $-a^{-1} < x \leq 0$. Thus $\frac{\partial f_n}{\partial p}(p, x) \geq 0$, since $\psi^n(p, 1-x) \geq 0$. This completes the proof of Lemma 3. \square