KK-equivalences and T^{2n}-duality type isomorphisms

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July 6, 2009

Outline of the talk:

- Topological T-duality.
- KK-equivalence of C^*-algebras.
- KK-equivalence $\Rightarrow$ T^{2n}-duality type isomorphisms.

- Topological T-duality.

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Conjecture (Homological Mirror Symmetry - Kontsevich)

Y and Y' mirror dual CYs $\Rightarrow D^b(Coh(Y)) \cong Fuk(Y')$.

Strominger-Zaslow-Yau said that whenever Y and Y' are mirror dual one can find a (generically) torus-fibration over a common base Z

$$\begin{array}{ccc} Y & \overset{\text{fiberwise T-duality}}{\dashrightarrow} & Y' \\ & \searrow \quad \swarrow & \\ & Z & \end{array} .$$

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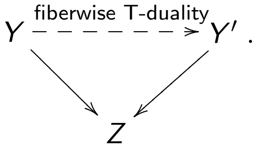

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Restricting to the topological sector we say that a topological homological T-duality transform is a DG functor between DG categories generalizing $D^b(Coh(Y))$ and $Fuk(Y')$ which changes (actually shifts) 'D-brane charges in a suitable manner'.

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Corr. between spaces $\rightsquigarrow$ Equiv. of DG categories.

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Definition

$A, B \in \mathtt{Sep}_{\mathrm{C}^*}$ are KK-equivalent iff they are isomorphic in $\mathtt{KK}_{C^*}$.

A functor $F : \mathtt{Sep}_{\mathtt{C}^*} \longrightarrow \mathcal{C}$ is a called C^*-stable (Morita invariant) if $F(\iota)$ is an isomorphism, where $\iota : A \to A \otimes \mathbb{K}$ is the corner embedding.

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Now let $\mathcal{C}$ be additive. We say F is <u>split exact</u> if whenever

$$0 \longrightarrow A \xrightarrow{\ i\ } B \underset{\curvearrowleft s}{\xrightarrow{\ j\ }} C \longrightarrow 0$$

is an exact sequence in $\mathtt{Sep}_{\mathrm{C}^*}$ one has $F(A) \cong F(I) \oplus F(A/I)$ in $\mathcal{C}$.

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Theorem (Higson)

The canonical functor $\mathtt{Sep}_{\mathrm{C}^*} \longrightarrow \mathtt{KK}_{C^*}$ is the universal C^*-stable and split exact functor on $\mathtt{Sep}_{\mathrm{C}^*}$. Such a functor is automatically homotopy invariant.

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Theorem (Cuntz)

$\mathtt{KK}_{C^*}(A, B) = [qA, B \otimes \mathbb{K}]$, *where [-,?] denotes homotopy classes of maps in* $\mathtt{Sep}_{\mathtt{C}^*}$. *Furthermore, Kasparov prod.* $=$ *comp. of* $*$*-homo.*

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$$
\begin{array}{ccccccc}
\mathbb{T} & \longrightarrow & (E, H) & \overset{\alpha}{\underset{BMRS}{\dashrightarrow}} & \hat{\mathbb{T}} & \longrightarrow & (\hat{E}, \hat{H}) \\
 & & \downarrow & & & & \downarrow \\
 & & M & & & & M
\end{array}
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where $\alpha \in \mathrm{KK}_1(CT(E, H), CT(\hat{E}, \hat{H}))$ is invertible.

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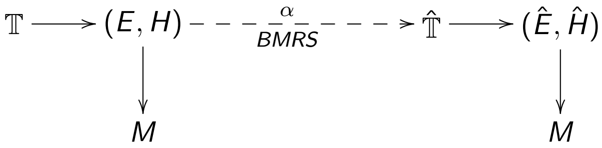

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Note: T-duality applied twice gives rise to an invertible KK_0-class.

- KK-equivalence $\Rightarrow$ T^{2n}-duality type isomorphisms.

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This category is enriched over monoids and we perform a naïve group completion of morphisms to obtain an additive category $\mathtt{NCC}^{\mathbf{K}}_{\mathrm{dg}}$ (noncommutative correspondence category).

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The category $\mathrm{Top}^{\mathbf{K}}_{\mathrm{fib}}(A)$ *is independent (up to homotopy) of the way in which one realizes* A *as a two sided ideal inside a unital* $\mathbb{C}$*-algebra* R*. Furthermore,* $\pi_0(\mathbf{K}(\mathrm{Top}^{\mathbf{K}}_{\mathrm{fib}}(A))) = \mathrm{K}_0(A)$*.*

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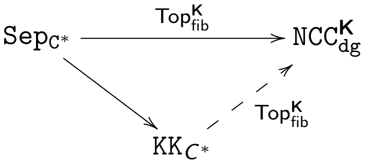
Theorem (-)

The functor $\mathsf{Top}^{\mathbf{K}}_{\mathsf{fib}} : \mathtt{Sep}_{C^*} \longrightarrow \mathtt{NCC}^{\mathbf{K}}_{\mathtt{dg}}$ *is* C^**-stable and split exact, i.e., one has*



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It seems that KK*-equivalences can account for them.*

Reference: Noncommutative correspondence categories, simplicial sets and pro C^*-algebras, *http://arxiv.org/abs/0906.5400*