Smooth actions of finite nonsolvable groups on spheres

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Theorem

Let G be a finite group not of prime power order. Then for any integer $k \geqslant 2$, there exists a smooth action of G on a disk with exactly k fixed points at which the tangential representations of G are not isomorphic to each other if and only if $k \equiv 1 \pmod{n_G}$ and $r_G \geqslant 2$.

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The integer n_G denotes the Oliver number of G. Recall that $n_G \geqslant 0$. Moreover, $n_G = 0$ if and only if there exists a normal subgroup P of G of order p^n for $n \geqslant 0$, such that G/P is cyclic.

Actions on spheres - Main Theorem

This is a joint work with Toshio Sumi.

Main Theorem (Pa.–Sumi 2009)

Let G be a finite nonsolvable group with $r_G \geqslant 2$, $G \ncong \mathrm{Aut}(A_6)$. Then for any integer $k \geqslant 2$, the following conclusions are true.

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- *If $k \geqslant 3$, the condition $r_G \geqslant 2$ is necessary and sufficient for the existence of such actions of G.*

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According to Morimoto (2006), if $G = \mathrm{Aut}(A_6)$ and $k = 2$, the two tangential representations of G are always isomorphic to each other. Recall $r_G = 2$ for $G = \mathrm{Aut}(A_6)$.

Smith's question

In 1960, Paul Althaus Smith posed the following question.

Smith Isomorphism Question

If a finite group G acts smoothly on a sphere with exactly two fixed points, is it true that the tangential representations of G at the two fixed points are always isomorphic to each other?

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In the real representation ring $RO(G)$, consider *the Smith set of G*,

$$Sm(G) := \{V_1 - V_2 \in RO(G) \mid V_i \cong T_{x_i}(S),\ i = 1, 2\}$$

for a smooth action of G the n-sphere S with $S^G = \{x_1, x_2\}$ and $n = \dim V_1 = \dim V_2$, where $V_i \cong T_{x_i}(S)$ as representations of G.

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Clearly, $V - V = 0 \in Sm(G)$ for every V. As $V_1 \cong V_2$ if and only if $V_1 - V_2 = 0$ in $RO(G)$, the Smith Isomorphism Question can be restated as follows: Is it true that $Sm(G) = 0$?

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- $Sm(\mathbb{Z}_p) = 0$ for any prime p (Atiyah and Bott).
- $Sm(\mathbb{Z}_{p^n}) = 0$ for any odd prime p, $n \geqslant 1$ (Sanchez).
- $Sm(S_3) = 0$, $Sm(\mathbb{Z}_n) = 0$ for $n \in \{2, 4, 6\}$ (character theory).

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- $Sm(\mathbb{Z}_n) \neq 0$ for $n = 4q$, $q \geqslant 2$. In particular, $Sm(\mathbb{Z}_8) \neq 0$ (Cappell and Shaneson, Petrie).
- $Sm(G) \neq 0$ for finite abelian groups G of odd order, which have four or more noncyclic Sylow subgroups (Petrie).
- $Sm(G) \neq 0$ for large families of cyclic groups G of odd order (Dovermann and Petrie).

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Problem

If $Sm(G) \neq 0$, is it true that $Sm(G)$ is a subgroup of $RO(G)$? If yes, what is the rank of $Sm(G)$? Find a subgroup of $RO(G)$ *'as big as possible'* contained in $Sm(G)$ and compute its rank.

Subgroups of $RO(G)$

Let G be a finite group. Define $PO(G) \leqslant RO(G)$ by setting

$$PO(G) = \{V_1 - V_2 \mid V_1 \cong_P V_2 \text{ for all } P \leqslant G\}$$

where P varies over all subgroups of G of prime power order.

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Let $d^G \colon RO(G) \to \mathbb{Z}$ be defined as follows: $d^G(V) = \dim V^G$. Then $\mathrm{Ker}\, d^G = \{V_1 - V_2 \mid \dim V_1^G = \dim V_2^G\}$.

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For $H \trianglelefteq G$, define $PO(G,H) \leqslant PO(G) \cap \mathrm{Ker}\, d^G$ by setting

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Lemma

For $H \trianglelefteq G$ and $K \trianglelefteq G$ with $H \leqslant K$, the following holds

$$PO(G, H) \leqslant PO(G, K) \leqslant PO(G, G) = PO(G) \cap \operatorname{Ker} d^G.$$

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Let $\mathcal{L}(G)$ denote the family of large subgroups of G. Set

$$LO(G) = \{V_1 - V_2 \mid V_i^H = 0 \text{ for all } H \in \mathcal{L}(G),\ i = 1, 2\}.$$

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Set $r_{(G,H)} = r_G - s_{G/H}$. As $r_{G/H} \leqslant s_{G/H} \leqslant r_G$ for any $H \trianglelefteq G$,

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For $H = G$, one gets $r_{(G,G)} = r_G - s_{G/G}$ and $s_{G/G} = 1$ for $r_G \geqslant 1$, and thus $r_{(G,G)} = r_G - 1$ for $r_G \geqslant 1$. Clearly, $r_{(G,G)} = 0$ for $r_G = 0$. If H is trivial, $s_{G/H} = r_G$ and thus $r_{(G,H)} = 0$.

Rank computations

The following two conclusions are true about the number $r_{(G,G)}$:

- $r_{(G,G)} = 0$ for $r_G = 0$ or 1.
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Lemma (Laitinen–Pa. 1999)

For a finite group G,

- $PO(G,G) = 0$ *for* $r_{(G,G)} = 0$.
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Oliver groups

In his Ph.D. thesis (1974), Bob Oliver proves that a finite group G has a smooth action on a disk without fixed points if and only if there does not exist any series of normal subgroups $P \trianglelefteq H \trianglelefteq G$, where P is a p-group, G/H is a q-group, and H/P is cyclic for two primes p and q, possibly $p = q$.

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- G has a smooth action on a disk without fixed points.
- G has a smooth action on a disk with two fixed points.
- G has a smooth action on a sphere with one fixed point.
- G is an Oliver group.

Three conjectures

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Schultz (1985)

- $Sm(G) \neq 0$ for any finite Oliver group G.

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Laitinen (1996)

- $Sm(G) \neq 0$ for any finite Oliver group G with $r_G \geqslant 2$.

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The sufficiency of $r_G \geqslant 2$

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- *G is a finite nonsolvable group with $r_G \geqslant 2$, and G is not isomorphic to $P\Sigma L(2,27)$ or $\mathrm{Aut}(A_6)$.*
- *G is a finite Oliver group satisfying the G^{nil}-condition (and thus $r_G \geqslant 2$).*

Gap condition and gap groups

Let G be a finite group. Consider

- the family $\mathcal{P}(G)$ of prime power order subgroups of G
- the family $\mathcal{L}(G)$ of large subgroups of G.

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Definition

A finite group G is called a *gap group* if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and there exists a real representation V of G such that

- V satisfies the *gap condition*: $\dim V^P > 2 \dim V^H$ for all subgroups $P < H \leqslant G$ with $P \in \mathcal{P}(G)$,
- V is an $\mathcal{L}(G)$*-free representation*: $V^H = 0$ for all $H \in \mathcal{L}(G)$.

Weak gap condition

Let G be a finite group and let V be a real representation of G. Set $d_V(P, H) = \dim V^P - 2\dim V^H$ for all subgroups $P < H \leqslant G$ with $P \in \mathcal{P}(G)$, the family of prime power order subgroups of G.

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Definition

We say that V satisfies the *weak gap condition* if $d_V(P,H) \geqslant 0$ for all subgroups $P < H \leqslant G$, and the following holds.

(1) If $d_V(P,H) = 0$ for some $P < H \leqslant G$, $[H:P] = 2$ and

- $\dim V^H > \dim V^K + 1$ for all K such that $H < K \leqslant G$,
- V^H is oriented in such a way that the map $g\colon V^H \to V^H$ is orientation preserving for any $g \in N_G(H)$.

(2) If $d_V(P,H) = d_V(P,H') = 0$ for some $P < H, H' \leqslant G$, then $\langle H, H' \rangle \notin \mathcal{L}(G)$, the family of large subgroups of G.

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Let $LO(G)^{\mathrm{wg}}$ consist of $V_1 - V_2 \in RO(G)$ such that $V_i^H = 0$ for all $H \in \mathcal{L}(G)$, and V_i satisfies the weak gap condition for $i = 1, 2$.

Algebraic Theorem

Definition

Let G be a finite group. For $H \leqslant G$, we say that G satisfies the *H-condition* if G has two elements x and y such that $xH = yH$ and the following two conclusions hold.

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We shall deal with finite Oliver groups G satisfying the H-condition for $H = G^{\mathrm{nil}}$ or G^{sol}.

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Lemma (Pa.–Sumi 2009)

Let G be a finite Oliver group.

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Algebraic Theorem (Pa.–Sumi 2009)

Let G be a finite Oliver group. If G satisfies the G^{nil}-condition, then $PO(G, G^{\mathrm{nil}}) \cap LO(G)^{\mathrm{wg}} \neq 0$.

Topological Theorem

The Topological Theorem describes some conditions under which an element of $RO(G)$ can be realized as the difference of two real Smith equivalent representations of G.

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Semi-direct products

For an integer $q = p^k$ with p prime, $k \geqslant 1$, let $\mathrm{Aut}(\mathbb{F}_q)$ be the group of all automorphisms of the field $\mathbb{F}_q$ of q elements.

The group $\mathrm{Aut}(\mathbb{F}_q)$ acts on $SL(n,q)$ and $PSL(n,q)$.

Definition

$$\Sigma L(n,q) := SL(n,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$$

$$P\Sigma L(n,q) := PSL(n,q) \rtimes \mathrm{Aut}(\mathbb{F}_q).$$

Nonsolvable groups

The two nonsolvable groups $G = P\Sigma L(2,27)$ and $\mathrm{Aut}(A_6)$, both with $r_G = 2$, do not satisfy the G^{nil}-condition. We wish to recall that $P\Sigma L(2,27)$ is a gap group while $\mathrm{Aut}(A_6)$ is not.

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- G satisfies the G^{sol}-condition.
- G satisfies the G^{nil}-condition.

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As $r_G \geqslant 2$, G satisfies the G^{nil}-condition by the proposition, and thus $PO(G, G^{\mathrm{nil}}) \cap LO(G)^{\mathrm{wg}} \neq 0$ by the Algebraic Theorem. Therefore, the Topological Theorem completes the proof of Main Theorem.

Simple nonabelian groups

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In this corollary, just four groups G have elements of order 8, namely $PSL(2,17)$, $PSL(3,3)$, M_{11}, and M_{22}, and in any case, $\dim V^g > 0$ for any irreducible representation V of G and any element $g \in G$ of order 2^n for $n \geqslant 3$.

Simple nonabelian groups

Definition

Let G be a finite group and let p be a prime. Two representations V_1 and V_2 of G are called

- *p-matched* if $V_1 \cong V_2$ when restricted to any cyclic subgroup of G of order p^n for $n \geqslant 1$.
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Any two Smith equivalent real representations V_1 and V_2 of G are:

- p-matched (p odd prime) by Atiyah–Bott 1968, Sanchez 1970,
- isomorphic when restricted to any cyclic subgroup of G of order 2 or 4, by elementary character theory arguments.

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Specific classes of groups

Atiyah–Bott 1968: $Sm(G) = 0$ for a finite abelian simple group G.

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The conclusion that $Sm(G) \neq 0$ if and only if $r_G \geqslant 2$ is true in either of the following cases.

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Pa.–Sumi 2009: $Sm(G) = 0$ and $r_G = 2$ for $G = \mathrm{Aff}(2, 3)$.