Applications of Elliptic genera

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1.Elliptic genus in non singular case.

2.Quasi-Jacobi forms

3.Push-forward formula, Pairs, Orbifolds and DMVV.

4.Real algebraic varieties.

5.Chern numbers of singular varieties.

6. Non simply connected case

7.Other applications.

Elliptic genus:

$\{Class\ of\ complex\ spaces\} \rightarrow \{functions\ on\ H \times \mathbf{C}\}$

Non Singular Case: (X almost complex:) Let x_i be the Chern roots of X, i.e. for the total Chern class we have $c(X) = \prod_i (1 + x_i)$, then

$$Ell(X; y, q) = \int_X \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)}$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ and $y = \mathrm{e}^{2\pi \mathrm{i} z}$ and

$$\theta(z, \tau) =$$

$$q^{\frac{1}{8}}(2\sin \pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l \mathrm{e}^{2\pi \mathrm{i} z})(1 - q^l \mathrm{e}^{-2\pi \mathrm{i} z})$$

For $z = 0$ $(y = 1), q = 0 \Rightarrow \int_X x_1 \cdots x_d = e(X)$

$q = 0$:

$$x \frac{e^{\pi i(\frac{x}{2\pi i} - z)} - e^{-\pi i(\frac{x}{2\pi i} - z)}}{e^{\pi i(\frac{x}{2\pi i})} - e^{-\pi i(\frac{x}{2\pi i})}} = y^{-\frac{1}{2}} \frac{x(1 - e^{-x} y)}{1 - e^{-x}} \Rightarrow \chi_y$$

$y^{dim/2} Ell(X) \Rightarrow (y = 0) \quad \frac{x}{1 - e^{-x}}\ Todd \quad genus$

$Ell(X,q,y)$ is Poincare series of holomorphic euler charactersitic of bigraded bundle:

$$\mathcal{E}=\sum E_{i,j}\Rightarrow\sum\chi(E_{i,j})y^{\frac{i}{2}}q^{j}$$

Typical graded bundles:

$$E\Rightarrow\Lambda_t(E)=\sum\Lambda^i(E)t^i\ \ S_t(E)=\sum\mathrm{Sym}^i(E)t^i$$

Let

$$\mathcal{ELL}=$$

$$y^{-\frac{d}{2}}\otimes_{n\geq 1}(\Lambda_{-yq^{n-1}}\Omega^1_X\otimes\Lambda_{-y^{-1}q^n}T_X\otimes S_{q^n}\Omega^1_X\otimes S_{q^n}T_X)$$

By Riemann Roch:

$$Ell(X,y,q)=\chi(\mathcal{ELL})=\int_X\prod_i x_i\frac{\theta(\frac{x_i}{2\pi\mathrm{i}}-z,\tau)}{\theta(\frac{x_i}{2\pi\mathrm{i}},\tau)}$$

Element of Chow ring under integral is call elliptic class.

Elliptic genus is the top degree component of elliptic class in Chow ring $A(X)$

Specializes ($q = 0$) into χ_y and hence into topological euler characteristic, signature,... (χ_y is specialization of Batyrev's $E(u,v)$-function). If $z = \frac{1}{2}$ ($y = -1$) it becomes one variable elliptic genus (Ochanine genus) depending only of Pontryagin classes. Generating series:

$$Q(x) = \frac{x/2}{\sinh(x/2)} \prod_{n=1}^{\infty} [\frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}]^{(-1)^n}$$

Alternatively:

$$Q(x) = \frac{x}{g^{-1}(x)} \quad g(x) = \int_0^x \frac{dt}{\sqrt{1-2\delta t^2+\epsilon t^4}}$$

δ, ϵ are combinations of Eisenstein series.

$$\delta = \frac{-1}{8} - 3 \sum_{n \geq 1} (\sum_{d|n \ d \text{ odd}} d) q^n$$

$$\epsilon = \sum_{n \geq 1} (\sum_{d|n, \frac{n}{d} \text{ odd}} d^3) q^n$$

$g(x)$-is logarithm of formal group associated with elliptic genus.

Totaro's theorem: The kernel of complex elliptic genus on $MSU \otimes \mathbf{Z}[\frac{1}{2}]$ is the ideal generated by $X_1 - X_2$ where X_1 and X_2 are related by classical SU-flop.

$$\begin{tikzcd}
& \hat{X} \arrow[dl] \arrow[dr] & \\
X_1 \arrow[dr] & & X_2 \arrow[dl] \\
& X_0 &
\end{tikzcd}$$

Hodge and Chern numbers

If dimension of a Calabi-Yau manifold is less than 12 or is equal to 13, then the numbers χ_p determine its elliptic genus uniquely. In all other dimensions there exist Calabi-Yau manifolds with the same $\{\chi_p\}$ but distinct elliptic genera. Non trivial relation between Hodge and Chern numbers (L. -Wood):

$$\sum_{p=2}^{d} (-1)^p \binom{p}{2} \chi^p = \frac{1}{12}\{\frac{1}{2}d(3d-5)c_d + c_{d-1}c_1\}[X]$$

Modular properties:

Ochanine genus of a manifold is a modular form for $\Gamma_0(2)$ and for Spin manifolds for Γ_θ (subgroup of $SL_2(\mathbf{Z})$ of index 3).

Non modularity is similar to non integrality of $\widehat{A}$-genus in non Spin case.

A Jacobi form of index $t \in \frac{1}{2}\mathbf{Z}$ and weight k is a holomorphic function χ on $H \times \mathbf{C}$ satisfying the following functional equations:

$$\chi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{\frac{2\pi itcz^2}{c\tau+d}} \chi(\tau, z)$$

$$\chi(\tau, z+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)} e^{-2\pi it(\lambda^2\tau+2\lambda z)} \chi(\tau, z)$$

Ellitpic genus of a Calabi Yau manifold is Jacobi form of weigth 0 and index $\frac{\mathrm{dim} X}{2}$.

Ring of Jacobi forms is a finitely generated bigraded algebra.

Quasi-jacobi forms

In non CY case one has a “quasi-modular” Jacobi form.

Problem: Find a finite dimensional algebra generated of functions on $\mathbf{H} \times \mathbf{C}$ which are elliptic genera of complex manifolds.

Here is example of “not quite” Jacobi forms.

$$E_n(z,\tau) = \sum_{(a,b)\in \mathbf{Z}^2} \frac{1}{(z+a\tau+b)^n}$$

For $n \geq 3$ one has absolute convergence and hence Jacobi property (index zero, weight n)

One has:

$$E_1(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=(c\tau+d)E_1(\tau,z)+\frac{\pi ic}{2}z$$

$$E_1(\tau,z+m\tau+n)=E_1(\tau,z)-2\pi im$$

and

$$E_2(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=(c\tau+d)^2E_2(\tau,z)-\frac{1}{2}\pi ic(c\tau+d)$$

$$E_2(\tau,z+a\tau+b)=E_2(\tau,z)$$

Elliptic genera of complex manifolds (after multiplying by $(\frac{\theta'(0)}{\theta(z)})^d$) are combination of $E_i(z,\tau)$ and ordinary Eisenstein series:

$$e_i=\sum_{(a,b)\in\mathbf{Z}^2,(a,b)\neq(0,0)}\frac{1}{(a\tau+b)^n}$$

Characterization of elliptic genera:

Recall quasi-modular forms (for $SL_2(\mathbf{Z})$):

Algebra of quasi-modular forms is algebra $\mathrm{C}[e_2, e_4, .$ generated by Eisenstein series.

One has

$$e_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2 e_2(\tau) - \frac{1}{2}\pi i c(c\tau+d)$$

but

$$e_2(\tau) - \frac{1}{4\pi Im\tau}$$

transforms as modular form of weight 2.

Definition Quasi-modular form of weight k and depth p is constant term of polynomial in $\frac{1}{4\pi Im\tau}$ of degree at most p which transforms as modular form of weigth k.

Ring of Quasimodular forms is closed under differentiation.

Solutions to enumeration problems (branched covering of torus) etc.

We let:

$$\lambda(z,\tau)=\frac{z-\bar{z}}{\tau-\bar{\tau}}, \qquad \mu(\tau)=\frac{1}{\tau-\bar{\tau}}$$

These real analytic functions have the following transformation properties:

$$\lambda(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d})=(c\tau+d)\lambda(z,\tau)-2icz$$

$$\lambda(z+m\tau+n,\tau)=\lambda(z,\tau)+m$$

$$\mu(\frac{a\tau+b}{c\tau+d})=(c\tau+d)^2\mu(\tau)-2ic(c\tau+d)$$

Definition *Almost meromorphic Jacobi form* of weight k, index zero and depth (s,t) is a (real) meromorphic function in $\mathbf{C}\{q^{\frac{1}{l}},z\}[z^{-1},\lambda,\mu]$, with λ,μ given above and which

a) satisfies the functional equations of Jacobi forms of weight k and index zero and

b) has degree at most s in λ and at most t in μ.

Definition *A quasi-Jacobi form* is a constant term of an almost meromorphic Jacobi form of index zero considered as a polynomial in the functions λ, μ i.e. a meromorphic function f_0 on $\mathbf{H} \times \mathbf{C}$ such that exist meromorphic functions $f_{i,j}$ such that $f_0 + \sum f_{i,j}\lambda^i\mu^j$ is almost meromorphic Jacobi form.

Theorem The algebra of quasi-Jacobi forms of depth $(k, 0), k \geq 0$ is isomorphic to the algebra of complex unitary cobordisms modulo flops with isomorphism given by

$$X \to Ell(X)(\frac{\theta'(0)}{\theta(z)})^d$$

Elliptic genera of manifolds of dimension at most d span the subspace of forms of depth $(d, 0)$ in the algebra of quasi-Jacobi forms.

If a complex manifold satisfies $c_1^k = 0, c_1^{k-1} \neq 0$ then its elliptic genus has depth at most $k-1$. In particular if X is CY elliptic genus is Jacobi form: depth is measure of deviation from being CY.

One can get formulas for the elliptic genus of specific examples in terms of Eisenstein series E_n. For example for a surface in $\mathbf{P}^3$ having degree d one has

$$(E_1^2(\frac{1}{2}d^2-4d+8)d+(E_2-e_2)(\frac{d^2}{2}-2)d)(\frac{\theta(z)}{\theta'(0)}^2)$$

In particular for $d=1$ one obtains:

$$(\frac{9}{2}E_1^2 - \frac{3}{2}(E_2-e_2))(\frac{\theta(z)}{\theta'(0)})^2$$

For toric varieties one has formula in terms of fan ⇒ non trivial identity: for $\mathbf{P}^2$:

$$\sum_{m\geq 1, n\geq 1} \frac{q^{m+n}}{(1+q^m)(1+q^n)(1+q^{m+n})} =$$

$$\sum_{r\geq 1} q^{2r} \sum_{k|r} k = \sum_{r\geq 1} \sigma_1(r) q^{2r}$$

Singular elliptic genus.

X be a $\mathbf{Q}$-Gorenstein variety with log-terminal singularities,

$\pi : Y \to X$ a desingularization of X whose exceptional divisor $E = \sum_k E_k$ has simple normal crossings.

The discrepancies α_k of the components E_k are determined by the formula

$$K_Y = \pi^* K_X + \sum_k \alpha_k E_k.$$

Chern roots y_l of Y are given by $c(TY) = \prod_l (1+y_l)$ and define cohomology classes $e_k := c_1(\nu(E_k))$.

Singular elliptic genus of X is given by

$$Ell_{sing}(X; z, \tau) :=$$

$$\int_Y (\prod_l \frac{(\frac{y_l}{2\pi \mathrm{i}})\theta(\frac{y_l}{2\pi \mathrm{i}} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi \mathrm{i}})})\times$$

$$(\prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} - (\alpha_k + 1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} - z)\theta(-(\alpha_k + 1)z)})$$

If resolution is crepant then elliptic genus of singular space is elliptic genus of resolution. Need to prove independence of resolution!!

The same defintion for $Ell_{sing}(X, D)$ provided that meaning of α_k is

$$K_Y = \pi^*(K_X + D) + \sum_k \alpha_k E_k.$$

Specializes into Batyrev's $\chi_y(X, D)$:

$$\widehat{Ell}(X, D; u, q = 0) = (u^{-\frac{1}{2}} - u^{\frac{1}{2}})^{\mathrm{dim} Z} E_{st}(X, D; u, 1)$$

Independece of resolution and push forward formula:

In definition of elliptic genus of pair one can look at the class

$$\mathcal{E}ll(X, E, z, \tau) \in A^*(Z)$$

before evaluation on the fundamental class.

Theorem

Let (X, D) be a Kawamata log-terminal pair and let Z be a smooth locus in X which is normal crossing to $Supp(D)$. Let $f : \hat{X} \to X$ denote the blowup of X along Z. We define $\hat{E}$ by $\hat{E} = -\sum_k \delta_k \hat{E}_k - \delta Exc(f)$ where $\hat{E}_k$ is the proper transform of E_k and δ is determined from $K_{\hat{X}} + \hat{E} = f^*(K_X + E)$. Then $(\hat{X}, \hat{E})$ is a Kawamata log-terminal and

$$f_* \mathcal{E}ll(\hat{X}, \hat{E}, z, \tau) = \mathcal{E}ll(X, E, z, \tau).$$

Weak factorization shows independence of resolution (connect two resolutions by sequence of blow ups and blow downs).

Another type of genus for singular varieties: Orbifold elliptic genus:

Let G be finite group acting on X via holomorphic transformations.

Let $g, h \in G$ be a pair of commuting elements,

$X^{g,h}$ be a connected component of the set of points in X fixed by both g and h,

x_λ be the Chern roots of a subbundle V_λ of $TX|_{X^{g,h}}$ on which both g and h act via the multiplication by $\exp(2\pi\mathrm{i}\lambda(g))$ and $\exp(2\pi\mathrm{i}\lambda(h))$ respectively. Let:

$$\Phi(g, h, \lambda, z, \tau, x) :=$$

$$\frac{\theta(\frac{x}{2\pi\mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x}{2\pi\mathrm{i}} + \lambda(g) - \tau\lambda(h))} e^{2\pi\mathrm{i}z\lambda(h)z}.$$

Then:

$$E_{orb}(X, G; z, \tau) =$$

$$\frac{1}{|G|} \sum_{gh=hg} \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big) \prod_\lambda \Phi(g, h, \lambda, z, \tau, x_\lambda)[X^{g,h}]$$

Theorem(B-L) (Ann. Math. 2005)

$$Ell_{orb}(X,G) = Ell_{sing}(X/G)$$

Corollary: If X/G has a crepant resolution $\widetilde{X/G}$ then:

$$e(\widetilde{X/G}) = \frac{1}{|G|} \sum_{gh=hg} e(X^{g,h})$$

This is a Mckay correspondence for elliptic genus.

Idea of the Proof:

There is elliptic genus in the category of triples (X, E, G) (similarly to Batyrev's E-function) where X is non singular G-invariant, E is NCD and isotropy subgroup of G acts trivaially on the irredicuble components of E containing x.

$Ell(X, E.G)$ specializes to singular (if $G = 1$) and orbifold (if $E = \emptyset$) genera.

Consider diagramm:

$$\begin{tikzcd}
\mu\colon & \widehat{Z} \arrow[r] \arrow[d] & Z \arrow[d] \\
\psi\colon & X \arrow[r] & X/G
\end{tikzcd}$$

where the vertical arrows are resolutions of singularities and μ is a G-equivariant toroidal morphism.

Push forward formula for quotients:

Let $(X; D_X)$ be a Kawamata log-terminal pair which is invariant under an effective action of G on X. Let $\psi: X \to X/G$ be the quotient morphism. Let $(X/G; D_{X/G})$ be the quotient pair. Then

$$\psi_* \mathcal{E}ll_{orb}(X, D_X, G; z, \tau) = \mathcal{E}ll(X/G, D_{X/G}; z, \tau).$$

(Definition of quotient pair: Let G be a finite group which acts effectively on a normal variety X and preserves a $\mathbf{Q}$-Weil divisor D. Let $g: X \to X/G$ be the quotient morphism. Then there is a unique divisor D/G on X/G such that

$$g^*(K_{X/G} + D/G) = K_X + D.$$

This and push forward formula for μ yileds the McKay correspondence formula.

Real algebraic varieties

Theorem(Totaro) Quotient of MSO by ideal generated by oriented real flops and complex flops is

$$\mathbf{Z}[\delta, 2\gamma, 2\gamma^2, 2\gamma^4..]$$

with $\mathbf{CP}^2$ (resp. $\mathbf{CP}^4$) corresponding to δ (resp. $2\gamma + \delta^2$). This quotient ring is the the image of MSO_* under the Ochanine genus.

Problem: Can Ochanine genus be defined for large class of singular varieties.

Ochanine genus of an oriented manifold X can be defined using as Hirzebruch characteristic power series the following series with coefficients in $\mathbf{Q}[[q]]$

$$(*) \quad Q(x) =$$

$$\frac{x/2}{\mathrm{sinh}(x/2)} \prod_{n=1}^{\infty} [\frac{(1-q^n)^2}{(1-q^n\mathrm{e}^x)(1-q^n\mathrm{e}^{-x})}]^{(-1)^n}$$

Evaluating genus using viewing the result as function of τ on the upper half plane (where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$) one obtains a modular form on $\Gamma_0(2) \subset SL_2(\mathbf{Z})$

Class of singularities:

A real algebraic variety X over $\mathbf{R}$ is $\mathbf{Q}$-Gorenstein log-terminal if its set of $\mathbf{C}$-points is $\mathbf{Q}$-Gorenstein log-terminal.

Example Affine variety $x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0$ in $\mathbf{R}^4$ is Gorenstein log-terminal and admit a crepant resolution.

Indeed it is well known that complexification of such Gorenstein singularity admits a small (and hence crepant) resolution having $\mathbf{P}^1$ as its exceptional set.

Example The 3-dimensional complex cone in $\mathbf{C}^4$ given by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ considered as codimension 2 sub-variety of $\mathbf{R}^8$ is a $\mathbf{Q}$-Gorenstein log-terminal variety over $\mathbf{R}$ and its complexification admits a crepant resolution.

Definition: Let X be a real algebraic manifold and D a divisor on complexification $X_{\mathbf{C}}$ of X. The Ochanine class $Ell_{\mathcal{O}}(X,D)$ of pair X,D is

$$\sqrt{Ell(X_{\mathbf{C}},D,q,\frac{1}{2})}$$

where $Ell(X_{\mathbf{C}},D,q,z)$ is the elliptic class constructed above.

Ochanine elliptic genus of pair (X,D) is

$$Ell(X_{\mathbf{R}},D)=\sqrt{Ell(X_{\mathbf{C}},D,q,\frac{1}{2})}\cup cl(X_{\mathbf{R}})[X_{\mathbf{C}}]$$

For $D=0$ one obtains the Ochanine genus of real locus:

$$Ell_{\mathcal{O}}(X_{\mathbf{R}}) = Ell_{\mathcal{O}}(T_{X_{\mathbf{R}}})[X_{\mathbf{R}}] =$$

$$\sqrt{Ell_{\mathcal{O}}(T_{X_{\mathbf{C}}})}|_{X_{\mathbf{R}}})[X_{\mathbf{R}}] =$$

$$\sqrt{Ell_{\mathcal{O}}(T_{X_{\mathbf{C}}})} \cup cl(X_{\mathbf{R}})[X_{\mathbf{C}}]$$

which is consequence of the exact sequence:

$$0 \to T_{X_{\mathbf{R}}} \to T_{X_{\mathbf{C}}}|_{X_{\mathbf{R}}} \to T_{X_{\mathbf{R}}} \to 0$$

(it yields $Ell(X_{\mathbf{R}})^2 = i^*(Ell_{X_{\mathbf{C}}})$ where $i : X_{\mathbf{R}} \to X_{\mathbf{C}}$).

The main result is:

Theorem Let $\tilde{X} \to X$ is a resolution of singularities of a real algebraic variety with $\mathbf{Q}$-Gorenstein log-terminal singularities and $\tilde{D}$ is the discrepancy. Then the elliptic genus of a pair $(\tilde{X}, \tilde{D})$ is independent of a resolution. In particular if real variety X has a crepant resolution then its elliptic genus is independent of a crepant resolution.

Proof For a blow up $f : (\tilde{X}, \tilde{D}) \to (X, D)$ we have

$$f_*(\sqrt{\mathcal{ELL}(\tilde{X}, \tilde{D}, q, \frac{1}{2})} = \sqrt{\mathcal{ELL}(X, D, q, \frac{1}{2})}$$

This is a special case of the push-forward formula. Hence

$$\mathcal{E}ll_{\mathcal{O}}(X_{\mathbf{R}}, D) = \sqrt{\mathcal{E}ll(X_{\mathbf{C}}, D, q, \frac{1}{2})} \cup cl(X_{\mathbf{R}})[X_{\mathbf{C}}] =$$

$$\sqrt{\mathcal{ELL}(\tilde{X}_{\mathbf{C}}, \tilde{D}, q, \frac{1}{2})} \cup f^*([X_{\mathbf{R}}] \cap [X_{\mathbf{C}}]) = \mathcal{ELL}(\tilde{X}_{\mathbf{R}}, \tilde{D})$$

as follows from projection formula since $f^*(cl[X_{\mathbf{R}}])$ $[cl\tilde{X}_{\mathbf{R}}]$ and since f_* is identity on H_0.

Applications of Elliptic Genus

- Chern classes of singular spaces.

Problem(Goreski-Macpherson): Which Chern numbers can be defined for singular varieties so that for varieties admitting IH-small resolution they coinside with the Chern numbers of resolution?

(An IH-small resolution of Z is a regular map $Y \to Z$ such that for every $i \geq 1$ the set of points $z \in Z$ such that $\dim(f^{-1}(z)) \geq i$ has codimension greater than $2i$ in Z)

Totaro: such Chern numbers are among linear combinations of coefficitents of elliptic genus: Ω^U/classical flops=image of elliptic genus.

Theorem (B-L) All coefficients of elliptic genus are such invariants of $\mathbf{Q}$-Gorenstein singular spaces with log-terminal singularities. Singular elliptic genus yields maximal collection of such chern numbers.

- Examples of invariants of singular spaces in terms of resolutions.

Igusa-zeta or topological zeta function.

Elliptic genus.

- Invariants of K-equivalence.

X and Y (smooth) are K-equivalent if they are birationally equivalent and Z, $f : Z \to X$ and $g : Z \to Y$ such that $f^*K_X = g^*K_Y$.

Theorem (B-L) Elliptic genus is an invariant of K-equivalence.

Other new results:

1.Waelder's Equivariant elliptic genus and rigidity theorem for elliptic genera of pairs with torus action.

2.Gorbunov and Malikov LG-CY correspondence for elliptic genera for hypersurfaces in projective space.

- Elliptic genera of Hilbert schemes.

(X, D) be a Kawamata log-terminal pair.

The quotient of $(X, D)^n$ by the symmetric group S_n, is $(X^n/S_n, D^{(n)}/S_n)$.

($D^{(n)}$ is the sum of pullbacks of D under n canonical projections to X)

Theorem (generalization of Dijkgraaf-Moore-Verlinde-Verlinde in smooth case):

$$\sum_{n \geq 0} p^n Ell(X^n/S_n, D^{(n)}/S_n; z, \tau) =$$

$$\prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}},$$

where the elliptic genus of (X, D) is

$$\sum_{m \geq 0} \sum_{l} c(m, l) y^l q^m$$

and $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$.

Higher elliptic genus.

Let X be a manifold and $\pi = \pi_1(X)$, $\alpha \in H^*(\pi, \mathbf{Q})$

Higher elliptic genus:

$$Ell_\alpha(X) = (\mathcal{ELL}(X) \cup f^*(\alpha))[X]$$

where

$$\mathcal{ELL}(X) = \prod_i x_i \frac{\theta(\frac{x_i}{2\pi \mathrm{i}} - z, \tau)}{\theta(\frac{x_i}{2\pi \mathrm{i}}, \tau)}$$

is the elliptic class.

Ell_α specialzies to Novikov's signature and higher Todd genera.

J.Rosenberg: is higher Todd genus a birational invariant?

Modularity

If X is a SU-manifold, $d = \mathrm{dim} X, \alpha \in H^k(\pi, \mathbf{Q})$ then the higher elliptic genus $(\mathcal{ELL}(X) \cup f^*(\alpha))[X]$ is a Jacobi form having index $\frac{d}{2}$ and weight $-k$

(i.e. is a function χ on $H \times \mathbf{C}$ satisfying:

$$\chi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi itcz^2}{c\tau+d}} \chi(\tau, z)$$

$$\chi(\tau, z+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)} e^{-2\pi it(\lambda^2\tau+2\lambda z)} \chi(\tau, z)$$

Beauville: a fundamental group of a Calabi Yau manifold is an extension of a free abelian group by a finite group so one does obtain new invariants if the rank of this abelian group is positive.

Let (X, D) be a Kawamata log terminal G-normal pair and $D = -\sum \delta_k D_k$. The *orbifold* elliptic class of (X, D, G) is the class in $H_*(X, \mathbf{Q})$ given by:

$$\mathcal{ELL}(X, D, G; z, \tau) :=$$

$$\frac{1}{|G|} \sum_{g,h,gh=hg} \sum_{X^{g,h}} [X^{g,h}] \Big(\prod_{\lambda(g)=\lambda(h)=0} x_\lambda \Big)$$

$$\times \prod_\lambda \frac{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z)}{\theta(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h))} \mathrm{e}^{2\pi \mathrm{i}\lambda(h)z}$$

$$\times \prod_k \frac{\theta(\frac{e_k}{2\pi \mathrm{i}} + \epsilon_k(g) - \epsilon_k(h)\tau - (\delta_k + 1)z)}{\theta(\frac{e_k}{2\pi \mathrm{i}} + \epsilon_k(g) - \epsilon_k(h)\tau - z)}$$

$$\frac{\theta(-z)}{\theta(-(\delta_k + 1)z)} \mathrm{e}^{2\pi \mathrm{i}\delta_k \epsilon_k(h) z}.$$

$$Ell_\alpha(X, D, G) = (\mathcal{ELL}(X, D, G) \cap f^*(\alpha))_0$$

If $m(K_X + D) = 0$ it is Jacobi form.

Let (X, D, G) and $(\hat{X}, \hat{D}, G)$ be G-normal and Kawamata log-terminal and let $\phi : (\hat{X}, \hat{D}) \to (X, D)$ is G-equivariant such that

$$\phi^*(K_X + D) = K_{\hat{X}} + \hat{D}$$

then

$$Ell_\alpha(\hat{X}, \hat{D}, G) = Ell_\alpha(X, D, G)$$

Corollaries:

Higher elliptic genera are invariants of crepant birational equivalences of CY manifolds.

higher signatures and $\hat{A}$-genera are invariant for crepant birational morphisms

Higher Todd genus is an invariant of arbitrary birational morphisms. (also Block-Weinberger).

Singular varieties:

Lemma(Takayama): Let X has only log-terminal singularities and let $f : X' \to X$ be a resolution of singularities of X. Then $\pi_1(X') = \pi_1(X)$.

Let X be a projective algebraic variety with $\mathbf{Q}$-Gorenstein log-terminal singularities. Let $\alpha \in H^*(\pi_1(X), \mathbf{Q})$ be the cohomology class of its fundamental group. If $\phi : \tilde{X} \to X$ is a resolution of singularities of X, $K_{\tilde{X}} = \phi^*(K_X) + \tilde{D}$ and α is viewed as the element in the $H^*(\pi_1(\tilde{X}), \mathbf{Q})$ identified with $H^*(\pi_1(X), \mathbf{Q})$ using ϕ.

Then:

$$Ell_\alpha(X) \overset{\text{dfn}}{=} Ell_\alpha(\tilde{X}, \tilde{D})$$

This is well defined as a consequence of invariance of higher ellptic genus under K-equivalence of pairs.

Further results:

1. McKay correspondence for higher elliptic genera.

2. Cobordisms: Let I_π (resp. I) be the ideal in $\Omega^U(B\pi)$ generated by the differences (X, f_X) and $(X', f_{X'})$ ($f_X : X \to \pi_1(X)$) where $(X', f_{X'})$ and (X, f_X) (resp. $X' - X$ where X' and X are differ by a classical flop Then

$$Hom(\Omega_d^{SU}(B\pi)/I_\pi \cap \Omega_d^{SU}(B\pi), \mathbf{Q}) =$$

$$\oplus_{k \in 2\mathbf{Z}} H^k(B\pi, \mathrm{Jac}_{-k,\frac{d}{2}})$$

where $\mathrm{Jac}_{-k,\frac{d}{2}}$ is the space of Jacobi forms having weight $-k$ and index $\frac{d}{2}$

MSV-complex.

$X \Longrightarrow$ sheaf of vertex operator algebras $\mathcal{MSV}(X)$ (Malikov-Schechtman-Vaintrob)

Can be constructed in terms of the loop space of X (Kapranov-Vasserot) Further clarified by Ben-Zvi-Heluani-Szczesny (math.AG 0601532)

Vertex operator algebra:

$$(V = V_{ev} \oplus V_{odd}, 0| > \in V_0,$$

$$V \to End(V)[z, z^{-1}], T : V \to V)$$

$a \to Y(a, z) = \sum_{n \in \mathbf{Z}} a_{(n)} z^{-n-1}$ satisfies axioms

Conformal vertex algebra: vertex algebra together with even element $L \in V$ such that

1. components $L(z) = \sum_n L_n z^{-n-2}$ satisfy Virasoro commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12} \cdot c \cdot \delta^n_{-m}$$

2. $L_{-1} = T$ infinitesimal translation operator.

3. L_0 is diagonalizable.

Has additional structure (topological vertex agebra) given by operator J_0 given another grading.

Theorem (MSV): Let X be a non singular compact complex manifold. There exist a sheaf Ω_X^{ch} of vector spaces on X with the properties:

a) For each Zariski open set U, $\Gamma(U, \Omega_X^{ch})$ has a structure of conformal vertex algebra, with restriction maps being morphisms of vertex algebras.

b) Ω_X^{ch} has two gradings with degrees called fermionic charge and conformal weight.

c) Ω_X^{ch} has deRham differential d_{DR}^{ch} of (fermionic) degree 1, $(d_{DR}^{ch})^2 = 0$.

d) Usual deRham complex Ω_X is isomorphic to conformal weight zero component of Ω_{DR}^{ch}.

e)The complex $(\Omega_X^{ch}, d_{DR}^{ch})$ is quasiisomorphic to (Ω_X, d_{DR}).

f) Each component of fixed conformal weight has canonical filtration with gr_F isomorphic to tensor product of exterior powers of tangent and cotangent bundles so that corresponding generating function is

$$\otimes_{n \geq 1}(\Lambda_{yq^{n-1}}\bar{T}_X \otimes \Lambda_{y^{-1}q^n}T_X \otimes S_{q^n}\bar{T}_X \otimes S_{q^n}T_X)$$

DEFINITION: Let X is a variety for which one can define a chiral DeRham complex $\mathcal{MSV}(X)$ with properties a)-f) as above.

Elliptic genus of X is

$$y^{-\frac{dimX}{2}} SuperTrace_{H^*(\mathcal{MSV}(X))} y^{J[0]} q^{L[0]}$$

- A test for mirror symmetry.

$$Ell(\hat{X}) = (-1)^{dim X} Ell(X)$$

Theorem: Let X be a generic hypersurface in the Gorenstein toric Fano variety defined by the combinatorial data above. Then

$$Ell(X,y,q) = y^{\frac{-d}{2}} \text{SuperTrace}_{H^*(\mathcal{MSV}(X))} y^{J[0]} q^{L[0]} =$$

$$y^{\frac{-d}{2}} \sum_{m \in M} \left(\sum_{n \in K^*} y^{n \cdot \text{deg} - m \cdot \text{deg}^*} q^{m \cdot n + m \cdot \text{deg}^*} G(y,q)^{d+2} \right)$$

where

$$G(y,q) = \prod_{k \geq 1} \frac{(1 - yq^{k-1})(1 - y^{-1}q^k)}{(1-q^k)^2}.$$

Corollary: Elliptic genera of toric hypersurfaces corresponding to dual poytopes satisfy mirror duality.

Discrete Torsion and Elliptic genus

Let $\alpha \in \mathsf{H}^2(\mathsf{G}, \mathsf{U}(1))$ and let

$$\delta(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)}$$

Definition:

$$Ell^{\alpha}_{orb}(X, G, q, y) :=$$

$$y^{-\mathrm{dim} X/2} \sum_{[g], X^g} y^{F(g, X^g \subseteq X)}.$$

$$\frac{1}{|C(g)|} \sum_{h \in C(h)} \delta(g,h) L(h, V_{g, X^g \subseteq X}).$$

(where:

$$V_{h, X^h \subseteq X} :=$$

$$\otimes_{k\geq 1}[(\Lambda^{\bullet}_{yq^{k-1}}V_0^*)\otimes(\Lambda^{\bullet}_{y^{-1}q^k}V_0)\otimes$$

$$(Sym^{\bullet}_{q^k}V_0^*)\otimes(Sym^{\bullet}_{q^k}V_0)\otimes$$

$$\otimes[\otimes_{\lambda\neq 0}(\Lambda^{\bullet}_{yq^{k-1+\lambda(h)}}V_\lambda^*)\otimes(\Lambda^{\bullet}_{y^{-1}q^{k-\lambda(h)}}V_\lambda)\otimes$$

$$(Sym^{\bullet}_{q^{k-1+\lambda(h)}}V_\lambda^*)\otimes(Sym^{\bullet}_{q^{k-\lambda(h)}}V_\lambda)]]$$

Alternative Form:

$$Ell^{\alpha}_{orb}(X,G;y,q)=$$

$$\frac{1}{|G|}\sum_{gh=hg}\delta(g,h)\prod_{\lambda(g)=\lambda(h)=0}x_\lambda\times$$

$$\prod_{\lambda}\Phi(g,h,\lambda,z,\tau,x_\lambda)[X^{g,h}]$$

$$\Phi(g,h,\lambda,z,\tau,x)=$$

$$\frac{\theta(\frac{x}{2\pi i}+\lambda(g)-\tau\lambda(h)-z)}{\theta(\frac{x}{2\pi i}+\lambda(g)-\tau\lambda(h))}e^{2\pi iz\lambda(h)}.$$

Specialization: Twisted E-function:

$$E^{\alpha}(u,v,G)=$$

$$\sum_{[g],X^g}(uv)^{F(g,X^g\subset X)}\sum_{p,q}\mathrm{dim}H^{p,q}(X^g,L_{\alpha})^{C(g)}u^pv^q$$

which for $u=1, v=-1$ yields:

$$e^{\alpha}(X,G)=\frac{1}{|G|}\sum_{fg=gf}\delta(f.g)e(X^{f,g})$$

The elliptic genus satisfies:

$$Ell^{\alpha}(0,y,G)=y^{\frac{\mathrm{dim}X}{2}}E^{\alpha}(y,-1,G)$$

Modularity :

$$Ell_{orb}^{\alpha}(X, G, q, y)$$

is Jacobi form (weight 0 index $\frac{d}{2}$) for a subgroup of the Jacobi group.

Vertex Algebra interpretation

$\alpha \in H^2(G, U(1))$ yields a character of $C(g)$

$$(\alpha_g(h) = \delta(g, h))$$

$\Omega_X^{ch,g}$ is sheaf of (g-twisted) Ω^{ch}-modules

(g-twisted is certain restriction on graded components of a VOA module with action of g)

Let

$$\mathcal{H}^{\alpha}_{orb}(X,G)=\bigoplus_{[g]} H^*(X,\Omega_X^{ch,g})^{C(g)_\alpha}$$

(for $\alpha = 0$ one has orbifold chiral deRham complex of Frenkel-Szczesny)

Then

$$Ell^{\alpha}_{orb}(X,G,q,y)=$$

$$y^{-\frac{\mathrm{dim} X}{2}}\mathrm{Supertrace}(q^{L_0}y^{J_0},\mathcal{H}^{\alpha}_{orb}(X,G))$$

Symmetric products and discrete torsion

One has

$$H^2(S_N, U(1)) = \mathbf{Z}_2 \quad (N \geq 4)$$

Let us before:

$$Ell(X; q, y) = \sum_{m,\ell} c(m,\ell) q^m y^\ell$$

Consider:

$$Z^\alpha(p, q, y) =$$

$$\sum_{N \geq 0} p^N Ell^\alpha_{orb}(X^N, S_N, q, y)$$

Then $(n > 0, m, l \geq 0)$:

$$Z^{\alpha}(p,q,y) = \frac{1}{2}\Big(Z_{++} + Z_{+-} + Z_{-+} + Z_{--}\Big)$$

. where:

$$Z_{++}(p,q,y) = \prod_{n,m,l} \frac{(1+p^{2n}q^{m-\frac{1}{2}}y^{\ell})^{c(n(2m-1),\ell)}}{(1-p^{2n-1}q^{m}y^{\ell})^{c((2n-1)m,\ell)}}$$

$$Z_{+-}(p,q,y) = \prod_{n,\, m,l} \frac{(1-p^{2n}q^{m-\frac{1}{2}}y^{\ell})^{c(n(2m-1),\ell)}}{(1-p^{2n-1}q^{m}y^{\ell})^{c((2n-1)m,\ell)}}$$

$$Z_{-+}(p,q,y) = \prod_{n,\, m,l} \frac{(1+p^{2n}q^{m}y^{\ell})^{c(2nm,\ell)}}{(1-p^{2n-1}q^{m}y^{\ell})^{c((2n-1)m,\ell)}}$$

$$Z_{--}(p,q,y) = -\prod_{n,m,l} \frac{(1-p^{2n}q^{m}y^{\ell})^{c(2nm,\ell)}}{(1-p^{2n-1}q^{m}y^{\ell})^{c((2n-1)m,\ell)}}$$

(there is G-equivariant form of this identity involving wreath products)

Specialization to euler characteristic case

Usual elliptic genus specialization (Goetsche):

$$\sum p^N e_{orb}(X^N, S_N) = \prod_{n>0} (1 - p^n)^{-e(X)}$$

(almost modular form)

In torsion case:

$$\sum p^N e^{\alpha}_{orb}(X^N, S_N) =$$

$$\prod_{n>0} (1 - p^{2n-1})^{-e(X)} \times$$

$$\{1 + \frac{1}{2}[\prod (1 + p^{2n})^{e(X)} - \prod (1 - p^{2n})^{e(X)}]\}$$

(modular properties?)