Triangulated categories: Enhancements, rigidity and exotic models

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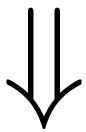

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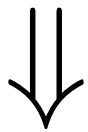

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- In how many ways?
- Obstructions?

Contents

I. Enhancements of triangulated categories

- Triangulated categories
- Algebraic enhancements
- Topological enhancements
- ‘Exotic’ example

Contents

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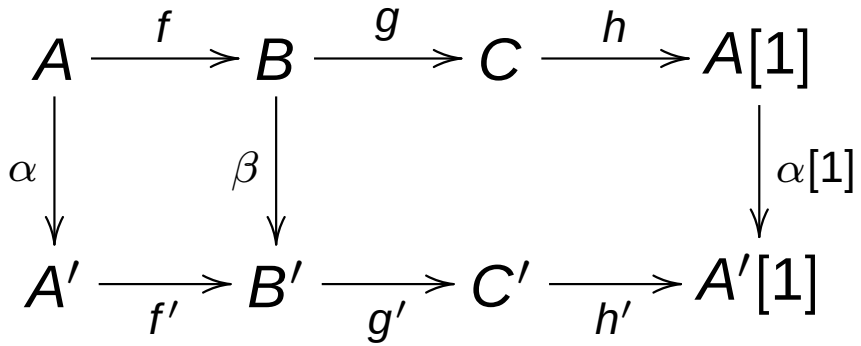
Triangulated categories

- every commutative diagram with exact rows



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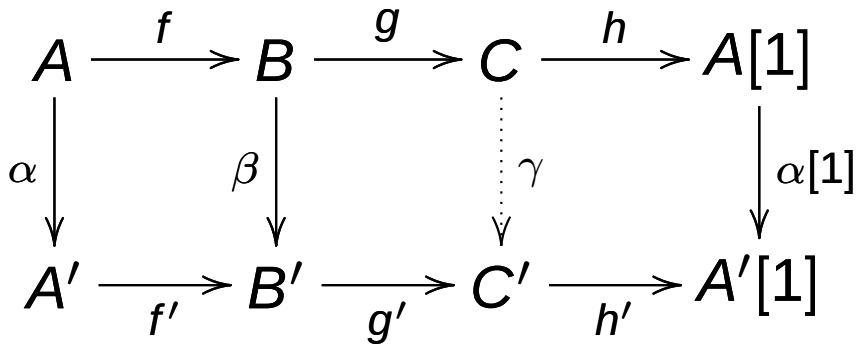

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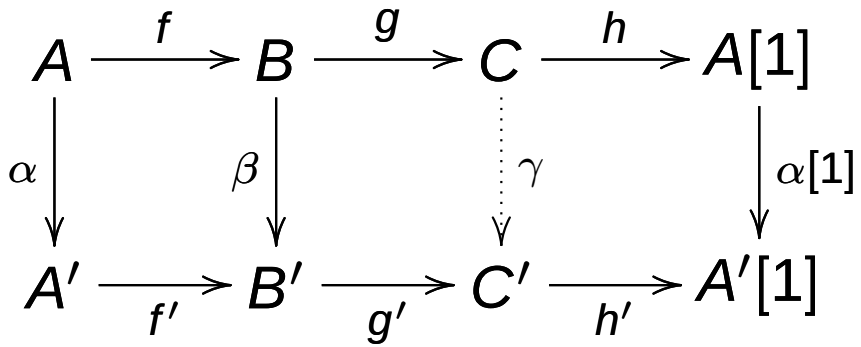

has a completion $\gamma : C \longrightarrow C'$ whose mapping cone

$$A' \oplus B \xrightarrow{\begin{pmatrix} f' & \beta \\ 0 & -g \end{pmatrix}} B' \oplus C \xrightarrow{\begin{pmatrix} g' & \gamma \\ 0 & -h \end{pmatrix}} C' \oplus A[1] \xrightarrow{\begin{pmatrix} h' & \alpha[1] \\ 0 & -f[1] \end{pmatrix}} A'[1] \oplus B[1]$$

is also exact.

Degenerate example: k-vector spaces

Any object C in an exact triangle

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is called a cone of f.

The cone C is a 'twisted version' of kernel and cokernel of f; it measures the deviation from f being an isomorphism.

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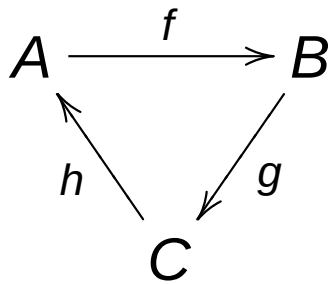

with $\operatorname{im}(f) = \ker(g)$,
$\operatorname{im}(g) = \ker(h)$ and
$\operatorname{im}(h) = \ker(f)$.

Algebraic example: $\mathbf{K}(A\text{-mod})$

For a ring A, the homotopy category $\mathbf{K}(A\text{-mod})$ has

- objects : $\mathbb{Z}$-graded chain complexes of A-modules

$$\cdots \xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \quad ; \; d \circ d = 0$$

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- shift : $C[1]_n = C_{n-1}$
- exact triangles: mapping cone sequences

$$C \xrightarrow{f} D \xrightarrow{\text{incl.}} \text{Cone}(f) \xrightarrow{\text{proj.}} C[1]$$

where $\text{Cone}(f) = D \oplus C[1]$, differential = $\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$

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- (k-vector spaces) for every field k
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- $K_{(p)}$-local stable homotopy category, p odd prime (Franke)

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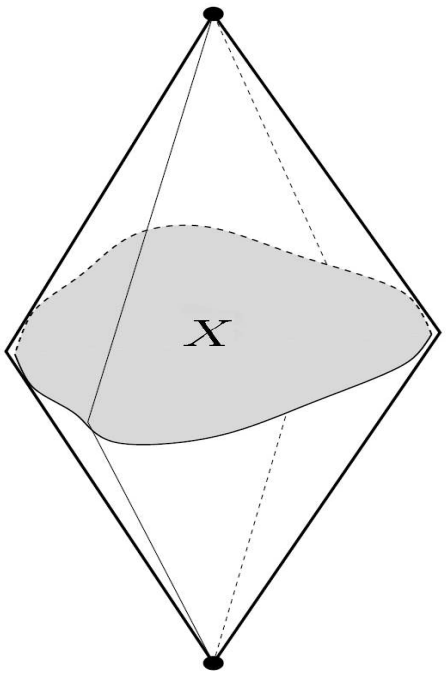
morphisms:

$$\mathcal{SW}((X,n),(Y,m)) \;=\; \mathrm{colim}_{k\to\infty}[\Sigma^{k+n}X, \Sigma^{k+m}Y]$$



$[-,-]$: pointed homotopy classes

$\Sigma X = \frac{X\times[0,1]}{X\times\{0,1\}\cup\{*\}\times[0,1]}$
reduced suspension

Triangulation of Spanier-Whitehead category

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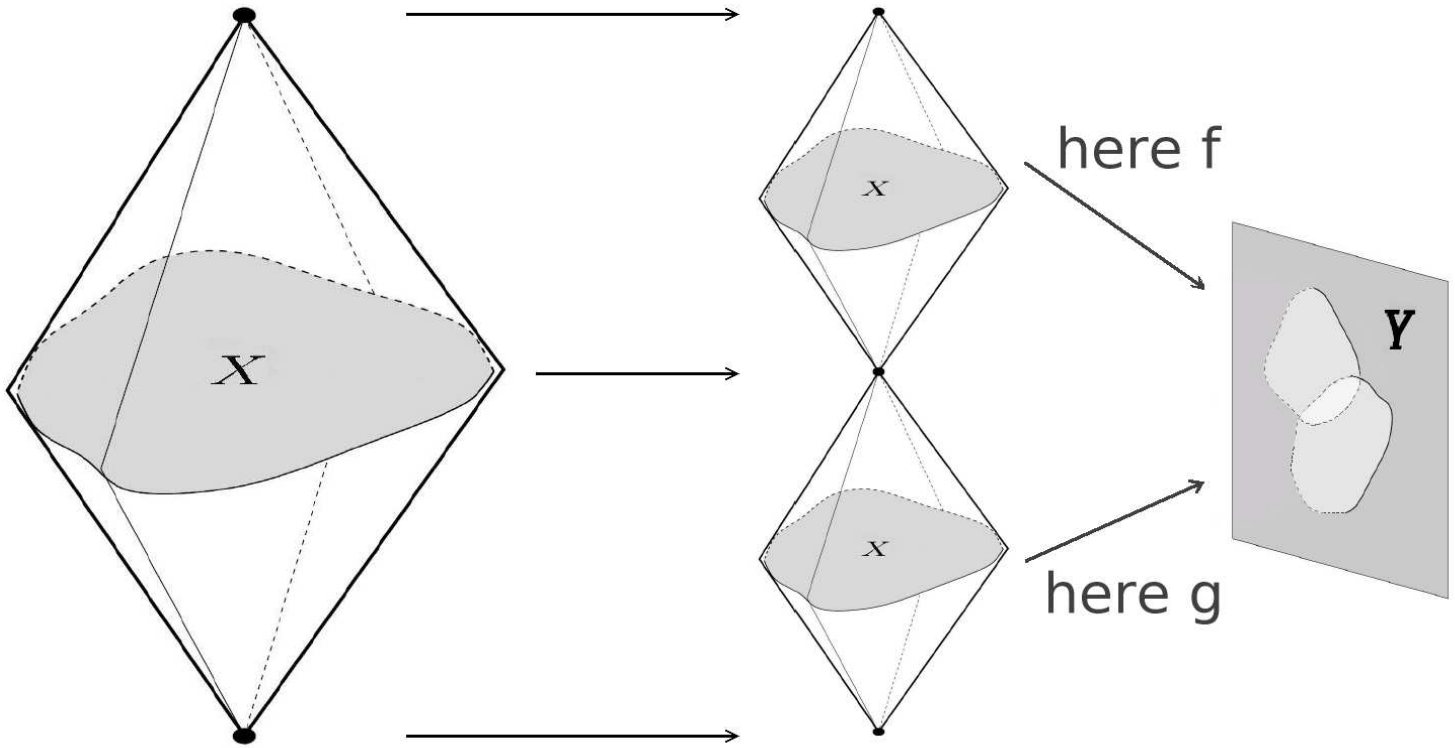

sum in $[\Sigma X, Y]$ of $f, g : \Sigma X \longrightarrow Y$

Triangulation of Spanier-Whitehead category

- exact triangles: mapping cone sequences

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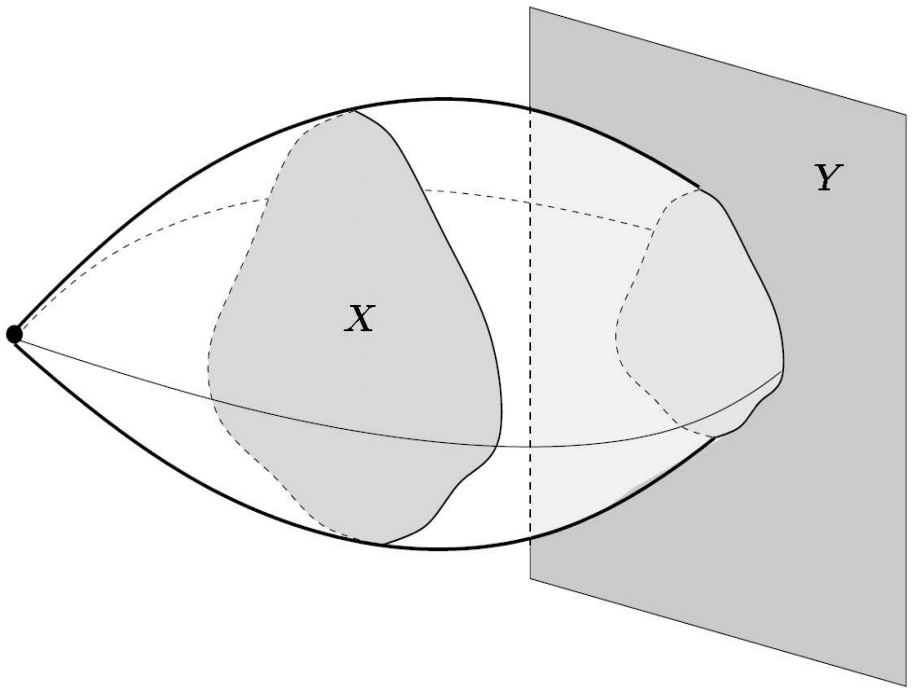

mapping cone

$$\mathrm{Cone}(f) = \frac{X \times [0,1] \cup_{X \times \{1\}} Y}{X \times \{0\} \cup \{x_0\} \times [0,1]}$$

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- homotopy category of stable infinity category (quasi-category)

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- all algebraic triangulated categories are also topological

‘Exotic’ example: $\mathbf{F}(\mathbb{Z}/4)$

$\mathbf{F}(\mathbb{Z}/4)$: finitely generated free modules over $\mathbb{Z}/4$

Example (Muro)

The category $\mathbf{F}(\mathbb{Z}/4)$ admits a unique triangulation with identical shift functor such that the triangle

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$\mathbf{F}(\mathbb{Z}/4)$ is neither algebraic nor topological

[Skip Hopf map]

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Proof.

In the 'universal example', the sphere spectrum in $\mathcal{SW}$, the class of the Hopf map $\eta : S^3 \longrightarrow S^2$ is a Hopf map. □

Hierarchy of enhancements

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A triangulated category **T** is rigid if:
given stable model categories $\mathcal{M}$, $\mathcal{M}'$ such that
$\mathbf{Ho}(\mathcal{M}) \cong \mathbf{T} \cong \mathbf{Ho}(\mathcal{M}')$ as triangulated categories,
then $\mathcal{M}$ and $\mathcal{M}'$ are Quillen equivalent as model categories.

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Interesting special case: 'rigid ring spectra' R, i.e., such that

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Theorem ('Tilting theory'; Rickard, Keller)

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Topological formulation:
the Eilenberg-Mac Lane ring spectrum HA is rigid

Finding rigid ring spectra

Topological proof:
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- so R is rigid

First example

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The stable homotopy category is rigid.

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Key ingredients:

- $\pi_*\mathbb{S}$ is 'generated' by η, ν, σ and α_1
- $\alpha_1\beta_1^{p-1} \in \langle p, \beta_1, p, \ldots, \beta_1, p\rangle$ ($(2p-1)$-fold bracket)

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More rigidity candidates: $ko_{(2)}$, $KO_{(2)}$, *tmf* and *TMF* (at $p = 2, 3$)

Local rigidity?

Theorem (Bousfield, Franke)

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Key ingredients:

- telescope conjecture
- periodicity
- $\pi_*(L_{K_{(2)}}\mathbb{S})$ is 'generated' by η, ν, σ
- all Toda brackets that could happen do happen

L_n-local stable homotopy

$E(n)$: Johnson-Wilson theory

$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, Honda formal group law

L_n-local: Bousfield localization with respect to $E(n)$

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Franke's 'exotic' model: twisted-periodic cochain complexes of $E(n)_*E(n)$-comodules

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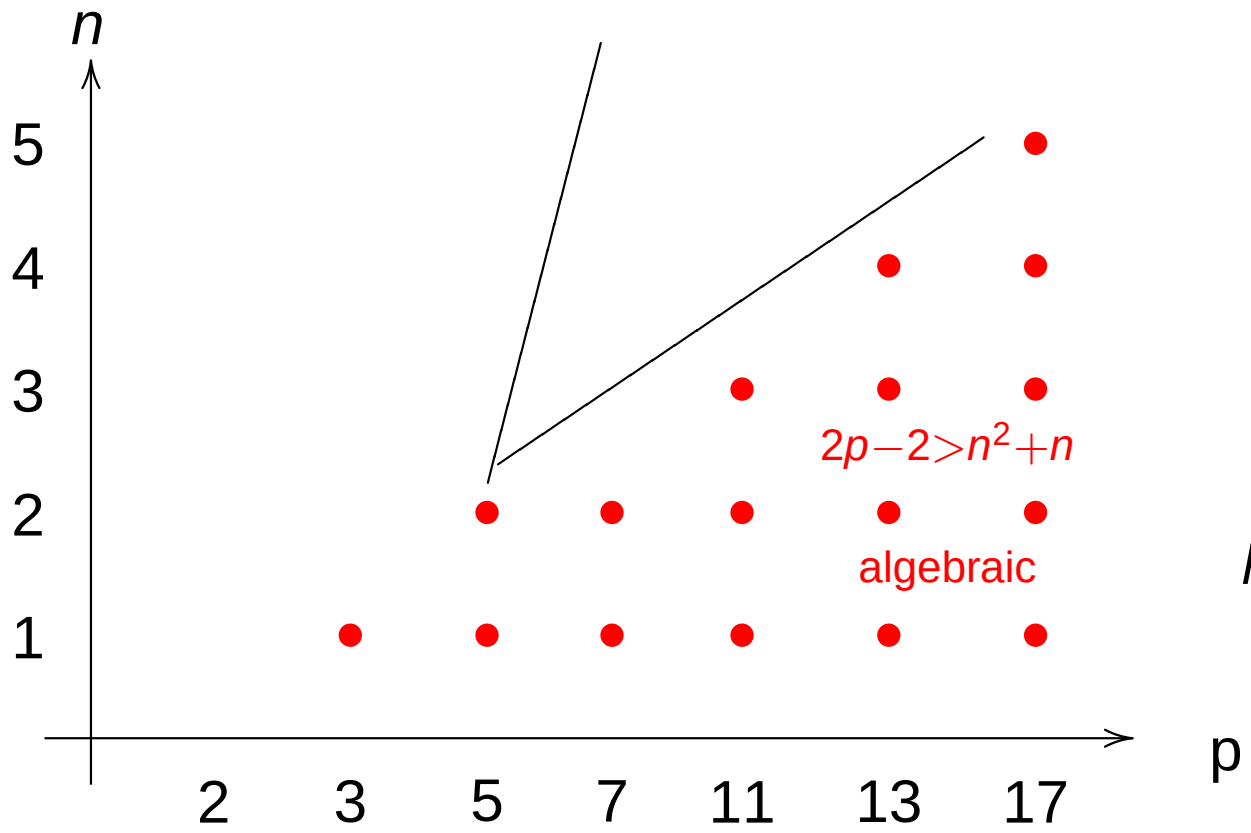

p 'large' $\Rightarrow$ algebraic

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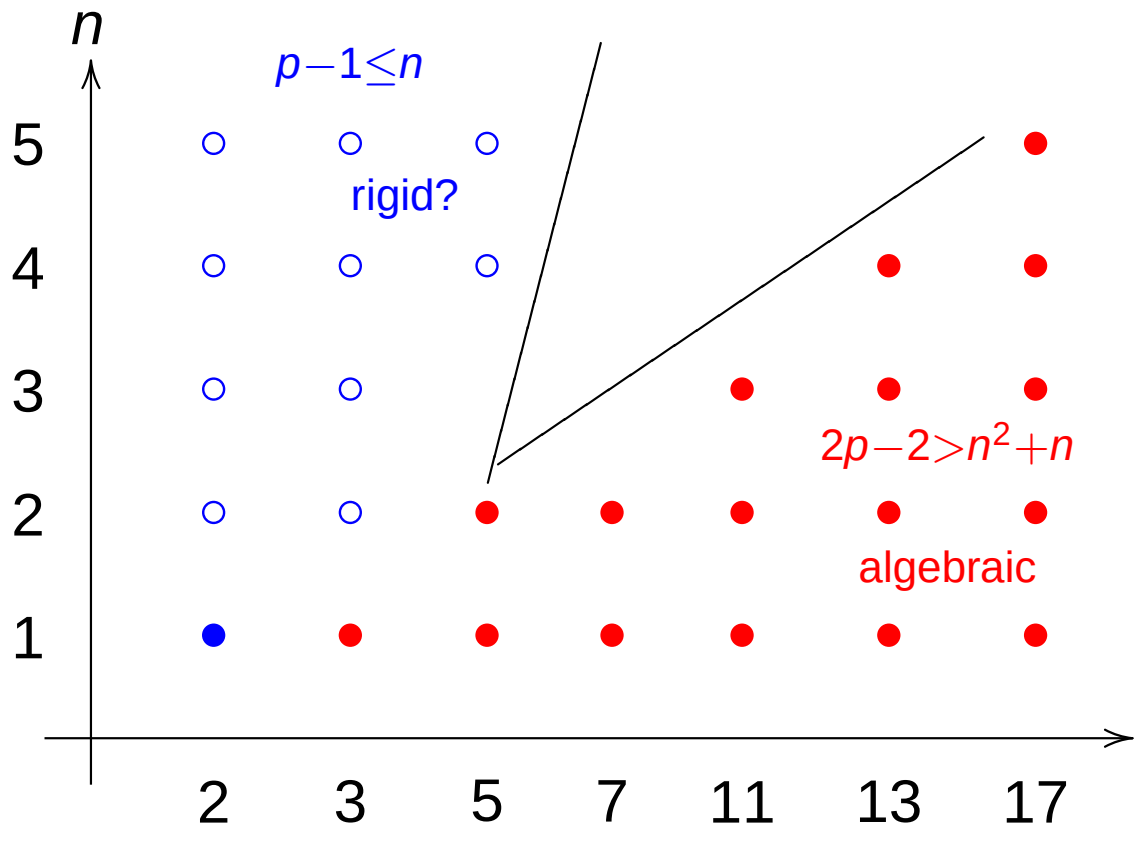

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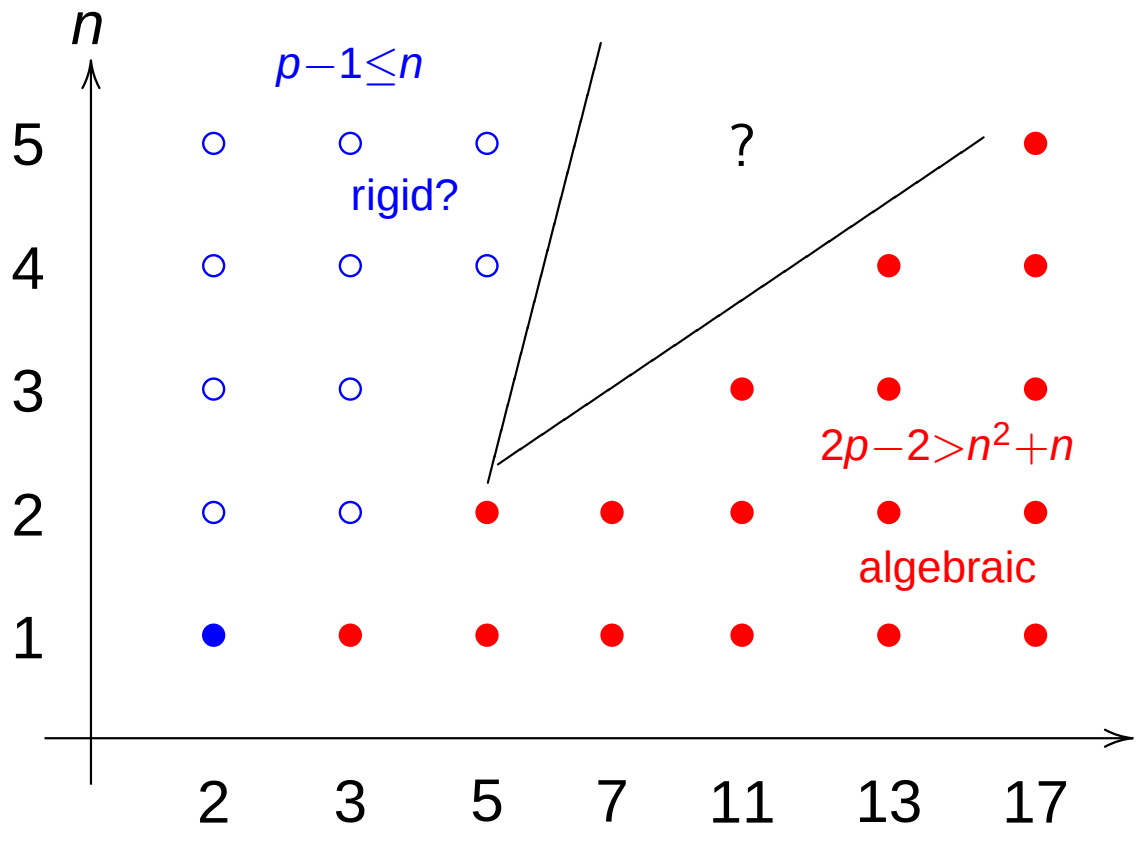

p 'small' $\Rightarrow$ rigid

p 'medium' $\Rightarrow$???

p 'large' $\Rightarrow$ algebraic