Massey products and classifying spaces of 2-groups

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Cohomology of groups

Structure in the cohomology of groups. Let G be a finite p-group, consider $H^*(G, \mathbb{F}_p)$:

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2. graded $\mathbb{F}_p$-algebra.
3. Algebra over the Steenrod algebra.
4. Higher Bockstein.
5. Massey products.

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Questions

If X is a topological space (p-completed and having the homotopy type of a CW-complex) having the same cohomology of BG, can we say $X \simeq BG$?
Which structures should we consider?

Introduction

Counterexample (with conditions 1,2 and 3)

Consider the dihedral family:

$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle \quad (n \geq 3)$$

and the cohomology (coefficients in $\mathbb{F}_2$):

$$H^*(BD_{2^n}) \cong \mathbb{F}_2[x, y, w]/(x^2 + xy)$$

where $\deg(x) = \deg(y) = 1$ and $\deg(w) = 2$.
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$\mathrm{Sq}^1(w) = wy$ and unstability.
The cohomology does not depend on n, and $BD_{2^n} \not\simeq BD_{2^m}$ when $n \neq m$.

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H^*BG is generated by H^1BG when considering (matric) Massey product, even if the Steenrod Algebra action is not considered!

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Definition

A defining system for the Massey product of order n $\langle \alpha_1, \ldots, \alpha_n \rangle$, where $\alpha_i \in H^*X$, is a matrix

$$M = \{m_{i,j} \mid 1 \leq i \leq n+1, i < j \leq n+1, (i,j) \neq (1, n+1)\},$$

where coefficients are in $C^*(X, \mathbb{F}_p)$ such that $m_{i,i+1}$ represents α_i and

$$dm_{i,j} = \sum_{k=i+1}^{j-1} m_{i,k} \cup m_{k,j} \ (j \neq i+1).$$

The value of $\langle \alpha_1, \ldots, \alpha_n \rangle$ relative to M is the element in H^*X represented by the cocycle:

$$\langle \alpha_1, \ldots, \alpha_n \rangle_M \stackrel{\text{def}}{=} \sum_{k=2}^{n} m_{1,k} \cup m_{k,n+1}$$

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3. If $f \colon Y \to X$ is a continuous map and $\alpha_1, \dots, \alpha_k \in H^*(X, R)$ such that $\langle \alpha_1, \dots, \alpha_k \rangle$ is defined, then $\langle f^*(\alpha_1), \dots, f^*(\alpha_k) \rangle$ is so and

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Problem

There are very few descriptions of H^*BG in terms of Massey products, none of them is complete.

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Then we have a central extension:

$$Z(\mathbb{F}_p, n) \to U(\mathbb{F}_p, n) \to \overline{U}(\mathbb{F}_p, n) \stackrel{\text{def}}{=} U(\mathbb{F}_p, n)/Z(\mathbb{F}_p, n)$$

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Given a group morphism

$$\begin{array}{rccc} \phi\colon & G & \to & U(\mathbb{F}_p, n) \\ & g & \mapsto & (\phi_{i,j}(g)) \end{array}$$

we have that $\phi_{i,i+1}(g_1 g_2) = \phi_{i,i+1}(g_1) + \phi_{i,i+1}(g_2) \in \mathbb{F}_p$.

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Massey products: computation tools

Theorem (Dwyer)

Let $\alpha_1, \ldots, \alpha_n \in H^1 BG$. Then there is a bijective correspondence $M \leftrightarrow \phi_M$ between defining systems M of $\langle \alpha_1, \ldots, \alpha_n \rangle$ and group morphisms $\phi_M \colon G \to \overline{U}(\mathbb{F}_p, n+1)$ such that $-\alpha_1, \ldots, -\alpha_n$ are near-diagonal of ϕ_M. Moreover, the pull-back diagram:

$$\begin{array}{ccccc} \mathbb{F}_p & \longrightarrow & \widetilde{G} & \longrightarrow & G \\ \| & & \downarrow & p.b. & \downarrow \phi_M \\ \mathbb{F}_p & \longrightarrow & U(\mathbb{F}_p, n+1) & \longrightarrow & \overline{U}(\mathbb{F}_p, n+1) \end{array}$$

identifies $\langle \alpha_1, \ldots, \alpha_n \rangle_M$ with the cohomology class that classifies the central extension of G by $\mathbb{F}_p$.

Massey products: computation tools

Remark

We cannot use Dwyer's result to calculate all Massey products in H^*BG, only those involving elements in H^1BG.

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In order to calculate Massey products that involve elements in $H^{>1}BG$ one has to work with projective resolutions and use Yoneda's complex.

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There are exact sequences (central extensions indeed) that connect these groups.

Dihedral groups

Theorem

$H^*BD_{2^n} \cong \mathbb{F}_2[x_1, y_1, w_2]/(x^2 + xy)$ and the Massey product $\langle x, x+y, \ldots, x, x+y \rangle$ of length m:

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- *is not defined if $m > 2^{n-1}$.*

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$H^*(BQ_8) \cong \mathbb{F}_2[x_1, y_1, v_4]/(x^2 + xy + y^2, x^2y + xy^2)$ and $H^*(BQ_{2^n}) \cong \mathbb{F}_2[x_1, y_1, v_4]/(x^2 + xy, y^3)$ for $n \geq 4$. The Massey product $\langle x, x + y, \dots, x, x + y \rangle$ of length m:

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Moreover $\langle y, y^2, y, y^2\rangle$ contains v.

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- *$\langle x, x^2, y\rangle = u + \mathbb{F}_2\{xy^2, y^3\}$.*

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Theorem

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Assume $|G| = 2^n$ and $\pi\colon G \to D_{2^{n-1}}$ the projection of G on the biggest dihedral quotient.

- Given any $f\colon X \to BD_{2^k}$, then either $f^*(w) = \pi^*(w)$ or $f^*(w) = 0$.

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- It has to stop at G (again length of Massey products).

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- Given any $f\colon X \to BD_{2^k}$, then either $f^*(w) = \pi^*(w)$ or $f^*(w) = 0$.
- Let $\phi_2\colon X \to BD_4$ the map that represents the classes x and y.
- Inductively we can lift ϕ_2 to a $\phi_{k-1}\colon X \to BD_{2^{k-1}}$ $(k \leq n)$.
- The process must stop since Massey products of type $\langle x, x+y, \dots, x, x+y\rangle$ in H^*X are defined till some length.
- It has to stop at G (again length of Massey products).
- It is an isomorphism in cohomology.