An Evolution of The Topological Spherical Space Form Problem

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Outline

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1. The Topological Spherical Space Form Problem
   - Group actions
   - Solution
2. The Topological Euclidean Space Form Problem
   - Historical background
   - Group cohomology
   - Cohomological dimension
   - Solution
3. Free and Proper Group Actions on $\mathbb{S}^n \times \mathbb{R}^k$
   - Current results
   - Talelli's conjecture
   - Groups with jump cohomology
   - General conjecture for solvable groups
   - Isometric actions

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Actions

Definition

Let G be a discrete group and X be a topological space. We say G acts on X if there exists a map

$$G \times X \to X, (g, x) \mapsto gx$$

such that

1. $ex = x$ for all $x \in X$ and the identity $e \in G$.
2. $(gh)x = g(hx)$ for all $x \in X$ and $g, h \in G$.

Ex. 1. $\mathbb{Z}$ acts on $\mathbb{R}$ by translations.

Ex. 2. The cyclic group C_2 acts by flips on the circle.

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Free actions

Definition

The action is said to be free if for any non-identity element $g \in G$, $gx \neq x$ for all $x \in X$.

- $\mathbb{Z}^n$ acts freely on $\mathbb{R}^n$ by translations.
- The cyclic group $C_m = \langle t | t^m = e \rangle$ acts freely on the sphere $S^{2k+1} = \langle z_o, \ldots, z_k | z_i \in \mathbb{C}, \sum |z_i|^2 = 1 \rangle$ by

$$t \cdot \langle z_o, \ldots, z_k \rangle = \langle e^{2\pi i/m} z_o, \ldots, e^{2\pi i/m} z_k \rangle.$$

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Properly discontinuous actions

Definition

G is said to act properly discontinuously on X if for any compact subset $C \subseteq X$, $\#\{g \in G | C \cap gC \neq \emptyset\} < \infty$.

- Any action of a finite group is properly discontinuous.
- In all of the previous examples actions are properly discontinuous.

Ex. 3. $\mathbb{Q}$, as a subgroup of $\mathbb{R}$, acts freely but not properly discontinuously on $\mathbb{R}$.

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Universal covering space

- A connected topological space is called simply connected if its fundamental group is trivial.
- Let M be a connected manifold. The universal covering space is the unique connected simply connected manifold $\widetilde{M}$ together with the covering map $p : \widetilde{M} \to M$.

Ex. 4. $\mathbb{R}$ is the universal cover of S^1 and $p : \mathbb{R} \to S^1, x \mapsto e^{xi}$.

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The fundamental group and the universal cover

Definition

Let G act on a space X. The quotient of the action is defined to be the space $X/G = \{\bar{x} \mid x \in X, \bar{x} = \bar{y} \text{ iff } \exists g \in G, gx = y\}$.

Fundamental characterization

Let M be a connected manifold. Then the fundamental group π of M acts freely and properly discontinuously on the universal cover $\widetilde{M}$ and $\widetilde{M}/\pi \cong M$.

Ex. 5. Let $T^2 = S^1 \times S^1$. Then $\pi_1(T^2) = \mathbb{Z}^2$ acts freely and properly discontinuously on $\widetilde{T^2} \cong \mathbb{R}^2$.

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The three main questions

(1) The topological spherical space form problem

When does a finite group act freely on a sphere S^n?

(2) The topological Euclidean space form problem

When does a countable group act freely and properly discontinuously on some Euclidean space $\mathbb{R}^k$?

(3) The hybrid problem

What countable groups act freely and properly discontinuously on some $S^n \times \mathbb{R}^k$?

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Theorems of Smith and Artin-Tate

Theorem (R.G. Smith, 1938)

If a finite group G acts freely on S^n, then every abelian subgroup of G is cyclic.

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Theorem (Artin-Tate, 1956)

A finite group has all abelian subgroups cyclic if and only if its cohomology is periodic.

- For instance, does the dihedral group $D_6 = \langle x, y | x^2 = y^3 = (xy)^2 = e \rangle$ act freely on a sphere?

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Milnor's condition

Theorem (Milnor, 1957)

Let $T : S^n \to S^n$ be a map such that $T \circ T = Id$ without fixed points. Then for every $f : S^n \to S^n$ of odd degree there exists a point $x \in S^n$ such that $Tf(x) = fT(x)$.

Corollary

If a finite group G acts freely on S^n, then any element of order 2 must be in the center $Z(G)$.

Proof. Let $t \in G$ be of order 2 and let $g \in G$. Then there exists $x \in S^n$ so that $tg(x) = gt(x)$. This implies $tg = gt$. □

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Classification

Fact: Milnor's condition together with periodicity is also sufficient for a group to act freely on some S^n.

- Let n be a positive integer. A group G is said to satisfy the n-condition if every subgroup of order n is cyclic.

Theorem (Madsen-Thomas-Wall, 1978)

A finite group G acts freely on some sphere S^n if and only if G satisfies p^2- and $2p$-conditions for all primes p.

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Euclidean space forms

Question 2. What countable groups act freely and properly discontinuously on $\mathbb{R}^k$?

Euclidean Space Form Problem. When does a group act freely, properly discontinuously, and isometrically on $\mathbb{R}^k$?

Definition

Let M be Riemannian manifold. A diffeomorphism $f : M \to M$ is said to be an isometry, if

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)},$$

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Cocompact actions

- $\mathrm{Isom}(\mathbb{R}^k) \cong \mathbb{R}^k \rtimes O(k)$.

Theorem (Bieberbach, 1911)

Let Γ act freely, properly discontinuously, and isometrically on $\mathbb{R}^k$ such that $\mathbb{R}^k/\Gamma$ is compact. Then Γ is torsion-free, $\Gamma \cap \mathbb{R}^k \cong \mathbb{Z}^k$, and $\Gamma/(\Gamma \cap \mathbb{R}^k)$ is finite.

Geometric Reformulation

Let M be a closed connected flat Riemannian manifold of dimension k. Then M admits a normal Riemannian covering by a flat k-dimensional torus.

- $\widetilde{M} = \mathbb{R}^k \to T^k \to M \Rightarrow \pi_1(T^k) \cong \mathbb{Z}^k \lhd \pi_1(M)$.

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Solution to the Euclidean Space Form Problem

Theorem

Let Γ act freely, properly discontinuously, and isometrically on $\mathbb{R}^k$. Then Γ acts freely, properly discontinuously, and isometrically on $\mathbb{R}^m$ with compact quotient for some $m \leq k$. Therefore, $\Gamma \cap \mathbb{R}^m \cong \mathbb{Z}^m$, and $\Gamma/\mathbb{Z}^m$ is finite.

Proof sketch. The quotient $\mathbb{R}^k/\Gamma$ can be deformation retracted onto a compact totally geodesic submanifold call it M. Let $m = \dim(M)$. Then $\pi_1(M) = \Gamma$ acts freely, properly discontinuously, and isometrically on $\widetilde{M} = \mathbb{R}^m$. □

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Proof sketch. The quotient $\mathbb{R}^k / \Gamma$ can be deformation retracted onto a compact totally geodesic submanifold call it M. Let $m = \dim(M)$. Then $\pi_1(M) = \Gamma$ acts freely, properly discontinuously, and isometrically on $\widetilde{M} = \mathbb{R}^m$. □

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Outline

$K(\Gamma, 1)$-complex

- For any discrete Γ there exists a CW-complex X which is a $K(\Gamma, 1)$-space. That is

$$\pi_i(X) = \begin{cases} \Gamma & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

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Group cohomology

Definition

Cohomology of a group Γ with coefficients in a Γ-module M is defined as

$$H^i(\Gamma, M) = H^i(X, M)$$

for any $i \geq 0$, where X is a $K(\Gamma, 1)$-complex.

Ex. 6. S^1 is a $K(\mathbb{Z}, 1)$-complex. $H^i(\mathbb{Z}, \mathbb{Z}) = H^i(S^1, \mathbb{Z}), \forall i \geq 0$.

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Cohomological dimension

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Cohomological dimension of Γ is defined by $cd(\Gamma) = \sup\{n : H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma\text{-module } M\}$.

- $cd(\mathbb{Z}^n) = n$.
- $\Gamma' < \Gamma \Rightarrow cd(\Gamma') \leq cd(\Gamma)$. $\Leftarrow H^*(\Gamma', M) \cong H^*(\Gamma, \mathrm{Coind}_{\Gamma'}^{\Gamma} M)$.
- If $cd(\Gamma) < \infty$, then Γ is tor-free. $\Leftarrow H^{2i}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$ for all $i > 0$.

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$gd(\Gamma) = \inf\{n : n = \dim(X)$ where X is a $K(\Gamma, 1)$-complex$\}$.

- If F is a free group, then $gd(F) = 1$. This is because F acts freely and properly discontinuously on its Cayley graph Y and Y/F is a $K(F, 1)$-complex.
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Outline

Solution to the space form problem

Question. What countable groups act freely and properly discontinuously on $\mathbb{R}^k$?

Theorem (Johnson, 1969)

Let Γ be a countable group. Then, $cd(\Gamma) < \infty$ if and only if Γ acts freely, properly discontinuously, and smoothly on some $\mathbb{R}^n$.

($\Leftarrow$): If Γ acts freely and properly discontinuously on some $\mathbb{R}^n$, then $\mathbb{R}^n/\Gamma$ has a structure of a $K(\Gamma, 1)$-complex. This shows $cd(\Gamma) \leq gd(\Gamma) \leq \dim(\mathbb{R}^n/\Gamma) = n$.

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The forward direction

($\Rightarrow$): Since Γ is countable, it admits a finite dimensional free-Γ-CW-complex such that X/Γ is countable. By a result of Milnor, we can assume X/Γ is l.f. and simplicial. It is therefore isomorphic to a closed simplicial subcomplex of some $\mathbb{R}^q$. Let Y be a smooth regular nbhd of this subcomplex. Then Y is a smooth submanifold of $\mathbb{R}^q$ with $\pi_1(Y) = \Gamma$. Let $W = \widetilde{Y}$, then

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End of proof

W is a contractible finite dim manifold with a free and properly discontinuous and smooth action of Γ. Let $n - 1 = \dim(W)$, then $W \times \mathbb{R}$ is simply connected at infinity. Let $D^n \subset W \times \mathbb{R}$. $W \times \mathbb{R} - D^n$ admits a boundary at infinity. By the h-cobordism theorem, $\overline{W \times \mathbb{R}} - D^n \cong \partial D^n \times [1, \infty]$. Hence, $W \times \mathbb{R} \cong \mathbb{R}^n$ and has the desired action of Γ. □

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Outline

Periodic cohomology

Question 3. When does a countable group act freely and properly discontinuously on $S^n \times \mathbb{R}^k$?

Lemma

If Γ acts freely and properly discontinuously on $S^n \times \mathbb{R}^k$, then Γ has periodic cohomology after dimension k.

Proof sketch. Let $X = (S^n \times \mathbb{R}^k)/\Gamma$. By the Gysin exact sequence,

$$\cdots \to H^{i+n}(X, M) \to H^i(\Gamma, M) \to H^{i+n+1}(\Gamma, M) \to H^{i+n+1}(X, M) \to \ldots$$

Thus, $H^i(\Gamma, M) \cong H^{i+n+1}(\Gamma, M)$ for all Γ-modules M and $i > k$. □

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Solution

Question. When does a countable group act freely and properly discontinuously on $S^n \times \mathbb{R}^k$?

Theorem (Adem-Smith, 2001)

Let Γ be countable. Then Γ acts freely, properly discontinuously, and smoothly on some $S^n \times \mathbb{R}^k$ if and only if Γ has periodic cohomology.

- Note that if Γ has periodic cohomology, then every subgroup of Γ also has periodic cohomology.

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1. The Topological Spherical Space Form Problem
   - Group actions
   - Solution
2. The Topological Euclidean Space Form Problem
   - Historical background
   - Group cohomology
   - Cohomological dimension
   - Solution
3. Free and Proper Group Actions on $\mathbb{S}^n \times \mathbb{R}^k$
   - Current results
   - Talelli's conjecture
   - Groups with jump cohomology
   - General conjecture for solvable groups
   - Isometric actions

Talelli's conjecture

Conjecture (Talelli, 2005)

Suppose Γ is torsion-free and it acts freely and properly discontinuously on some $S^n \times \mathbb{R}^k$. Then $cd(\Gamma) \leq k$.

forward

Implied from the action: Let $\Gamma' < \Gamma$, $cd(\Gamma') = m < \infty$. Then, by periodicity, for all $i > k$

$$H^i(\Gamma', M) \cong H^{i+m(n+1)}(\Gamma', M) = 0.$$

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Outline

Jumps

Definition (Petrosyan)

A group Γ has jump cohomology of height k, if for any subgroup $\Gamma' < \Gamma$, $cd(\Gamma') \leq k$ or $cd(\Gamma') = \infty$.

- If Γ has periodic cohomology after dimension k, then Γ has jump cohomology of height k.
- Jump cohomology is a subgroup closed property.

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A group Γ has jump cohomology of height k, if for any subgroup $\Gamma' < \Gamma$, $cd(\Gamma') \leq k$ or $cd(\Gamma') = \infty$.

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Generalizing Talelli's conjecture

Talelli's conjecture

Suppose Γ is torsion-free and it acts freely and properly discontinuously on some $S^n \times \mathbb{R}^k$. Then $cd(\Gamma) \leq k$.

General conjecture

The following are equivalent for a torsion-free group Γ.

1. $cd(\Gamma) \leq k$.
2. Γ has periodic cohomology after dimension k.
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- General conjecture $\Rightarrow$ Talelli's conjecture i.e. (1) $\Leftrightarrow$ (2) for torsion-free Γ.

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More general

Theorem (Petrosyan)

Suppose Γ acts freely and properly discontinuously on $M \times N$, where M is a closed, connected and orientable manifold and N is a contractible manifold. Then Γ has jump cohomology of height $\dim(N)$.

- If, in addition, Γ is torsion-free and the general conjecture holds, then $cd(\Gamma) \leq \dim(N)$.
- This conjecture holds for all solvable groups.
- It also holds for Kropholler groups, $H\mathcal{F}$. This groups, among others, contain all countable linear groups and all countable elementary amenable groups.

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Solvable groups

Theorem (Petrosyan)

Let Γ be torsion-free solvable group. Γ has jump cohomology of height k if and only if $cd(\Gamma) \leq k$.

For a solvable group G and its derived series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

set $h_i = \dim(G_i/G_{i-1} \otimes \mathbb{Q})$ for all i. The Hirsch length of G is defined as $h(G) = \sum h_i$.

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The conjecture for solvable groups

Let Γ be torsion-free solvable group. Γ has jump cohomology of height k if and only if $cd(\Gamma) \leq k$.

Proof sketch. It is enough to show that $cd(\Gamma) < \infty$. Suppose $cd(\Gamma) = \infty$. Since Γ is torsion-free, $h(\Gamma) \leq cd(\Gamma) \leq h(\Gamma) + 1$. Therefore, $h(\Gamma) = \infty$ and we can find $\Gamma' < \Gamma$ with $k < h(\Gamma') < \infty$. Then $k < cd(\Gamma') < \infty$, a contradiction. □

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Work in progress

Question. When does a group act freely, properly discontinuously, and isometrically on some $S^n \times \mathbb{R}^k$?

- (Cheeger & Gromoll, 1972) $\mathrm{Isom}(S^n \times \mathbb{R}^k) \cong \mathrm{Isom}(S^n) \times \mathrm{Isom}(\mathbb{R}^k)$

Theorem (Dreesen-Petrosyan,'09)

Let M be a closed, connected n-dim. Riemannian manifold and N be a Riemannian manifold s.t. $\pi : M \times N \to M$ induces an isomorphism $\pi^* : H^n(M, \mathbb{Z}_2) \to H^n(M \times N, \mathbb{Z}_2)$, then $\mathrm{Isom}(M \times N) = \mathrm{Isom}(M) \times \mathrm{Isom}(N)$.

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Theorem (Dreesen-Petrosyan)

Let M be closed, connected Riemannian manifold and N be a contractible Riemannian manifold. If Γ is torsion-free and acts freely, properly discontinuously and fiberwise volume decreasingly on $M \times N$, then Γ acts freely and properly discontinuously on N. In particular $cd(\Gamma) \leq \dim(N)$.

Corollary

With M and N as above. If Γ is torsion-free and acts properly discontinuously and isometrically on $M \times N$, then $cd(\Gamma) \leq \dim(N)$.

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Theorem (Dreesen-Petrosyan)

Let M be closed, connected Riemannian manifold and N simply connected, connected, nilpotent Lie group with a left-invariant metric. If Γ is acting properly discontinuously, cocompactly and isometrically on $M \times N$, then Γ contains a finite index subgroup isomorphic to a uniform lattice of N.

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Γ_1 must be finite as it maps $M \times \{1\}$ to itself. $p(\Gamma)$ is almost crystallographic. Hence, with group theoretical arguments . . .

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Thank You!

Dziękuję!

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