Indexed Categories of Fragments*

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Abstract. Algebraic signatures fail to behave like sets – they usually contain too much structure that makes it impossible to arbitrarily remove some of their components. In other words, one can not, for instance, represent as a single signature the result of the following operation: $\{\textbf{sort } s; \textbf{ops } f\colon s \to s\} \setminus \{\textbf{sort } s\}$. We address this issue in a general way by working with arbitrary indexed categories. To obtain the notion of a fragment we inductively use the structure of the category of indices. For a given indexed category we define the corresponding indexed category of fragments. We show the embedding of a flattened base category into a flattened category of fragments and prove that it has both left and right adjoint. We also prove some facts including the (co)completeness of flattened indexed categories of fragments. We present examples and applications of fragments of algebraic signatures, algebras, theories, and institutions. To our knowledge this is the first systematic approach to define a category of sub-object structures (fragments) that corresponds to the indexed category given.

1 Introduction

Working with sets is easy, it is always a set that remains when we remove something from a set. Unfortunately, it is not so when we consider objects that have more structure than sets. Taking algebraic signatures as an example, we do not know the result of removing the sort name s from a signature $\{\textbf{sort } s; \textbf{ops } f\colon s \to s, a\colon s\}$.

In category theory there is a standard notion of *subobjects* of an object o defined as an equivalence class with respect to the natural preorder of monomorphisms into o. They capture the fact that one object is a part of another one. We push this idea a bit further and define *fragments* that are not objects of the original category but of a category that is an extension of the original one and gives us a nice property that a fragment that remains after taking some parts of an object away is still an object of this category. Our definition uses the inductive structure of indexed categories; however, this doesn't seem to limit its applications because many categories related to formal specifications are isomorphic to some flattened (via Grothendieck construction) indexed categories. The list includes most categories of typical signatures, models, sentences, theories, institutions etc.

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In literature, from the very first works on algebraic specifications, one can find some ad-hoc definitions of entities that have similar purpose to fragments. E.g. there is a notion of enrichment in CLEAR [BG80], ACT ONE [EM85], and CASL [CoF04]. The later contains the set-theoretic definition of "signature fragments" (cf. Sect. III:2.1 in [CoF04]). However, our generic definition allows one to prove some fragment-related facts once-for-all and gives a broader view on concepts that so far have been examined only in separation or not at all. As shown in Sect. 3.1 and Sect. 3.2, our concept has been proved useful when applied to some particular indexed categories related to formal specification theory.

The paper is organized as follows. First, in Sect. 2 we briefly go trough preliminaries. Then Sect. 3 contains the main theoretical part of our work with the definition of fragments and a number of theorems and lemmas. Section 3.1 contains examples and Sect. 3.2 discusses applications. In Sect. 4 we introduce *pullback-pushout complements*, finally Sect. 5 presents conclusion and future work.

2 Preliminaries and Notation

We assume that the reader is familiar with the basic category theory. We try to follow the notation used e.g. in [TBG91].

Indexed categories are categories over a collection of indices (see [TBG91] for definition, basic facts and many examples). Formally an indexed category $\mathbf{C}$ over an index category $\mathbf{Ind}$ is a functor $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$[1], so $\mathbf{C}(i)$ is the i^{th} component category, and $\mathbf{C}(\sigma)$ is the translation functor $\mathbf{C}(\sigma)\colon \mathbf{C}(j) \to \mathbf{C}(i)$, for $\sigma\colon i \to j$. A *flattened indexed category* $\mathbf{Flat}(\mathbf{C})$ has pairs $\langle i, a\rangle$ as objects (where $i \in \mathbf{Ind}$ and $a \in \mathbf{C}(i)$) and pairs $\langle \sigma, f\rangle\colon \langle i, a\rangle \to \langle j, b\rangle$ as morphisms (where $\sigma\colon i \to j$, $f\colon a \to \mathbf{C}(\sigma)(b) \in \mathbf{C}(i)$). Moreover we define a *projection functor* $\mathbf{Proj_C}\colon \mathbf{Flat}(\mathbf{C}) \to \mathbf{Ind}$ as $\mathbf{Proj_C}(\langle i, a\rangle) = i$, $\mathbf{Proj_C}(\langle \sigma, f\rangle) = \sigma$.

A *comma category* $(\mathbf{F}, \mathbf{G})$ for two functors $\mathbf{F}\colon \mathbf{B} \to \mathbf{D}$ and $\mathbf{G}\colon \mathbf{C} \to D$ has triples $\langle b, f, c\rangle$ as objects (for $b \in \mathbf{B}$, $c \in \mathbf{C}$ and $f\colon \mathbf{F}(b) \to \mathbf{G}(c) \in \mathbf{D}$) and pairs $\langle g, h\rangle\colon \langle b_1, f_1, c_1\rangle \to \langle b_2, f_2, c_2\rangle$ as morphisms (for $g\colon b_1 \to b_2 \in \mathbf{B}$ and $h\colon c_1 \to c_2 \in \mathbf{C}$) such that $f_1; \mathbf{G}(h) = \mathbf{F}(g); f_2$. To denote an object $\langle a, f, b\rangle \in (\mathbf{F}, \mathbf{G})$ we usually explicitly write $f\colon \mathbf{F}(a) \to \mathbf{G}(b) \in (\mathbf{F}, \mathbf{G})$.

Institutions (cf. [BG92]) formalize a notion of a logical system; many classical logics have been represented as institutions. Formally, an institution is a tuple $\langle \mathbf{Sig}, \mathit{Mod}, \mathit{Sen}, \models\rangle$, where $\mathbf{Sig}$ is a category of *signatures*; $\mathit{Mod} : \mathbf{Sig}^{op} \to \mathbf{Cat}$ is a *model functor*; $\mathit{Sen} : \mathbf{Sig} \to \mathbf{Set}$ is a *sentence functor*; $\models$ is a family $\{\models_\Sigma \subseteq |\mathit{Mod}(\Sigma)| \times \mathit{Sen}(\Sigma)\}_{\Sigma \in |\mathbf{Sig}|}$ of *satisfaction relations* such that the following satisfaction condition holds: $\mathit{Mod}(\sigma)(M') \models_\Sigma \varphi$ iff $M' \models_{\Sigma'} \mathit{Sen}(\sigma)(\varphi)$ for all $\sigma\colon \Sigma \to \Sigma'$, $M' \in \mathit{Mod}(\Sigma'), \varphi \in \mathit{Sen}(\Sigma)$.

3 Indexed Categories of Fragments

Indexed categories, as described in the previous section, group objects into categories using indices from **Ind**. If it happens that a category **Ind** is a flattened

[1] **Cat** is a category of all small categories and functors

indexed category, then there are two levels of indexation, if the index category of this indexed category is a flattened indexed category, then there are three levels etc. The idea behind fragments is to take advantage of this fact. We define what is inside the fragment, layer by layer, in an independent manner, using morphisms to connect subsequent layers. The number of layers of indexed category is called *a rank*.

Definition 1 (Indexed Categories with Ranks) Indexed categories with ranks *are defined by induction:*

1. *for every category* $\mathbf{C}$ *define the corresponding* indexed category of rank 0*,* $\mathbf{C_0}\colon \mathbf{1}^{op} \to \mathbf{Cat}$*, as* $\mathbf{C_0}(*) = \mathbf{C}$*,* $\mathbf{C_0}(id_*) = \mathbf{Id_C}$*, where* $\mathbf{1}$ *is a final object in* $\mathbf{Cat}$*, i.e. the category that contains one object* $*$ *and the identity* id_* *on it as the only morphism;*
2. *every indexed category* $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$*, where* $\mathbf{Ind}$ *is a flattened indexed category of rank* n*, for a natural number* n*, is* an indexed category of rank $n+1$.

In what follows we add a subscript to the category name to indicate its rank. We use a letter, e.g. n, to indicate ranks of some natural number. Let us mention that nothing is either gained or lost by representing a category as an indexed category of rank zero.

Lemma 2 *A category* $\mathbf{C}$ *and a flattened corresponding category of rank zero* $\mathbf{Flat}(\mathbf{C_0})$ *are isomorphic.*

Proof: Obvious. □

The higher rank of a category, the higher granularity of its fragments because there are more layers of independent definitions of fragment contents. Let us now define the main concept of this paper.

Definition 3 (Fragments in Indexed Categories with Ranks) *Given an indexed category* $\mathbf{C_n}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *of rank* n*, we define* an indexed category of its fragments *as an indexed category*

$$\mathbf{C_n}^{frag}\colon \mathbf{IndFrag}^{op} \to \mathbf{Cat}$$

together with three functors

- $Frag\colon \mathbf{Flat}(\mathbf{C_n}) \to \mathbf{Flat}(\mathbf{C_n}^{frag})$ *that is a full and faithful embedding (of complete objects to a category of fragments),*
- $Compl\colon \mathbf{Flat}(\mathbf{C_n}^{frag}) \to \mathbf{Flat}(\mathbf{C_n})$ *that is a completion functor (that gives the smallest complete object including the fragment),*
- $Sub\colon \mathbf{Flat}(\mathbf{C_n}^{frag}) \to \mathbf{Flat}(\mathbf{C_n})$ *that is a functor (that gives the biggest complete object included inside a fragment).*

The definition goes by induction on rank n. For $n = 0$ define $\mathbf{IndFrag} = \mathbf{1}$ *and*

$$\mathbf{C_n}^{frag} = \mathbf{C_n}$$

and define $Frag = Compl = Sub = \mathbf{Id}_{\mathbf{Flat}(\mathbf{C_n})} \colon \mathbf{Flat}(\mathbf{C_n}) \to \mathbf{Flat}(\mathbf{C_n})$.
For $n > 0$ the category $\mathbf{Ind}$ *is a flattened indexed category of rank $n-1$ (cf. Def. 1), i.e.* $\mathbf{Ind} = \mathbf{Flat}(\mathbf{C'_{n-1}})$ *for some* $\mathbf{C'_{n-1}}$ *of rank $n-1$. Let* $\mathbf{C'_{n-1}}^{frag}$ *be the category of its fragments and* $Compl' \colon \mathbf{Flat}(\mathbf{C'_{n-1}}) \to \mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})$, $Frag' \colon \mathbf{Flat}(\mathbf{C'_{n-1}}^{frag}) \to \mathbf{Flat}(\mathbf{C'_{n-1}})$, $Sub' \colon \mathbf{Flat}(\mathbf{C'_{n-1}}) \to \mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})$ *be three functors associated with* $\mathbf{C'_{n-1}}^{frag}$. *In this case we define* $\mathbf{IndFrag} = (\mathbf{Id}_{\mathbf{Flat}(\mathbf{C'}^{frag})}, Frag')$ *as the comma category for functors* $\mathbf{Id}_{\mathbf{Flat}(\mathbf{C'}^{frag})}$ *and* $Frag'$.
For any object $\gamma_i \colon i' \to Frag'(i) \in \mathbf{IndFrag}$ *define*

$$\mathbf{C_n}^{frag}(\gamma_i) = \mathbf{C}(i)$$

For any morphism $\langle \sigma', Frag'(\sigma) \rangle \colon \gamma_i \to \gamma_j \in \mathbf{IndFrag}$ *define*

$$\mathbf{C_n}^{frag}(\langle \sigma', Frag'(\sigma) \rangle) = \mathbf{C_n}(\sigma)$$

Moreover, define functors

- $Frag \colon \mathbf{Flat}(\mathbf{C_n}) \to \mathbf{Flat}(\mathbf{C_n}^{frag})$
  - $Frag(\langle i, a \rangle) = \langle id_{Frag'(i)}, a \rangle$, *for object* $\langle i, a \rangle \in \mathbf{Flat}(\mathbf{C_n})$,
  - $Frag(\langle \sigma, f \rangle) = \langle \langle Frag'(\sigma), Frag'(\sigma) \rangle, f \rangle$, *for morphism* $\langle \sigma, f \rangle \in \mathbf{Flat}(\mathbf{C_n})$,
- $Compl \colon \mathbf{Flat}(\mathbf{C_n}^{frag}) \to \mathbf{Flat}(\mathbf{C_n})$
  - $Compl(\langle \gamma_i, a \rangle) = \langle i, a \rangle$, *for object* $\langle \gamma_i \colon i' \to Frag'(i), a \rangle \in \mathbf{Flat}(\mathbf{C_n}^{frag})$,
  - $Compl(\langle \langle \sigma', Frag'(\sigma) \rangle, f \rangle) = \langle \sigma, f \rangle$,
    for morphism $\langle \langle \sigma', Frag'(\sigma) \rangle, f \rangle \in \mathbf{Flat}(\mathbf{C_n}^{frag})$,
- $Sub \colon \mathbf{Flat}(\mathbf{C_n}^{frag}) \to \mathbf{Flat}(\mathbf{C_n})$
  - $Sub(\langle \gamma_i, a \rangle) = \langle Sub'(i'), \mathbf{C_n}(Sub'(\gamma_i))(a) \rangle$,
    for object $\langle \gamma_i \colon i' \to Frag'(i), a \rangle \in \mathbf{Flat}(\mathbf{C_n}^{frag})$,
  - $Sub(\langle \langle \sigma', Frag'(\sigma) \rangle, f \rangle) = \langle Sub'(\sigma'), \mathbf{C_n}(Sub'(\gamma_i))(f) \rangle$,
    for morphism $\langle \langle \sigma', Frag'(\sigma) \rangle, f \rangle \colon \gamma_i \to \gamma_j \in \mathbf{Flat}(\mathbf{C_n}^{frag})$.

Objects of $\mathbf{Flat}(\mathbf{C_n}^{frag})$ *are called* fragments*.*

By construction it is obvious that in both cases *Compl*, *Frag*, *Sub* are functors. It is also visible that in both cases functor *Frag* is full and faithful and that $Frag; Sub = Frag; Compl = \mathbf{Id}_{\mathbf{Flat}(C_n)}$. Moreover we have the following fact that shows how close these categories are one to another.

Theorem 4 *The three functors defined by Def. 3 are adjoint*

$$Compl \dashv Frag \dashv Sub$$

The proof of the theorem is in Appendix A.

Lemma 5 *Given an indexed category* $\mathbf{C_n}$ *of rank n and its category of fragments as in Theorem 4 above, the counit of* $Compl \dashv Frag$ *is identity, i.e., for every object* $\langle \gamma_i, a \rangle \in \mathbf{Flat}(\mathbf{C_n}^{frag})$, $\epsilon_{\langle \gamma_i, a \rangle} = id_{\langle \gamma_i, a \rangle}$.

Proof: As *Compl* $\dashv$ *Frag* we have $\epsilon_{\langle\gamma_i,a\rangle} = (id_{Compl(\langle\gamma_i,a\rangle)})^{\#} = \langle id_{\gamma_i}, id_a\rangle = id_{\langle\gamma_i,a\rangle}$. □

Lemma 6 *Given an indexed category* $\mathbf{C_n}$ *of rank* n *and its category of fragments as in Theorem 4 above, the unit of Frag* $\dashv$ *Sub is identity, i.e., for every object* $\langle i,a\rangle \in \mathbf{Flat}(\mathbf{C_n}), \eta_{\langle i,a\rangle} = id_{\langle i,a\rangle}$.

Proof: Proof by induction on rank n. For $n = 0$ it is obvious. Let us assume that $n > 0$, so $\mathbf{Ind} = \mathbf{Flat}(\mathbf{C'_{n-1}})$ and for any $i \in \mathbf{Ind}$ we have $Sub'(Frag'(i)) = i$. Given $\langle i,a\rangle \in \mathbf{Flat}(\mathbf{C_n})$ we check by Def. 3 that $Sub(Frag(\langle i,a\rangle)) = \langle Sub'(Frag'(i)), a\rangle = \langle i,a\rangle$. □

The above lemmas give hint to the following observation that a flattened category of fragments is a kind of extension of a flattened base category. We formalize it by the theorem.

Theorem 7 *A flattened indexed category of rank* n, $\mathbf{Flat}(\mathbf{C_n})$, *for some* $\mathbf{C_n}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$, *is fully embeddable into* $\mathbf{Flat}(\mathbf{C_n}^{frag})$ *as a reflective and coreflective subcategory (cf. Sect. 4 in [AHS90]).*

Proof: The embedding functor is *Frag*. The proof follows from Def. 3 and Theorem 4. Functor *Compl* induces a reflector for *Frag*($\mathbf{Flat}(\mathbf{C_n})$). Functor *Sub* induces a coreflector for *Frag*($\mathbf{Flat}(\mathbf{C_n})$). □

We prove completeness and cocompleteness of flattened indexed categories of fragments using the properties of the base indexed categories with ranks. Before we do so, we introduce the notion of *(co)completeness conditions* that directly come from Theorems 1–2 in [TBG91].

Definition 8 *An indexed category* $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *meets the* completeness condition *iff the category* $\mathbf{Ind}$ *is complete,* $\mathbf{C_n}(i)$ *is complete for all objects* $i \in \mathbf{Ind}$*, and* $\mathbf{C_n}(\sigma)\colon \mathbf{C_n}(j) \to \mathbf{C_n}(i)$ *is continuous for all morphisms* $\sigma\colon i \to j \in \mathbf{Ind}$.

Definition 9 *An indexed category* $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *meets the* cocompleteness condition *iff the category* $\mathbf{Ind}$ *is cocomplete,* $\mathbf{C_n}(i)$ *is cocomplete for all objects* $i \in \mathbf{Ind}$*, and* $\mathbf{C_n}(\sigma)\colon \mathbf{C_n}(j) \to \mathbf{C_n}(i)$ *has a left adjoint for all morphisms* $\sigma\colon i \to j \in \mathbf{Ind}$.

Definition 10 *An indexed category of rank* n, $\mathbf{C_n}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$*, meets the* full (co)completeness condition *iff* $\mathbf{C_n}$ *meets the (co)completeness condition and if* $n > 0$*, thus by Def. 1* $\mathbf{Ind} = \mathbf{Flat}(\mathbf{C'_{n-1}})$ *for some indexed category* $\mathbf{C'_{n-1}}$ *of rank* $n-1$*, the category* $\mathbf{C'_{n-1}}$ *meets the full (co)completeness condition.*

Theorem 11 ((Co)completness of $\mathbf{Flat}(\mathbf{C_n}^{frag})$) *If* $\mathbf{C_n}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *is an indexed category of rank* n *that meets the full (co)completeness condition, then* $\mathbf{Flat}(\mathbf{C_n}^{frag})$ *is (co)complete.*

Proof: First we prove the completeness. Let us assume that $\mathbf{C_n}$ meets the full completeness condition. It implies that $\mathbf{C_n}$ meets the completeness condition that is sufficient for $\mathbf{Flat}(\mathbf{C_n})$ to be complete (cf. Theorem 1 in [TBG91]). The proof of completeness of $\mathbf{Flat}(\mathbf{C_n}^{frag})$ goes by induction on ranks. For $n = 0$ we have $\mathbf{Flat}(\mathbf{C_n}^{frag}) = \mathbf{Flat}(\mathbf{C_n})$ that is complete. Now, for the induction step $\mathbf{Ind} = \mathbf{Flat}(\mathbf{C'_{n-1}})$, for some $\mathbf{C'_{n-1}} \colon \mathbf{Ind}'^{op} \to \mathbf{Cat}$, and $\mathbf{C'_{n-1}}$ meets the full completeness condition, so as an induction step assumption we have that $\mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})$ is complete. Full completeness condition met by $\mathbf{C_n}$ implies a completeness condition met by $\mathbf{C'_{n-1}}$, thus $\mathbf{Flat}(\mathbf{C'_{n-1}})$ is complete. We show that $\mathbf{C_n}^{frag} \colon (\mathbf{Id}_{\mathbf{Flat}(\mathbf{C}'^{frag})}, \mathit{Frag}')^{op} \to \mathbf{Cat}$ also meets the completeness condition (cf. Def. 8). For a comma category to be complete it is enough that both source categories are complete and both functors are continuous (cf. I. 2.16.1 in [Bor94]). In fact, both $\mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})$ and $\mathbf{Flat}(\mathbf{C'_{n-1}})$ are complete (see above), functor $\mathbf{Id}_{\mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})}$ is trivially continuous, and so is functor Frag' because it is a right adjoint (cf. Theorem 4). For an object $\gamma_i \colon i' \to \mathit{Frag}'(i) \in (\mathbf{Id}_{\mathbf{Flat}(\mathbf{C}'^{frag})}, \mathit{Frag}')$, category $\mathbf{C_n}^{frag}(\gamma_i) = \mathbf{C_n}(i)$ is complete by assumption, similarly, for a morphism $\langle \sigma', \mathit{Frag}'(\sigma)\rangle \in (\mathbf{Id}_{\mathbf{Flat}(\mathbf{C}'^{frag})}, \mathit{Frag}')$ functor $\mathbf{C_n}^{frag}(\langle \sigma', \mathit{Frag}'(\sigma)\rangle) = \mathbf{C_n}(\sigma)$ is continuous also by assumption (equalities come from Def. 3). By Theorem 1 in [TBG91] $\mathbf{Flat}(\mathbf{C_n}^{frag})$ is complete.

We omit the proof of cocompleteness. It is very similar to the proof of completeness above. The only difference is that we use the cocompleteness condition, Theorem 2 from [TBG91], and a well known fact that for a comma category to be cocomplete it is enough that both source categories are cocomplete and the first functor is cocontinuous. □

There are two types of fragments that deserve to be distinguished. These are *empty fragments* and *complete fragments*. The former represent such entities that do not contain anything non-trivial. The later are such fragments that contain everything needed to be an object of the original category. The definitions follow.

Definition 12 (Empty Fragments) *Given some indexed category $\mathbf{C_n} \colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ of rank n we define* empty fragments *in $\mathbf{Flat}(\mathbf{C_n}^{frag})$:*

1. *for $n = 0$ a fragment $\langle *, x\rangle$ is empty iff x is initial in $\mathbf{C}_0(*)$;*
2. *for $n > 0$, where a category $\mathbf{Ind}$ is a flattened indexed category $\mathbf{Flat}(\mathbf{C'_{n-1}})$ for some indexed category $\mathbf{C'_{n-1}} \colon \mathbf{Ind}'^{op} \to \mathbf{Cat}$ of rank $n-1$, the fragment $\langle \gamma_i, \emptyset_{\mathbf{C}(i)}\rangle \in \mathbf{Flat}(C^{frag})$ for some $i \in \mathbf{Ind}$ is empty iff $\emptyset_{\mathbf{C}(i)}$ is initial in $\mathbf{C}(i)$ and $\gamma_i \colon i' \to \mathit{Frag}'(i)$ is such that the fragment i' is an empty fragment in $\mathbf{Flat}(\mathbf{C'_{n-1}}^{frag})$.*

The first empty fragment that comes to our mind is the initial object (provided it exists), thus the following lemma.

Lemma 13 (Initial Object is Empty) *The initial object $\emptyset \in \mathbf{Flat}(\mathbf{C_n}^{frag})$, for some indexed category of rank n, is an empty fragment.*

Proof: Obvious. Observe that the initial object in a flattened indexed category is a pair of an initial index and initial object of its component category. □

Definition 14 (Complete Fragments) *A fragment* $p \in \mathbf{Flat}(\mathbf{C}^{frag})$ *for some indexed category* $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *of some rank* n *is* complete *iff the unit of* $Compl \dashv Frag$, $\eta_p\colon p \to Frag(Compl(p))$, *is an isomorphism.*

Lemma 15 *Given a complete fragment* $p \in \mathbf{Flat}(C^{frag})$ *for some indexed category* $\mathbf{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ *of some rank* n, *there exists an object* $o \in \mathbf{Flat}(C^{frag})$ *such that* $Frag(o)$ *is isomorphic to* p.

Proof: Let $o = Compl(p)$. Object p is complete thus, by Def. 14, $Frag(Compl(p))$ is isomorphic to p. □

3.1 Examples

Example 1 (Set Fragments). We define *a category of set fragments* simply as the category of sets itself. We present a category **Set** as $\mathbf{SET_0}\colon \mathbf{1}^{op} \to \mathbf{Cat}$, an indexed category of rank 0. By Lemma 2 we know that $\mathbf{Set_0} = \mathbf{Flat}(\mathbf{SET_0})$ is isomorphic to **Set**. A category of set fragments is defined by Def. 3 as $\mathbf{SET_0}^{frag} = \mathbf{SET_0}$ and $Frag_{\mathbf{Set_0}} = Compl_{\mathbf{Set_0}} = Sub_{\mathbf{Set_0}} = \mathbf{Id}_{\mathbf{Set_0}}$ that is isomorphic to $\mathbf{Id}_{\mathbf{Set}}$. We name $\mathbf{Set_0}^{frag} = \mathbf{Flat}(\mathbf{SET_0}^{frag})$.

Example 2 (Many-sorted Set Fragments). An indexed category of many-sorted sets $\mathbf{SSET}\colon \mathbf{Set}^{op} \to \mathbf{Cat}$ is defined (cf. [TBG91]) as a category of functors $\mathbf{SSET}(S) = [S \to \mathbf{Set}]$ for a set $S \in \mathbf{Set}$ (interpreted as a discrete category) and a functor $\mathbf{SSET}(f)\colon \mathbf{SSET}(S_2) \to \mathbf{SSET}(S_1)$ for a function $f\colon S_1 \to S_2 \in \mathbf{Set}$. Note that an object $X\colon S \to \mathbf{Set} \in \mathbf{SSET}(S)$ may be presented as $\langle X\rangle_{s\in S}$ and a morphism $g\colon X \to Y \in \mathbf{SSET}(S)$ may be presented as $\langle g_s\colon X_s \to Y_s\rangle_{s\in S}$. A functor $\mathbf{SSET}(f)$ is defined on objects as $\mathbf{SSET}(f)(X) = f;X$ for $X\colon S_2 \to \mathbf{Set}$ and on morphisms as $\mathbf{SSet}(f)(g) = \langle g_{f(s_1)}\colon X_{f(s_1)} \to Y_{f(s_1)}\rangle_{s_1\in S_1}$ for $g = \langle g_{s_2}\colon X_{s_2} \to Y_{s_2}\rangle_{s_2\in S_2}\colon X \to Y \in \mathbf{SSET}(S_2)$. We name $\mathbf{SSet} = \mathbf{Flat}(\mathbf{SSET})$. A category **SSET** may be presented as an indexed category of rank 1 as follows:

$$\mathbf{SSET_1}\colon \mathbf{Set_0}^{op} \to \mathbf{Cat}$$

where $\mathbf{Set_0} = \mathbf{Flat}(\mathbf{SET_0})$ is the category defined in the example above. We define $\mathbf{SSet_1} = \mathbf{Flat}(\mathbf{SSET_1})$. Using Def. 3 we obtain *the indexed category of many-sorted set fragments*

$$\mathbf{SSET_1}^{frag}\colon \langle \mathbf{Id}_{\mathbf{Set_0}^{frag}}, Frag_{\mathbf{Set_0}}\rangle^{op} \to \mathbf{Cat}$$

Since $\mathbf{SET_0}^{frag} = \mathbf{SET_0}$, a functor $Frag_{\mathbf{Set_0}}$ is equal to $\mathbf{Id}_{\mathbf{Set_0}}$, and $\mathbf{Set_0}$ is isomorphic to **Set**, $\mathbf{SSET_1}^{frag}\colon \langle \mathbf{Id}_{\mathbf{Set}}, \mathbf{Id}_{\mathbf{Set}}\rangle^{op} \to \mathbf{Cat}$ is defined as $\mathbf{SSET_1}^{frag}(i) = \mathbf{SSET}(b)$ for $i\colon a \to b \in \mathbf{Set}$ and $\mathbf{SSET_1}^{frag}(\langle f, g\rangle) = \mathbf{SSET}(g)$ for $\langle f, g\rangle\colon i \to j$, for some $i\colon a \to b$ and $j\colon c \to d$, $f\colon a \to c$ and $g\colon b \to d$ such that $i;g = f;j$. Many-sorted set fragments are objects of $\mathbf{SSet_1}^{frag} = \mathbf{Flat}(\mathbf{SSET_1}^{frag})$, so a fragment $\langle i, X\rangle$ is a pair of a function $i\colon S' \to S$ and an S-sorted set $X = \langle X_s\rangle_{s\in S}$. One should interpret S' as a set of sorts and X as an S-sorted set of elements that *are in* the fragment. Notice that it may happen that a fragment contains elements of a sort that *is not in* the fragment.

Below we give some examples of many-sorted set fragments:

1. $e_1 = \langle i\colon \{s_1\} \to \{s_1\}, X_{s_1} = \{a,b\}\rangle$, where $i(s_1) = s_1$, is the fragment that contains one sort s_1 and elements $a, b\colon s_1$ (in fact it is equivalent to a many-sorted set, we call such fragments *complete fragments*, cf. Def. 14);
2. $e_2 = \langle i\colon \{s_1\} \to \{s_1, s_2\}, X_{s_1} = \{a,b\}, X_{s_2} = \{c\}\rangle$, where $i(s_1) = s_1$, is the fragment that contains one sort s_1 and elements of both sorts s_1 and s_2, however, the sort s_2 itself is not in the fragment;
3. $e_3 = \langle \emptyset_{s_1}\colon \emptyset \to \{s_1\}, X_{s_1} = \emptyset\rangle$ is the fragment that has no sorts and no elements (fragments that do not contain sorts and do not contain elements are called *empty fragments*, cf. Def. 12);
4. $e_4 = \langle \emptyset_{s_1}\colon \emptyset \to \{s_1\}, X_{s_1} = \{a\}\rangle$ is the fragment that has no sorts but it contains an element a of sort s_1, element a causes that this is not an empty fragment;
5. $e_5 = \langle i\colon \{s_1, s_2\} \to \{s\}, X_s = \{a,b\}\rangle$, where $i(s_1) = s, i(s_2) = s$, is the fragment that contains two sorts s_1 and s_2 that share the set of elements $\{a, b\}$, the function i is not injective.

Definition 3 applied to $\mathbf{SSET_1}$ describes not only a category of fragments, but also the three adjoint functors $\mathit{Compl}_{\mathbf{SSet_1}} \dashv \mathit{Frag}_{\mathbf{SSet_1}} \dashv \mathit{Sub}_{\mathbf{SSet_1}}$ that allow one to move between the worlds of many-sorted sets and many-sorted set fragments. There is an embedding of sets to fragments $\mathit{Frag}_{\mathbf{SSet_1}}\colon \mathbf{SSet_1} \to \mathbf{SSet_1}^{\mathit{frag}}$ defined as

$$\mathit{Frag}_{\mathbf{SSet_1}}(\langle S, X\rangle) = \langle id_S, X\rangle, \quad \mathit{Frag}_{\mathbf{SSet_1}}(\langle \sigma, f\rangle) = \langle\langle \sigma, \sigma\rangle, f\rangle$$

Then there is a completion functor $\mathit{Compl}_{\mathbf{SSet_1}}\colon \mathbf{SSet_1}^{\mathit{frag}} \to \mathbf{SSet_1}$ defined as

$$\mathit{Compl}_{\mathbf{SSet_1}}(\langle i\colon S' \to S, X\rangle) = \langle S, X\rangle, \quad \mathit{ComplDecl}_{\mathbf{SSet_1}}(\langle\langle \sigma', \sigma\rangle, f\rangle) = \langle \sigma, f\rangle$$

Finally there is a functor $\mathit{Sub}_{\mathbf{SSet_1}}\colon \mathbf{SSet_1}^{\mathit{frag}} \to \mathbf{SSet_1}$ that gives the biggest many-sorted set that is included in a given fragment, it is defined as

$$\mathit{Sub}_{\mathbf{SSet_1}}(\langle i\colon S' \to S, X\rangle) = \langle S', \mathbf{SSET_1}(i)(X)\rangle$$

$$\mathit{Sub}_{\mathbf{SSet_1}}(\langle\langle \sigma', \sigma\rangle\colon i \to j, f\rangle) = \langle \sigma', \mathbf{SSET_1}(i)(f)\rangle$$

Let us present the effect of application of the three functors to the examples given above:

1. e_1, as every complete fragment, doesn't change much when it is transformed by the three functors, we have $\mathit{Compl}_{\mathbf{SSet_1}}(e_1) = \mathit{Sub}_{\mathbf{SSet_1}}(e_1) = \langle \{s_1\}, X_{s_1} = \{a, b\}\rangle$; moreover $\mathit{Frag}_{\mathbf{SSet_1}}(\mathit{Compl}_{\mathbf{SSet_1}}(e_1)) = e_1$;
2. e_2 is a strict fragment with an injective i, the completion functor just gives the whole many-sorted set $\mathit{Compl}_{\mathbf{SSet_1}}(e_2) = \langle \{s_1, s_2\}, X_{s_1} = \{a, b\}, X_{s_2} = \{c\}\rangle$, the functor $\mathit{Sub}_{\mathbf{SSet_1}}$ extracts only these elements that are of the sorts that are in the fragment, i.e., $\mathit{Sub}_{\mathbf{SSet_1}}(e_2) = \langle \{s_1\}, X_{s_1} = \{a, b\}\rangle$
3. e_3 is an empty fragment, the functor $\mathit{Sub}_{\mathbf{SSet_1}}$ gives the initial many-sorted set, $\mathit{Sub}_{\mathbf{SSet_1}}(e_3) = \langle \emptyset, \emptyset\rangle$, the completion functor gives an empty $\{s_1\}$-sorted set, $\mathit{Compl}_{\mathbf{SSet_1}}(e_3) = \langle \{s_1\}, X_{s_1} = \emptyset\rangle$;

4. e_4 does not contain sorts but contains some elements, we have $Sub_{\mathbf{SSet_1}}(e_4) = \langle\emptyset, \emptyset\rangle$, $Compl_{\mathbf{SSet_1}}(e_4) = \langle\{s_1\}, X_{s_1} = \{a\}\rangle$;
5. e_5 is somewhat strange fragment because the function i is not injective and as a consequence we have $Sub(e_5) = \langle\{s_1, s_2\}, X_{s_1} = \{a, b\}, X_{s_2} = \{a, b\}\rangle$ and $Compl(e_5) = \langle\{s\}, X_s = \{a, b\}\rangle$; it may be surprising because intuition says that the result of *Compl* shall be bigger than the result of *Sub*, however here it is the other way around.

Example 3 (FO Algebraic Signature Fragments). Category of first order algebraic signatures $\mathbf{ALGSIG}\colon \mathbf{Set}^{op} \to \mathbf{Cat}$ is defined as $\mathbf{ALGSIG} = ((\_)^+)^{op}; \mathbf{SSET}$ (cf. [TBG91]) where $(\_)^+\colon \mathbf{Set} \to \mathbf{Set}$ is the nonempty sequence endofunctor induced by a map $S \mapsto S^+$ in $\mathbf{Set}$. We name $\mathbf{AlgSig} = \mathbf{Flat}(\mathbf{ALGSIG})$. A category $\mathbf{ALGSIG}$ may be presented as an indexed category of rank 1 as follows:

$$\mathbf{ALGSIG_1}\colon \mathbf{Set_0}^{op} \to \mathbf{Cat}$$

where again, as above, category $\mathbf{Set_0}$ comes from Example 1. We use Def. 3 to obtain *the indexed category of first order algebraic signature fragments*

$$\mathbf{ALGSIG_1}^{frag}\colon \langle \mathbf{Id}_{\mathbf{Set_0}^{frag}}, Frag_{\mathbf{Set_0}}\rangle^{op} \to \mathbf{Cat}$$

The definition of $\mathbf{ALGSIG_1}^{frag}$ and the three adjoint functors $Compl_{\mathbf{AlgSig_1}} \dashv Frag_{\mathbf{AlgSig_1}} \dashv Sub_{\mathbf{AlgSig_1}}$ is pretty much the same as in Example 2 above. The only difference is that sorted elements are not only given for sorts but also for finite sort sequences. Fragments are like $\langle i\colon S' \to S, \Omega\rangle$ for the sort names sets $S, S' \in \mathbf{Set}$ and the set operation names $\Omega \in \mathbf{ALGSIG_1}(S)$. We omit the details here. We name $\mathbf{AlgSig_1}^{frag} = \mathbf{Flat}(\mathbf{ALGSIG_1}^{frag})$. Let us give some examples of signature fragments, i.e. objects of $\mathbf{AlgSig_1}^{frag}$:

1. $\Delta_1 = \langle i\colon \{s_1, s_2\} \to \{s_1, s_2\}, \{h\colon s_1 \times s_2 \to s_1, a : s_1\}\rangle$, where $i(s_1) = s_1, i(s_2) = s_2$, is a complete signature fragment; completing a fragment gives a signature $Compl_{\mathbf{AlgSig_1}}(\Delta_1) = \langle\{s_1, s_2\}, \{h\colon s_1 \times s_2 \to s_1, a : s_1\}\rangle$, that is also the biggest signature included in the fragment because $Sub_{\mathbf{AlgSig_1}}(\Delta_2) = \langle\{s_1, s_2\}, \{h\colon s_1 \times s_2 \to s_1, a : s_1\}\rangle$;
2. $\Delta_2 = \langle j\colon \{s_1\} \to \{s_1, s_2\}, \{f\colon s_1 \times s_2 \to s_1, a : s_1\}\rangle$, where $i(s_1) = s_1$, is such a signature fragment that sort s_2 is not included in, nevertheless, there is an operation f that takes a parameter of sort s_2; completing the fragment gives signature $Compl_{\mathbf{AlgSig_1}}(\Delta_2) = \langle\{s_1, s_2\}, \{f\colon s_1 \times s_2 \to s_1, a : s_1\}\rangle$, the biggest signature included in the fragment contains only the constant a, $Sub_{\mathbf{AlgSig_1}}(\Delta_2) = \langle\{s_1\}, \{a : s_1\}\rangle$;
3. $\Delta_3 = \langle\emptyset_{s_1} : \emptyset \to \{s_1\}, \emptyset\rangle$ is a signature fragment that has neither sorts nor operation symbols, it is an *empty signature fragment* (cf. Def. 12); we have $Compl_{\mathbf{AlgSig_1}}(\Delta_3) = \langle\{s_1\}, \emptyset\rangle$ and $Sub_{\mathbf{AlgSig_1}}(\Delta_3) = \langle\emptyset, \emptyset\rangle$;
4. $\Delta_4 = \langle\emptyset_{s_1} : \emptyset \to \{s_1\}, \{g : s_1 \to s_1\}\rangle$ is a signature fragment that has no sorts but contains one operation symbol g which makes it a non-empty fragment; $Compl_{\mathbf{AlgSig_1}}(\Delta_4) = \langle\{s_1\}, \{g : s_1 \to s_1\}\rangle$ and $Sub_{\mathbf{AlgSig_1}}(\Delta_4) = \langle\emptyset, \emptyset\rangle$;

The following is an example of a morphism between signature fragments, i.e. a morphism of $\mathbf{AlgSig_1}^{frag}$. Let $\delta_1 = \langle\langle\sigma', \sigma\rangle, \gamma\rangle\colon \Delta_2 \to \Delta_1$ be a morphism of signature fragments where $\sigma'\colon \{s_1\} \to \{s_1, s_2\}$ is an inclusion, $\sigma = id_{\{s_1,s_2\}}$ is an identity and $\gamma\colon \{f, a\} \to \{h, a\}$ is a function such that $\gamma(f) = h, \gamma(a) = a$; it is easy to show that $j;\sigma = \sigma';i$; the completion functor gives the following morphism between signatures $Compl_{\mathbf{AlgSig_1}}(\delta_1) = \langle\sigma, \gamma\rangle$, whereas the functor $Sub_{\mathbf{AlgSig_1}}$ reduces δ_1 to the signature morphism $Sub_{\mathbf{AlgSig_1}}(\delta_1) = \langle\sigma', \mathbf{ALGSIG_1}(j)(\gamma)\rangle$ where $\mathbf{ALGSIG_1}(j)(\gamma)\colon \{a\} \to \mathbf{ALGSIG_1}(\sigma')(\{h, a\})$ is an identity $id_{\{a\}}$, one can easily check that $\mathbf{ALGSIG_1}(\sigma')(\{h, a\}) = \{a\}$;

Example 4 (FO Algebra Fragments). First, let us define a category of first order algebraic structures $\mathbf{ALG}\colon \mathbf{Flat}(\mathbf{ALGSIG})^{op} \to \mathbf{Cat}$. Given a signature $\Sigma \in \mathbf{Flat}(\mathbf{ALGSIG})$, $\mathbf{ALG}(\Sigma)$ is a class of all Σ-algebras; given a signature morphism $\sigma\colon \Sigma_1 \to \Sigma_2$, $\mathbf{ALG}(\sigma)\colon \mathbf{ALG}(\Sigma_2) \to \mathbf{ALG}(\Sigma_1)$ is the σ-reduct. We name $\mathbf{Alg} = \mathbf{Flat}(\mathbf{ALG})$. Category $\mathbf{ALG}$ may be presented as an indexed category of rank 2

$$\mathbf{ALG_2}\colon \mathbf{AlgSig_1}^{op} \to \mathbf{Cat}$$

where $\mathbf{AlgSig_1} = \mathbf{Flat}(\mathbf{ALGSIG_1})$ is the flattened indexed category of rank 1 defined in Example 3. Using Def. 3 we define *the indexed category of algebra fragments*

$$\mathbf{ALG_2}^{frag}\colon \langle \mathbf{Id}_{\mathbf{AlgSig_1}^{frag}}, Frag_{\mathbf{AlgSig_1}}\rangle^{op} \to \mathbf{Cat}$$

We name $\mathbf{Alg_2}^{frag} = \mathbf{Flat}(\mathbf{ALG_2}^{frag})$. An algebra fragment $E \in \mathbf{Alg_2}^{frag}$ is defined as

$$E = \langle \delta\colon \Delta_1 \to Frag_{\mathbf{AlgSig_1}}(\Sigma_2), A\rangle$$

for some signature fragment $\Delta_1 = \langle i\colon S_1' \to S_1, \Omega_1\rangle \in \mathbf{AlgSig_1}^{frag}$, some signature $\Sigma_2 = \langle S_2, \Omega_2\rangle \in \mathbf{AlgSig_1}$, and some algebra $A \in \mathbf{ALG_2}(\Sigma_2)$. One should think that an algebra fragment, having the above fragment as an example, contains sort names S_1', operation symbols Ω_1, and carriers and functions as in algebra A.
An algebra fragment morphism $\epsilon\colon E_1 \to E_2 \in \mathbf{Alg_2}^{frag}$ between algebra fragments $E^1 = \langle \delta^1\colon \Delta_1^1 \to Frag_{\mathbf{AlgSig_1}}(\Sigma_2^1), A^1\rangle$ and $E^2 = \langle \delta^2\colon \Delta_1^2 \to Frag_{\mathbf{AlgSig_1}}(\Sigma_2^2), A^2\rangle$ is defined as

$$\epsilon = \langle\langle \gamma'\colon \Delta_1^1 \to \Delta_1^2, Frag_{\mathbf{AlgSig_1}}(\sigma\colon \Sigma_2^1 \to \Sigma_2^2)\rangle, f\colon A_1 \to \mathbf{ALG_2}(\sigma)(A_2)\rangle$$

for some signature fragment morphism $\gamma' \in \mathbf{AlgSig_1}^{frag}$, signature morphism $\sigma \in \mathbf{AlgSig_1}$, and homomorphism $f \in \mathbf{ALG_2}(\Sigma_2^1)$.
Following Def. 3 there are three adjoint functors $Compl_{\mathbf{Alg_2}} \dashv Frag_{\mathbf{Alg2_1}} \dashv Sub_{\mathbf{Alg_2}}$ that allow to move between categories of algebras and algebra fragments. First we present the embedding of algebras to fragments $Frag_{\mathbf{Alg_2}}\colon \mathbf{Alg_2} \to \mathbf{Alg_2}^{frag}$ defined as

$$Frag_{\mathbf{Alg_2}}(\langle \Sigma, A\rangle) = \langle id_{Frag_{\mathbf{AlgSig_1}}(\Sigma)}, A\rangle$$

$$Frag_{\mathbf{Alg_2}}(\langle \sigma, f\rangle) = \langle\langle Frag_{\mathbf{AlgSig_1}}(\sigma), Frag_{\mathbf{AlgSig_1}}(\sigma)\rangle, f\rangle$$

Then there is a completion functor $Compl_{\mathbf{Alg_2}} \colon \mathbf{Alg_2}^{frag} \to \mathbf{Alg_2}$ defined as

$$Compl_{\mathbf{Alg_2}}(\langle \delta \colon \Delta_1 \to Frag_{\mathbf{AlgSig_1}}(\Sigma_2), A\rangle) = \langle \Sigma_2, A\rangle$$

$$Compl_{\mathbf{Alg_2}}(\langle\langle \gamma', Frag_{\mathbf{AlgSig_1}}(\sigma)\rangle, f\rangle) = \langle \sigma, f\rangle$$

Finally there is a functor $Sub_{\mathbf{Alg_2}} \colon \mathbf{Alg_2}^{frag} \to \mathbf{Alg_2}$ that gives the largest many-sorted algebra that is included in a given fragment. It is defined as

$$Sub_{\mathbf{Alg_2}}(\langle \delta \colon \Delta_1 \to Frag_{\mathbf{AlgSig_1}}(\Sigma_2), A\rangle) = \langle S', \mathbf{ALG_2}(Sub_{\mathbf{AlgSig_1}}(\delta))(A)\rangle$$

$$Sub_{\mathbf{Alg_2}}(\langle\langle \gamma', Frag_{\mathbf{AlgSig_1}}(\sigma)\rangle \colon \delta_1 \to \delta_2, f\rangle) = \langle Sub_{\mathbf{AlgSig_1}}(\gamma'), \mathbf{ALG_2}(Sub_{\mathbf{AlgSig_1}}(\delta_1))(f)\rangle$$

To give the example of an algebra fragment, i.e. an object of $\mathbf{Alg_2}^{frag}$, we take the algebraic signature fragment Δ_2 from Example 3 and define the algebra fragment E_1 using the notation that shall be understandable, even though not defined formally (underlined parts <u>*are in*</u> the fragment):

$$
\begin{aligned}
E_1 = \qquad \mathbf{sorts} &: \underline{s_1}, s_2 \\
\mathbf{ops} &: \underline{f \colon s_1 \times s_2 \to s_1}, \\
& \underline{a \colon s_2}, \\
& b \colon s_2 \\
\mathbf{carriers} &: \underline{s_1^A} = \{*, \cdot, \diamond\}, \\
& \underline{s_2^A} = \{+, \triangle\}, \\
& \underline{s_3^A} = \emptyset \\
\mathbf{functions} &: \underline{f^A} \colon s_1^A \times s_2^A \to s_1^A = \{* \mapsto +, \cdot \mapsto +, \diamond \mapsto \triangle\}, \\
& \underline{a^A} \colon s_1^A = \diamond, \\
& \underline{b^A} \colon s_2^A = \triangle, \\
& \underline{empty^A} \colon s_3^A \to s_3^A = \emptyset.
\end{aligned}
$$

The largest algebra included in E_2, as given by the functor $Sub_{\mathbf{Alg_2}}$, is $Sub_{\mathbf{Alg_2}}(E_1) = \{\{s_1\}, s_1^A = \{*, \cdot, \diamond\}, a^A = \diamond\}$.

Example 5 (Theory Fragments). Let us assume that there is an institution $\mathbf{I} = \langle \mathbf{Sig_n}, Mod, Sen, \models\rangle$ (cf. Sect. 2), with the category of signatures of rank n. Given a signature $\Sigma \in \mathbf{Sig_n}$ we can define a Σ-presentation simply as a set of Σ-sentences, $\Phi \subseteq Sen(\Sigma)$. A Σ-theory is a Σ-presentation Φ that is closed under semantic consequence, i.e. $T_\Phi = Cl^{\mathbf{I}}(\Phi) = \{\varphi \in Sen \mid \text{for all } m \in Mod(\Sigma), m \models \varphi \text{ whenever } m \models \Phi\}$ (cf. [TBG91]). Let us define a category of theories in $\mathbf{I}$ as an indexed category of rank $n+1$.

$$\mathbf{TH^I}_{n+1} \colon \mathbf{Sig_n}^{op} \to \mathbf{Cat}$$

Given a signature $\Sigma \in \mathbf{Sig_n}$ define $\mathbf{TH^I}_{n+1}(\Sigma)$ as a poset category of Σ-theories ordered by inclusion. Given a signature morphism $\sigma \colon \Sigma_1 \to \Sigma_2$ and a Σ_2-theory $T_2 \in \mathbf{TH^I}_{n+1}(\Sigma_2)$ define $\mathbf{TH^I}_{n+1}(\sigma)(T_2) = \{\varphi \in Sen(\Sigma_1) \mid \sigma(\varphi) \in T_2\}$. Let us name $\mathbf{Th^I}_{n+1} = \mathbf{Flat}(\mathbf{TH^I}_{n+1})$.

Using Def. 3 we can define *the indexed category of fragments of theories in* $\mathbf{I}$

$$\mathbf{TH^I}_{n+1}{}^{frag} \colon \langle \mathbf{Id}_{\mathbf{Sig_n}^{frag}}, Frag_{\mathbf{Sig_n}} \rangle^{op} \to \mathbf{Cat}$$

where $\mathbf{Sig_n}^{frag}$ is a flattened indexed category of $\mathbf{Sig_n}$-fragments. We name $\mathbf{Th^I_{n+1}}^{frag} = \mathbf{Flat}(\mathbf{TH^I}_{n+1}{}^{frag})$. A theory fragment $TF \in \mathbf{Th^I_{n+1}}^{frag}$ is defined as

$$TF = \langle \delta \colon \Delta_1 \to Frag_{\mathbf{Sig_n}}(\Sigma_2), T_2 \rangle$$

for some signature fragment $\Delta_1 \in \mathbf{Sig_n}^{frag}$, some signature $\Sigma_2 \in \mathbf{Sig_n}$, and a Σ_2-theory $T_2 \in \mathbf{TH^I}_{n+1}(\Sigma_2)$. Definition 3 describes also the three adjoint functors $Compl_{\mathbf{Th^I_{n+1}}} \dashv Frag_{\mathbf{Th^I_{n+1}}} \dashv Sub_{\mathbf{Th^I_{n+1}}}$ that allow to convert theories to theory fragments and backwards.

Let us give an example of a theory fragment in the institution $\mathbf{AlgI} = \langle \mathbf{AlgSig_1}, AlgMod, AlgSen, \models_{Alg} \rangle$ of first order algebras and first order logic, that is an object in $\mathbf{Th_2^{AlgI}}^{frag}$. Theory fragment

$$TF = \langle \iota \colon \langle \emptyset \to \{s\}, \langle a \colon s, f \colon s \to s \rangle \rangle \to Frag_{\mathbf{AlgSig_1}} \langle s, \langle a \colon s, b \colon s, f \colon s \to s \rangle \rangle, Cl^{\mathbf{AlgI}}(\{b = f(a)\}) \rangle$$

consists of a signature fragment $\langle \mathbf{ops} : a \colon s, f \colon s \to s \rangle$ and a theory over a signature $\langle \mathbf{sort} : s; \mathbf{ops} : a \colon s, b \colon s, f \colon s \to s \rangle$ that contains a theory that is the closure of a single sentence "$b = f(a)$".

Example 6 (Institution of Fragments). Institutions (cf. Sect. 2) provide an abstract framework for describing logical systems. Given an institution $\mathbf{I} = \langle \mathbf{Sig_n}, \mathbf{MOD_{N+1}}, \mathbf{SEN_{N+1}}, \models \rangle$, where $\mathbf{Sig_n}$ is a flattened indexed category of rank n, one can define the corresponding *institution of fragments*

$$\mathbf{FI} = \langle (\mathbf{Id}_{\mathbf{Sig_n}^{frag}}, Frag_{\mathbf{Sig_n}}), \mathbf{Mod_{n+1}}^{frag}, \mathbf{Sen_{n+1}}^{frag}, (\pi_3; Compl_{\mathbf{Sig_n}}; \models) \rangle$$

Where $(\mathbf{Id}_{\mathbf{Sig_n}^{frag}}, Frag_{\mathbf{Sig_n}})$ is the comma category for functors $\mathbf{Id}_{\mathbf{Sig_n}^{frag}}$ and $Frag_{\mathbf{Sig_n}}$. Most constructions needed to do so follow directly from definitions presented in Examples 3–5. Specifications over this institution are theory fragments from Example 5.

Example 7 (Institution Fragments). As shown in [TBG91] a category of generalized institutions (generalization concerns a satisfaction relation that needs not be interpreted in **Bool**, instead it can take logical values from any arbitrary category $\mathbf{V}$) may be presented as an indexed category

$$\mathbf{INST} \colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$

where $\mathbf{INST}(\mathbf{Sig}) = [\mathbf{Sig}^{op} \to \mathbf{Room}(\mathbf{V})]$ and given a translation functor $\Phi \colon \mathbf{Sig_1} \to \mathbf{Sig_2}$ and institution $\mathbf{I_2} \in \mathbf{INST}(\mathbf{Sig_2})$ define $\mathbf{INST}(\Phi) \colon \mathbf{INST}(\mathbf{Sig_2}) \to \mathbf{INS}(\mathbf{Sig_1})$ by $\mathbf{INST}(\Phi)(\mathbf{I_2}) = \Phi; \mathbf{I_2}$. A category of $\mathbf{V}$-*rooms* is defined as a comma category $\mathbf{Room}(\mathbf{V}) = \langle |\_|, \mathbf{FUNC}_{Disc}(\mathbf{V}) \rangle$ where $|\_| \colon \mathbf{Cat} \to \mathbf{Cat}$ is a discretization functor and $\mathbf{FUNC}_{Disc}(\mathbf{V}) \colon \mathbf{DCat}^{op} \to \mathbf{Cat}$ is the indexed category of functors into $\mathbf{V}$ restricted to discrete categories in **DCat** as source. Objects of $\mathbf{Room}(\mathbf{V})$ are called $\mathbf{V}$-rooms. A $\mathbf{V}$-room is a triple $\langle \mathbf{M}, R, \mathbf{S} \rangle$ where $\mathbf{M}$

is a category (of models), S is a discrete category (of sentences) and $R\colon |\mathbf{M}| \to [\mathbf{S} \to \mathbf{V}]$ (is a satisfaction "relation"). In the above setting the satisfaction condition is a natural requirement of commutativity of morphisms in comma categories. Let us name $\mathbf{Inst} = \mathbf{Flat}(\mathbf{INST})$. Every objects of $\mathbf{Inst}$

$$\mathbf{I} = \langle \mathbf{Sig}, \mathbf{F}\colon \mathbf{Sig^{op}} \to \langle |\_|, \mathbf{FUNC}_{Disc}(\mathbf{V})\rangle\rangle$$

can be equally represented as

$$\mathbf{I} = \langle \mathbf{Sig}, Mod\colon \mathbf{Sig}^{op} \to \mathbf{Cat}, Sen\colon \mathbf{Sig} \to \mathbf{DCat}, \models\colon \mathbf{Sig}^{op} \to \langle |Mod|, Sen; \mathbf{FUNC}_{Disc}(\mathbf{V})\rangle\rangle$$

that looks a bit more like standard representation of institutions as presented in Sect. 2.

Following Def. 1 we define the corresponding indexed category of rank 1

$$\mathbf{INST_1}\colon {\mathbf{Cat_0}}^{op} \to \mathbf{Cat}$$

where category $\mathbf{Cat_0} = \mathbf{Flat}(\mathbf{CAT_0})$ is the flattened indexed category of rank 0 isomorphic to $\mathbf{Cat}$ (cf. Def. 1). We name $\mathbf{Inst_1} = \mathbf{Flat}(\mathbf{INST_1})$. Now, we are ready to define *the indexed category of institution fragments*

$${\mathbf{INST_1}}^{frag}\colon \langle \mathbf{Id}_{{\mathbf{Cat_0}}^{frag}}, Frag_{\mathbf{Cat_0}}\rangle^{op} \to \mathbf{Cat}$$

where, similarly to definition of ${\mathbf{SSET_1}}^{frag}$ in Example 2, ${\mathbf{Cat_0}}^{frag} = \mathbf{Cat_0}$ and $Frag_{\mathbf{Cat_0}} = \mathbf{Id}_{\mathbf{Cat_0}}$ that is naturally isomorphic to $\mathbf{Id}_{\mathbf{Cat}}$. All what it means is that in fact the category of fragments is a category indexed by functors (between signature categories) ${\mathbf{INST_1}}^{frag}\colon \langle \mathbf{Id}_{\mathbf{Cat}}, \mathbf{Id}_{\mathbf{Cat}}\rangle^{op} \to \mathbf{Cat}$.

To give an example of an institution fragment we define an embedding functor $Emb\colon \mathbf{Set_0} \to \mathbf{AlgSig_1}$ that takes a set of sort names and gives an algebraic signature that contains these sort names only (without any operation names). We define it as $Emb(A) = \langle A, \emptyset\rangle$, $Emb(f\colon A \to B) = \langle f, \emptyset\rangle$. Now, the following pair

$$\mathbf{AlgIF} = \langle Emb\colon \mathbf{Set_0} \to \mathbf{AlgSig_1}, \langle AlgMod, AlgSen, \models_{Alg}\rangle\rangle$$

is an institution fragment that contains a category of signatures with merely sort names ($\mathbf{Set_0}$). When we apply the completion functor $Compl_{\mathbf{Inst}}$ we obtain the institution of the first order logic

$$Compl_{\mathbf{Inst}}(\mathbf{AlgIF}) = \langle \mathbf{AlgSig_1}, AlgMod, AlgSen, \models_{Alg}\rangle$$

The application of functor $Sub_{\mathbf{Inst}}$ gives the institution of sets and the first order logic to describe them

$$Sub_{\mathbf{Inst}}(\mathbf{AlgIF}) = \langle \mathbf{Set_0}, (Emb^{op}; AlgMod), (Emb; AlgSen), (Emb^{op}; \models_{Alg})\rangle$$

The next example is an institution fragment of the institution $\mathbf{FI}$ from Example 6. There is an embedding translation functor

$$\Phi^F\colon ((Compl_{\mathbf{Sig_n}}; Frag_{\mathbf{Sig_n}}), Frag_{\mathbf{Sig_n}}) \to (\mathbf{Id}_{{\mathbf{Sig_n}}^{frag}}, Frag_{\mathbf{Sig_n}})$$

that selects only complete fragments as source of morphisms of comma category. Now, we define a fragment of institution **FI** as

$$\mathbf{IFI} = \langle \Phi^F, \langle \mathbf{MOD_{N+1}}^{frag}, \mathbf{SEN_{N+1}}^{frag}, (\pi_3; \mathit{Compl}_{\mathbf{Sign_n}}; \models)\rangle\rangle$$

Finally, we use functor $\mathit{Sub}_{\mathbf{Inst}}$ to obtain the largest institution included in **IFI**. Reader is asked to check that $\mathit{Sub}_{\mathbf{Inst}}(\mathbf{IFI})$ is exactly *an institution of extended models* as defined e.g. in Sect. 6 of [SML05].

3.2 Applications to Architectural Specifications

Signature fragments naturally represent signature extensions, i.e. a difference between the target signature and the start signature. They (or rather entities of their kind defined in an ad-hoc manner) have been used for this purpose in e.g. CASL to represent specification enrichments (cf. Sect.III:2.1 in [CoF04]). Here, however, we would like to discuss another application of fragments that have not been used yet.

Signature fragments give rise to the definition of the generic units sharing analysis within the framework of architectural specifications (cf. [CoF04]). Let us consider the category **Sig** of CASL signatures (taken from cf. Sect. III:2 of [CoF04] and represented as an indexed category) defined as tuples $\langle S, F, P\rangle$ of sets of sort, function (total and partial[2]), and predicate symbols respectively. Using Def. 3 define a corresponding category of CASL signature fragments $\mathbf{Sig}^{frag}$. To demonstrate how fragments can be used for sharing analysis we transform a signature of a generic architectural unit[3] to a signature fragment in the following way (cf. Sect. 4 for a discussion on how to make it in more general terms). Given a generic unit signature $\mathcal{U}\colon \Sigma_F \to \Sigma_R$ we define a signature fragment $\Delta_{\mathcal{U}}$ as a pair of the inclusion of all sort symbols of Σ_R that are not in Σ_F, $S_R \setminus S_F$, into sorts from Σ_R, S_R, and the difference between sets of function and predicate symbols of Σ_R and Σ_F

$$\Delta_{\mathcal{U}} = \langle \iota\colon (S_R \setminus S_F) \to S_R, \langle (F_R \setminus F_F), (P_R \setminus P_F)\rangle\rangle$$

In fact a signature of a generic unit is an embedding of a signature fragment, as defined above, to the result signature $\mathcal{UF}\colon \Delta_{\mathcal{U}} \to \mathit{Frag}_{\mathbf{Sig}}(\Sigma_R)$. Notice the difference between $\mathcal{U}$ where Σ_F is contravariantly included in Σ_R and $\mathcal{UF}$ where $\Delta_{\mathcal{U}}$ is covariantly included in $\mathit{Frag}_{\mathbf{Sig}}(\Sigma_R)$. Notice also that $\mathcal{UF}$ is an object of a category of signatures of an institution **IF** from Example 6.

Now, to prove that two unit signatures given as parameters to the generic unit application operation[4] don't contain conflicting symbols, it is enough to

[2] Names of total and partial functions are separated in CASL. We joined these two sets to make the presentation more concise.

[3] Non-generic units may be defined as generic units with an empty parameter signature

[4] In CASL semantics, as described in [CoF04], the application of a generic unit to a generic unit is not allowed. We perceive it as a serious disadvantage of CASL and disregard it here.

take the pullback of corresponding signature fragments wrt. inclusions into the global signature (signatures are analyzed as inclusion of unit signatures into the global signature obtained by a pushout, cf. Sect. III:5.6 in [CoF04]). If the pullback object is an empty fragment (cf. Def. 12), then there are no conflicting symbols.

For example the pullback of a signature fragment for a unit $N\colon$ **Nat** and the one for an identity generic unit $I\colon$ **Nat** $\rightarrow$ **Nat** is empty. So is the pullback of the signature fragment corresponding to the construction $P\colon$ **Nat** $\rightarrow$ **NatPlus** and the one for $M\colon$ **Nat** $\rightarrow$ **NatMul** (where P and M are construction that add addition and multiplication operations to the natural numbers respectively). However, a pullback of a signature fragment for P, as above, and $O\colon$ **Nat** $\rightarrow$ **NatOpers** (a construction that adds all arithmetic operations at once) is not empty and contains exactly one operation symbol *plus*.

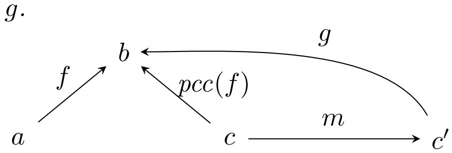
The pushout operation on signature fragments may be used to obtain the signature fragment of the result of application of generic units to generic units. Since pushout is a symmetric operation, so becomes the application operation in the framework that uses fragments to sharing analysis. We are currently working on details of such approach that are beyond the scope of this paper.

4 Differences of Fragments

In the previous section we had to manually define how to obtain a signature fragment that is a difference of two signatures. To avoid this kind of definitions there is a need for a generic categorical notion of some sort of subtraction operation. We believe that the following definition of *pullback-pushout complement (ppc)* captures this issue.

Definition 16 (Pullback-pushout Condition) *In category* **C***, given a morphism* $f\colon a \rightarrow b$*, we say that* $g\colon c \rightarrow b$ meets the pullback-pushout condition wrt. f *iff the pullback square of* f *and* g *is a pushout square (i.e. it is bicartesian).*

Definition 17 (Pullback-pushout Complement) *In category* **C***, a pullback-pushout complement (ppc) of morphism* $f\colon a \rightarrow b$ *is such an object* c *and a monomorphism* $pcc(f)\colon c \rightarrow b$ *that* $pcc(f)$ *meets the pullback-pushout condition wrt.* f*. Moreover, for every object* c' *and a monomorphism* $g\colon c' \rightarrow b$ *meeting the pullback-pushout condition wrt.* f*, there exists a unique morphism* $m\colon c \rightarrow c'$*, such that* $pcc(f) = m; g$*.*



The use of the ppc to obtain a signature fragment that represents all components that are defined in a generic architectural unit (cf. Sect. 3.2) is as follows. Provided ppc is defined in $\mathbf{Sig}^{frag}$, let $\sigma\colon \Sigma_F \rightarrow \Sigma_R \in$ **Sig** be an inclusion (or just a monomorphism) of the formal parameter signature Σ_F into the result signature Σ_R. A morphism $pcc(Frag_{\mathbf{Sig}}(\sigma))\colon \Delta \to Frag_{Sig}(\Sigma_R)$ corresponds to $\mathcal{UF}$ above and Δ is the difference between Σ_R and Σ_F.

The work on pullback-pushout complements is in progress. We already have some results concerning ppcs in various categories. However, this is again beyond the scope of this paper.

5 Conclusion and Future Work

In the first part of our paper we have defined a notion of *a fragment*. To our knowledge this is the first systematic approach to define such entities. The problem we started with concerned the category of algebraic signatures. However, our work has been carried out in more general terms, for arbitrary indexed categories. We have defined the indexed category of fragments that corresponds to a given indexed category of some rank. Ranks are attributed to indexed categories and correspond to number of levels of indexation present in the given category. The higher rank of a category, the higher granularity of its fragments. Our construction uses the inductive structure of indexed categories with ranks to distinguish subsequent levels of their construction and allow to separately define components on each level without restricting anything that is above. For example, referring to the problem given at the beginning of introduction, the result of removing the sort name s from signature $\{\mathbf{sort}\ s; \mathbf{ops}\ f\colon s \to s, a\colon s\}$ is fragment $\langle \gamma\colon \emptyset \to \{\mathbf{sort}\ s\}, \{\mathbf{ops}\ f\colon s \to s, a\colon s\}\rangle$ that doesn't contain sort names but contains names of some operations on them. The sort name s present in the target of γ is auxiliary, needed only to represent the next level of definition. Within the definition of a category of fragments we have also defined the embedding functor that shows how objects of a flattened original category are represented as fragments. We have also defined the left and right adjoint to the embedding functor. Informally speaking, they represent two very intuitive operations of taking the smallest object containing the given fragment, and the largest object included in the given fragment respectively. Moreover, we have given a notion of *an empty fragment* and *a complete fragment*. Empty fragments are such that don't really have any non-trivial content, however some auxiliary elements at some levels of definition may be present inside them. This makes such fragments different from the initial fragment. Complete fragments are such that all their auxiliary elements have counterparts in their real content.

The second part of the paper is devoted to examples and applications. We have explicitly defined categories of fragments for indexed categories of different ranks that are typical for the area of algebraic specifications. We believe that we have proved fragments to be a useful contribution to this field. The plans for the future work include attempts to successfully use fragments to enhance the theory of architectural specifications.

As already mentioned in Sect. 4 we also plan to work on the concept of *pullback-pushout complements* that may lead to a definition of some kind of a subtraction operation on fragments.

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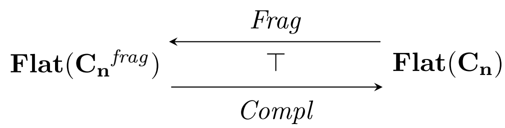
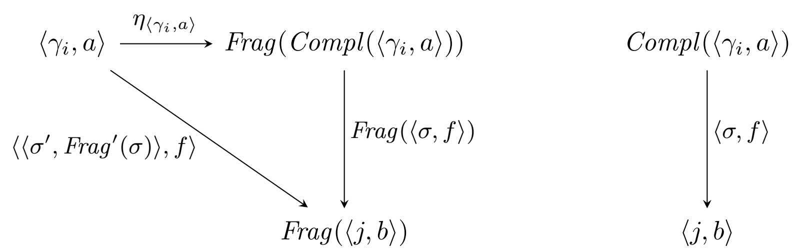
A Proof of Theorem 4

Proof: The proof goes by induction on rank n. In the base case, when $n = 0$, all three functors are identities, thus they are indeed adjoint.

In an induction step case, category **Ind** is a flattened indexed category of rank $n-1$, i.e. $\mathbf{Ind} = \mathbf{Flat}(\mathbf{C'_{n-1}})$ for some $\mathbf{C'_{n-1}}$ and ${\mathbf{C'_{n-1}}}^{frag}$ is the category of its fragments and we assume that $Compl' \dashv Frag' \dashv Sub'$ are three adjoint functors associated with ${\mathbf{C'_{n-1}}}^{frag}$.

First we show that $\mathit{Compl} \dashv \mathit{Frag}$:





where $\eta_{\langle\gamma_i,a\rangle} = \langle\langle\gamma_i, id_{\mathit{Frag}'(i)}\rangle, id_a\rangle$. We read the above diagram as: given an object $\langle\gamma_i, a\rangle \in \mathbf{Flat}(\mathbf{C_n}^{\mathit{frag}})$, for every object $\langle j, b\rangle \in \mathbf{Flat}(\mathbf{C_n})$ and a morphism $\langle\langle\sigma', \mathit{Frag}'(\sigma)\rangle, f\rangle \colon \langle\gamma_i, a\rangle \to \mathit{Frag}(\langle j, b\rangle) \in \mathbf{Flat}(\mathbf{C_n}^{\mathit{frag}})$, the unique morphism $\langle\sigma, f\rangle \colon \mathit{Compl}(\langle\gamma_i, a\rangle) \to \langle j, b\rangle$ is such that $\eta_{\langle\gamma_i,a\rangle}; \mathit{Frag}(\langle\sigma, f\rangle) = \langle\langle\sigma', \mathit{Frag}'(\sigma)\rangle, f\rangle$. We check that indeed

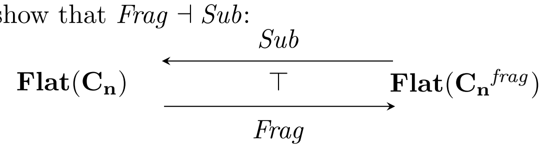
$$\begin{aligned}
\eta_{\langle\gamma_i,a\rangle}; \mathit{Frag}(\langle\sigma, f\rangle) &= \langle\langle\gamma_i, id_{\mathit{Frag}'(i)}\rangle, id_a\rangle; \mathit{Frag}(\langle\sigma, f\rangle) \\
&= \langle\langle\gamma_i, id_{\mathit{Frag}'(i)}\rangle, id_a\rangle; \langle\langle \mathit{Frag}'(\sigma), \mathit{Frag}'(\sigma)\rangle, f\rangle \\
&= \langle\langle\gamma_i; \mathit{Frag}'(\sigma), id_{\mathit{Frag}'(i)}; \mathit{Frag}'(\sigma)\rangle, id_a; f\rangle \\
&= \langle\langle\sigma', \mathit{Frag}'(\sigma)\rangle, f\rangle
\end{aligned}$$

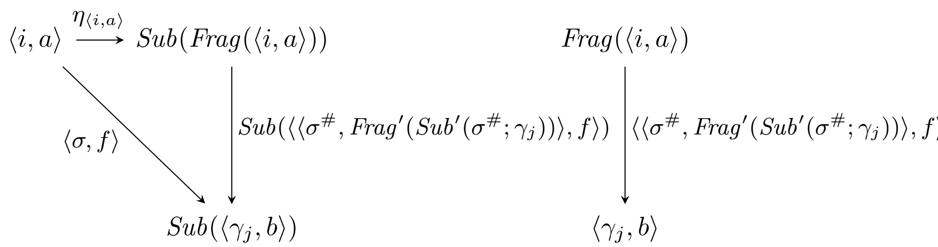
We show the uniqueness of $\langle\sigma, f\rangle$ by contradiction. Assume that there exists $\langle\sigma_1, f_1\rangle \colon \mathit{Compl}(\langle\gamma_i, a\rangle) \to \langle j, b\rangle$ such that $\langle\sigma_1, f_1\rangle \neq \langle\sigma, f\rangle$ and $\eta_{\langle\gamma_i,a\rangle}; \mathit{Frag}(\langle\sigma_1, f_1\rangle) = \langle\langle\sigma', \mathit{Frag}'(\sigma)\rangle, f\rangle$. We have

$$\begin{aligned}
\langle\langle\sigma', \mathit{Frag}'(\sigma)\rangle, f\rangle &= \eta_{\langle\gamma_i,a\rangle}; \mathit{Frag}(\langle\sigma_1, f_1\rangle) \\
&= \langle\langle\gamma_i, id_{\mathit{Frag}'(i)}\rangle, id_a\rangle; \mathit{Frag}(\langle\sigma_1, f_1\rangle) \\
&= \langle\langle\gamma_i, id_{\mathit{Frag}'(i)}\rangle, id_a\rangle; \langle\langle \mathit{Frag}'(\sigma_1), \mathit{Frag}'(\sigma_1)\rangle, f_1\rangle \\
&= \langle\langle\gamma_i; \mathit{Frag}'(\sigma_1), \mathit{Frag}'(\sigma_1)\rangle, f_1\rangle
\end{aligned}$$

which implies that $\mathit{Frag}'(\sigma) = \mathit{Frag}'(\sigma_1)$ and $f = f_1$. So, since Frag' is faithful, we know that $\sigma = \sigma_1$ and finally $\langle\sigma_1, f_1\rangle = \langle\sigma, f\rangle$ which contradicts the assumption and proves the uniqueness of $\langle\sigma, f\rangle$.

Now, we show that $\mathit{Frag} \dashv \mathit{Sub}$:





where $\eta_{\langle i,a\rangle} = \langle id_i, id_a\rangle$ which is well defined because one can easily prove by induction (cf. Lemma 6) that for any $i \in \mathbf{Ind}$, $\mathit{Sub}'(\mathit{Frag}'(i)) = i$ thus $\mathit{Sub}(\mathit{Frag}(\langle i,a\rangle)) = \langle \mathit{Sub}'(\mathit{Frag}'(i)), a\rangle = \langle i, a\rangle$. Morphism $\sigma^{\#}\colon \mathit{Frag}'(i) \to j'$ is the morphism "adjoint" to $\sigma\colon i \to \mathit{Sub}'(j')$ wrt. $\mathit{Frag}' \dashv \mathit{Sub}'$. Let us read the above diagram as: given an object $\langle i,a\rangle \in \mathbf{Flat}(\mathbf{C_n})$, for every object $\langle \gamma_j, b\rangle \in \mathbf{Flat}(\mathbf{C_n}^{\mathit{frag}})$ and a morphism $\langle \sigma, f\rangle\colon \langle i,a\rangle \to \mathit{Sub}(\langle \gamma_j, b\rangle) \in \mathbf{Flat}(\mathbf{C_n})$, the unique morphism

$$\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle\colon \mathit{Frag}(\langle i,a\rangle) \to \langle \gamma_j, b\rangle$$

is such that $\eta_{\langle i,a\rangle}; \mathit{Sub}(\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle) = \langle \sigma, f\rangle$. And we check that indeed

$$\begin{aligned} \eta_{\langle i,a\rangle}; \mathit{Sub}(\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle) &= \langle id_i, id_a\rangle; \mathit{Sub}(\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle) \\ &= \langle \mathit{Sub}'(\sigma^{\#}), \mathbf{C}_n(\mathit{Sub}'(id_{\mathit{Frag}'(i)}))(f)\rangle \\ &= \langle \sigma, \mathbf{C}_n(id_i)(f)\rangle \\ &= \langle \sigma, f\rangle \end{aligned}$$

Finally we prove the uniqueness of $\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle$ by contradiction. Assume that there exists $\langle\langle \sigma_1', \mathit{Frag}'(\sigma_1)\rangle, f_1\rangle$ such that $\langle\langle \sigma_1', \mathit{Frag}'(\sigma_1)\rangle, f_1\rangle \neq \langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle$ and $\eta_{\langle i,a\rangle}; \mathit{Sub}(\langle\langle \sigma_1', \sigma_1\rangle, f_1\rangle) = \langle \sigma, f\rangle$. The following holds

$$\begin{aligned} \langle \sigma, f\rangle &= \eta_{\langle i,a\rangle}; \mathit{Sub}(\langle\langle \sigma_1', \sigma_1\rangle, f_1\rangle) \\ &= \langle id_i, id_a\rangle; \mathit{Sub}(\langle\langle \sigma_1', \sigma_1\rangle, f_1\rangle) \\ &= \langle \mathit{Sub}'(\sigma_1'), \mathbf{C}_n(\mathit{Sub}'(id_{\mathit{Frag}'(i)}))(f_1)\rangle \\ &= \langle \mathit{Sub}'(\sigma_1'), f_1\rangle \end{aligned}$$

The above implies that $f = f_1$ and $\sigma = \mathit{Sub}'(\sigma_1')$ from which we conclude that $\sigma^{\#} = \sigma_1'$ because for all $i \in \mathbf{Ind}, \eta_i' = id_i$. Morphism $\langle \sigma_1', \sigma_1\rangle$ is a morphism in a comma category, so we have $\sigma_1'; \gamma_j = \mathit{Frag}'(\sigma_1)$ which implies $\mathit{Sub}'(\sigma_1';\gamma_j) = \sigma_1$ and $\mathit{Sub}'(\sigma^{\#};\gamma_j) = \sigma_1$. Therefore we have $\mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j)) = \mathit{Frag}'(\sigma_1)$. Finally we have all we need to show that $\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle = \langle\langle \sigma_1', \mathit{Frag}'(\sigma_1)\rangle, f_1\rangle$ which contradicts the assumption and proves the uniqueness of $\langle\langle \sigma^{\#}, \mathit{Frag}'(\mathit{Sub}'(\sigma^{\#};\gamma_j))\rangle, f\rangle$.

□