# Gottlieb groups: their applications and generalizations

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# 1. Gottlieb groups

Given a map $f : X \longrightarrow X$, what sort of information about $X$ and $f$ do we need in order to compute $N(f)$, the Nielson number of $f$?

The main tool in these computations is the *Jiang subgroup* $J(f, x_0) \subseteq \pi_1(X, f(x_0))$ (in honor of Bo-Ju Jiang). Given $x_0 \in X$, the subgroup

$$J(f, x_0) = \mathrm{Im}(ev_* : \pi_1(\mathrm{Map}(X, X), f) \longrightarrow \pi_1(X, f(x_0))),$$

where $ev : (\mathrm{Map}(X, X), f) \longrightarrow (X, f(x_0))$ denotes the evaluation map. Because of the inclusion $J(\mathrm{id}_X, x_0) \subseteq J(f, x_0)$, the subgroup $J(\mathrm{id}_X, x_0) = G_1(X, x_0)$, called also the *Gottlieb subgroup* plays a central role in the study of Jiang subgroups.

**Theorem 1.** **(Jiang).** (1) *If* $J(f, x_0) = \pi_1(X, f(x_0))$ *then all the fixed point classes of* $f$ *have the same index. In particular,* $N(f) = 0$ *provided* $L(f) = 0$, *the Lefschetz number.*

(2) $J(f) \leq N(f)$, *where* $J(f)$ *denotes the cardinality of the set of* $f_*$*-equivalence classes of* $J(f, x_0)$ *for the induced map* $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$.

The higher Gottlieb groups $G_k(X)$ of a pointed space $X$ have been defined by D. Gottlieb, Amer. J. of Math. **91** (1969) as evaluation subgroups

$$G_k(X, x_0) = \mathrm{Im}(ev_* : \pi_k(\mathrm{Map}(X, X), \mathrm{id}_X) \to \pi_k(X, x_0)).$$

For a wide class of spaces, $G_k(X)$ is the subgroup of the $k$-th homotopy group $\pi_k(X)$ containing all elements which can be represented by a map $f : \mathbb{S}^k \to X$ such that $f \vee \mathrm{id}_X : \mathbb{S}^k \vee X \to X$ extends (up to homotopy) to a map $F : \mathbb{S}^k \times X \to X$ i.e., the diagram

$$\begin{tikzcd}
\mathbb{S}^k \vee X \arrow[r] \arrow[d, hook] & X \\
\mathbb{S}^k \times X \arrow[ur, dashed] &
\end{tikzcd}$$

commutes up to homotopy, where $\mathbb{S}^k$ denotes the $k$-th sphere. Define the $k$-th *Whitehead center group* $P_k(X) \subseteq \pi_k(X)$ of elements $\alpha \in \pi_k(X)$ such that the Whitehead product $[\alpha, \beta] = 0$ for all $\beta \in \pi_l(X)$ and $l \geq 1$. Then, $G_k(X) \subseteq P_k(X)$.

Those groups are related to the problem of sectioning fibrations with the fibre $X$.

**Theorem 2. (Gottlieb).** *For any fibration*

$$X \to E \to B,$$

$d(\pi_{n+1}(B)) \subseteq G_n(X)$ *where* $d : \pi_{n+1}(B) \to \pi_n(X)$ *arises from the homotopy exact sequence of the fibration.*

*In particular,* $G_n(X) = 0$ *implies that every fibration* $X \to E \to \mathbb{S}^{n+1}$ *has a cross-section.*

Given $\alpha \in \pi_k(\mathbb{S}^n)$ for $k \geq 1$, we deduce that $\alpha \in G_k(\mathbb{S}^n)$ if and only if $[\iota_n, \alpha] = 0$. The paper **M.G., J. Mukai, Topology 47 (2008)** takes up the systematic study of the Gottlieb groups $G_{n+k}(\mathbb{S}^n)$ of spheres for $k \leq 13$ by means of the classical homotopy theory methods (e.g., H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies **49**, Princeton (1962)). We fully determine the groups $G_{n+k}(\mathbb{S}^n)$ for $k \leq 13$ except for the $2$-primary components in the cases: $k = 9$, $n = 53$; $k = 11$, $n = 115$. In particular, we show $[\iota_n, \eta_n^2 \sigma_{n+2}] = 0$ if $n = 2^i - 7$ for $i \geq 4$.

**Methods:** Write $\eta_2 \in \pi_3(\mathbb{S}^2)$, $\nu_4 \in \pi_7(\mathbb{S}^4)$ and $\sigma_8 \in \pi_{15}(\mathbb{S}^8)$ for the Hopf maps, respectively. We set $\eta_n = \Sigma^{n-2}\eta_2 \in \pi_{n+1}(\mathbb{S}^n)$ for $n \geq 2$, $\nu_n = \Sigma^{n-4}\nu_4 \in \pi_{n+3}(\mathbb{S}^n)$ for $n \geq 4$ and $\sigma_n = \Sigma^{n-8}\sigma_8 \in \pi_{n+7}(\mathbb{S}^n)$ for $n \geq 8$. Write $\eta_n^2 = \eta_n \circ \eta_{n+1}$, $\nu_n^2 = \nu_n \circ \nu_{n+3}$ and $\sigma_n^2 = \sigma_n \circ \sigma_{n+7}$.

First, we describe of $G_{n+k}(\mathbb{S}^n)$ for $k \leq 7$. To reach that for $G_{n+6}(\mathbb{S}^n)$, we make use of result **(L. Kristensen and I. Madsen, Math. Scand. 21 (1967), 301-314.)**

$[\iota_n, \nu_n^2] = 0$ *if and only if* $n \equiv 4, 5, 7 \pmod 8$ *or* $n = 2^i - 5$ *for* $i \geq 4$.

Then, **Mahowald's result:**

$$[\iota_n, \sigma_n] \neq 0$$

is shown for $n \equiv 7 \pmod{16}$ and $n \geq 23$.

Given an element $\alpha$ of a group, write $\sharp\alpha$ for its order. We have founded that the order of $[\iota_n, \alpha]$ for $\alpha = \iota_n, \eta_n, \eta_n^2, \nu_n, \nu_n^2$ and $\sigma_n$ is given as follows:

$$\sharp[\iota_n, \iota_n] = \begin{cases} 1, & \text{if } n = 1, 3, 7, \\ 2, & \text{if } n \text{ is odd and } n \neq 1, 3, 7, \\ \infty, & \text{if } n \text{ is even;} \end{cases} \tag{1}$$

$$\sharp[\iota_n, \eta_n] = \begin{cases} 1, & \text{if } n \equiv 3 \pmod 4,\ n = 2 \text{ or } n = 6, \\ 2, & \text{if otherwise;} \end{cases} \tag{2}$$

$$\sharp[\iota_n, \eta_n^2] = \begin{cases} 1, & \text{if } n \equiv 2, 3 \pmod 4, \\ 2, & \text{if otherwise;} \end{cases} \tag{3}$$

$$\sharp[\iota_n, \nu_n] = \begin{cases} 1, & \text{if } n \equiv 7 \pmod 8),\ n = 2^i - 3 \geq 5, \\ 2, & \text{if } n \equiv 1, 3, 5 \pmod 8) \geq 9,\ n \neq 2^i - 3, \\ 12, & \text{if } n \equiv 2 \pmod 4) \geq 6, n = 4, 12, \\ 24, & \text{if } n \equiv 0 \pmod 4) \geq 8,\ n \neq 12; \end{cases} \tag{4}$$

$$\sharp[\iota_n, \nu_n^2] = 1 \text{ if and only if } n \equiv 4, 5, 7 \pmod 8) \text{ or } n = 2^i - 5 \text{ for } i \geq 4; \tag{5}$$

$$\sharp[\iota_n, \sigma_n] = \begin{cases} 1, & \text{if } n = 11,\ n \equiv 15 \pmod{16}, \\ 2, & \text{if } n \equiv 1 \pmod 2) \geq 9,\ n \neq 11,\ n \not\equiv 15 \pmod{16}, \\ 120, & \text{if } n = 8, \\ 240, & \text{if } n \equiv 0 \pmod 2) \geq 10. \end{cases} \tag{6}$$

Next, we take up computations of $G_{n+k}(\mathbb{S}^n)$ for $10 \leq k \leq 13$ and partial ones of

$G_{n+k}(\mathbb{S}^n)$ for $k = 8, 9$. We have found out the triviality of the Whitehead product:

$$[\iota_n, \eta_n^2 \sigma_{n+2}] = 0, \text{ if } n = 2^i - 7 \ (i \geq 4),$$

which corrects Mahowald's result for $n = 2^i - 7$.

More generally. Given a pointed space $X$, define

$$P : \pi_n(\Sigma X) \longrightarrow [\Sigma(X \wedge \mathbb{S}^{n-1}), \Sigma X]$$

given by $P(\alpha) = [\mathrm{id}_{\Sigma X}, \alpha]$, the generalized Whitehead product for $\alpha \in \pi_n(\Sigma X)$. Then, we can state:

**Proposition 3.** $G_k(\Sigma A) = \ker P$.

But $\mathbb{S}^m = M(\mathbb{Z}, m)$, the Moore space. Hence, we can ask for $G_n(M(A, m))$ for any Moore space $M(A, m)$ of type $(A, m)$. In particular, for the real projective plane $\mathbb{R}P^2$, the $(n-2)$-suspension $\Sigma^{n-2}\mathbb{R}P^2 = M(\mathbb{Z}_2, n-1)$ for $n \geq 2$.

By **J. Wu, Memoirs AMS, 2003**, the homotopy groups $\pi_{n+k}(\Sigma^n\mathbb{R}P^2)$ are known for $n \geq 2$ and $k \leq 8$. Hence, we could follow the above to search for $G_{n+k}(\Sigma^n\mathbb{R}P^2)$ as well.

Now, let $\mathbb{F}P^n$ be the $n$-projective space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and put $d = \dim_{\mathbb{R}} \mathbb{F}$.

**Problem:** *What about looking for $P_k(\mathbb{F}P^n)$ and $G_k(\mathbb{F}P^n)$?*

Here, $G_k(\mathbb{F}P^n) \subsetneqq P_k(\mathbb{F}P^n)$ in general!

**Example 4.** (1) $P_1(\mathbb{R}P^n) = \frac{1+(-1)^{n-1}}{2}\pi_1(\mathbb{R}P^n)$, $P_2(\mathbb{C}P^n) = \frac{3+(-1)^n}{2}\pi_2(\mathbb{C}P^n)$ and $P_4(\mathbb{H}P^n) = [[12, \frac{24}{(24,n+1)}]]\pi_4(\mathbb{H}P^n)$ for $n \geq 2$;

**(Pak-Woo)** $G_n(\mathbb{R}P^n) = \gamma_{n*}G_n(\mathbb{S}^n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \pi_n(\mathbb{R}P^n), & \text{if } n = 1,3,7; \\ 2\pi_n(\mathbb{R}P^n), & \text{if } n \text{ is odd and } n \neq 1,3,7. \end{cases}$

(2) $P_1(\mathbb{R}P^n) = \frac{1+(-1)^{n-1}}{2}\pi_1(\mathbb{R}P^n)$,
$P_2(\mathbb{C}P^n) = \frac{3+(-1)^n}{2}\pi_2(\mathbb{C}P^n)$ and $P_4(\mathbb{H}P^n) = [[12, \frac{24}{(24,n+1)}]]\pi_4(\mathbb{H}P^n)$ for $n \geq 2$, where $[[-,-]]$ is the least common multiple.
$G_2(\mathbb{C}P^n) = G_4(\mathbb{H}P^n) = 0$.

**Methods:** Certainly, we have the fibration $\gamma_n : \mathbb{S}^{d(n+1)-1} \longrightarrow \mathbb{F}P^n$ and some knowledge on $G_k(\mathbb{S}^n) = P_k(\mathbb{S}^n)$. But, we need much more!

**(1)** The result due to **J. Siegel, Pacific J. Math. 31-1 (1969), 209-214:**

**Theorem 5.** *Let Y be a Lie group and G any closed subgroup. If* $p : Y \longrightarrow Y/G$ *is the quotient map then* $p_*\pi_n(Y) \subseteq G_n(Y/G)$ *for all* $n \geq 1$.

**(2)** The homotopy exact sequences

$$(\mathcal{F}_k^n)\ \cdots \longrightarrow \pi_{k+1}(\mathbb{S}^{d(n+1)-1}) \xrightarrow{\Delta_{\mathbb{F}}} \pi_k(SO_{\mathbb{F}}(n)) \xrightarrow{i_*} \pi_k(SO_{\mathbb{F}}(n+1)) \xrightarrow{p_*} \cdots ,$$

determined by $SO_{\mathbb{F}}(n+1) \overset{SO_{\mathbb{F}}(n)}{\longrightarrow} \mathbb{S}^{d(n+1)-1}$, where $SO_{\mathbb{F}}(n) = SO(n), SU(n), Sp(n)$, respectively for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Further, by **W.D. Barcus and M.G. Barratt, Trans. Amer. Math. Soc. 88 (1958), 57-74.)** and **M.G. Barratt, M. James and N. Stein, J. Math. Mech. 9 (1960), 813-819)** we obtain a key formula determining the Whitehead center groups of $\mathbb{F}P^n$.

**Lemma 6.** *Let* $h_0\alpha \in \pi_k(\mathbb{S}^{2d(n+1)-3})$ *be the* $0$*-th Hopf-Hilton invariant for* $\alpha \in \pi_k(\mathbb{S}^{d(n+1)-1})$*. Then:*

(1) $[\gamma_n\alpha, i_{\mathbb{R}}] = \begin{cases} 0, & if\ n\ is\ \textit{odd}, \\ (-1)^k\gamma_n(-2\alpha + [\iota_n, \iota_n] \circ h_0\alpha), & if\ n\ is\ \textit{even}; \end{cases}$

(2) $[\gamma_n\alpha, i_{\mathbb{C}}] = \begin{cases} 0, & if\ n\ is\ \textit{odd}, \\ \gamma_n(\eta_{2n+1} \circ E\alpha + [\iota_{2n+1}, \eta_{2n+1}] \circ Eh_0\alpha), & if\ n\ is\ \textit{even}; \end{cases}$

(3) $[\gamma_n\alpha, i_{\mathbb{H}}] = \pm(n+1)\gamma_n(\nu_{4n+3} \circ E^3\alpha + [\iota_{4n+3}, \nu_{4n+3}] \circ E^3h_0\alpha)$.

Let

We set

$$P'_k(\mathbb{F}P^n) = P_k(\mathbb{F}P^n) \cap (\gamma_{n*}\pi_k(\mathbb{S}^{d(n+1)-1}))$$

and

$$P''_k(\mathbb{F}P^n) = P_k(\mathbb{F}P^n) \cap i_{\mathbb{F}*}(\Sigma\pi_{k-1}(\mathbb{S}^{d-1})),$$

where $i_{\mathbb{F}} : \mathbb{S}^d \hookrightarrow \mathbb{F}P^n$.

Then, by **(G.M.-J. Mukai)**, we can state the following:

**Theorem 7.** $P_{4n+3+k}(\mathbb{H}P^n) = P'_{4n+3+k}(\mathbb{H}P^n) \oplus P''_{4n+3+k}(\mathbb{H}P^n)$ *for* $0 \leq k \leq 10$, *where*

(1) $P'_{4n+3}(\mathbb{H}P^n) = [[\frac{24}{(24,n+1)}, 2]]\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3})$ *for* $n \geq 2$;

(2) $P'_{4n+3+k}(\mathbb{H}P^n) = \gamma_{n*}\pi_{4n+3+k}(\mathbb{S}^{4n+3})$ *for* $k = 1, 2, 4, 5, 8, 9, 10$;

(3) $P'_{4n+6}(\mathbb{H}P^n) = \frac{3+(-1)^n}{2}\gamma_{n*}\pi_{4n+6}(\mathbb{S}^{4n+3})$;

(4) $P'_{4n+9}(\mathbb{H}P^n) = \frac{1-(-1)^n}{2}\gamma_{n*}\pi_{4n+9}(\mathbb{S}^{4n+3})$;

(5) $P'_{4n+10}(\mathbb{H}P^n) = \begin{cases} 2\gamma_{n*}\pi_{4n+10}(\mathbb{S}^{4n+3}), & \textit{if } n \equiv 0, 1, 2 \pmod 4; \\ \gamma_{n*}\pi_{4n+10}(\mathbb{S}^{4n+3}), & \textit{if } n \equiv 3 \pmod 4 \textit{ or } n = 2. \end{cases}$

## The real case:

**Theorem 8.** *The equality* $G_{k+n}(\mathbb{R}P^n) = \gamma_{n*}G_{k+n}(\mathbb{S}^n)$ *holds if* $k \leq 7$ *except the following pairs:* $(k,n) = (3,4),(4,4)$, $(5,4),(6,4),(5,6),(7,8),(7,11),(3,2^i-3)$ *with* $i \geq 4$ *and* $(6,2^i-5)$ *with* $i \geq 5$.

*Furthermore,*

(1) $G_7(\mathbb{R}P^4) \supseteq 12\pi_7(\mathbb{R}P^4)$;

(2) $G_{10}(\mathbb{R}P^4) \supseteq 3\pi_{10}(\mathbb{R}P^4)$;

(3) $G_{11}(\mathbb{R}P^6) \supseteq 30\pi_{11}(\mathbb{R}P^6)$;

(4) $G_{15}(\mathbb{R}P^8) \supseteq 2520\pi_{15}(\mathbb{R}P^8)$;

(5) $G_{18}(\mathbb{R}P^{11}) \supseteq 2\pi_{18}(\mathbb{R}P^{11})$;

(6) $G_{2^i}(\mathbb{R}P^{2^i-3}) \supseteq 2\pi_{2^i}(\mathbb{R}P^{2^i-3})$ *for* $i \geq 4$.

## The complex case:

**Theorem 9.** (1) *Let* $k = 1, 2$*. Then:*

$$G_{k+2n+1}(\mathbb{C}P^n) = \begin{cases} 0, & if\ n\ is \text{ even}; \\ \pi_{k+2n+1}(\mathbb{C}P^n) \cong \mathbb{Z}_2, & if\ n\ is \text{ odd}. \end{cases}$$

(2)

$$G_{2n+4}(\mathbb{C}P^n) \supseteq \begin{cases} (24, n)\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{24}{(24,n)}}, & \text{if } n \text{ is even}; \\ \frac{(24,n+3)}{2}\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{48}{(24,n+3)}}, & \text{if } n \geq \text{ is odd}. \end{cases}$$

*In particular,* $G_{2n+4}(\mathbb{C}P^n) = 2\pi_{2n+4}(\mathbb{C}P^n)$ *if* $n \equiv 2, 10 \pmod{12} \geq 10$ *except* $n = 2^{i-1} - 2$ *or* $n \equiv 1, 17 \,(\mathrm{mod}\, 24) \geq 17$ *and* $G_{2n+4}(\mathbb{C}P^n) = \pi_{2n+4}(\mathbb{C}P^n)$ *if* $n \equiv 7, 11 \pmod{12}$.

(3) $G_{2n+7}(\mathbb{C}P^n) = \pi_{2n+7}(\mathbb{C}P^n)$ *if* $n \equiv 2, 3 \pmod{4}$.

## The quaternionic case:

**Theorem 10.** (1) $G_{4n+3}(\mathbb{H}P^n) \supseteq \frac{3+(-1)n+1}{2}(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3})$;

(2) $G_{4n+6}(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_{n*}\pi_{4n+6}(\mathbb{S}^{4n+3}) \cong \mathbb{Z}_{\frac{24}{(24,n+2)}}$ *for* $n \geq 2$;

(3) $G_k(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_n\nu_{4n+3} \circ \pi_k(\mathbb{S}^{4n+6}) \cong \mathbb{Z}_2$.

# 2. Various evaluation groups.

Using the modern language of homotopy theory, we reintroduce so-called torus homotopy groups by **R. Fox, Ann. Math. 49 (1948)**.

Let $X$ be a pointed space. For $n \geq 1$, the $n$-th Fox group of $X$ is defined to be

$$\tau_n(X) = [\Sigma(\mathbb{T}^{n-1} \sqcup *), X],$$

where $\mathbb{T}^k$ denotes the $k$-dimensional torus and $\Sigma$ the reduced suspension.

The obvious map $(\mathbb{T}^{n-1} \sqcup *) \to \mathbb{T}^{k-1}/(\mathbb{T}^{k-1})^{(k-2)} = \mathbb{S}^{k-1}$ leads to imbeddings

$$\pi_k(X) \longrightarrow \tau_n(X)$$

for $1 \leq k \leq n$, where $(\mathbb{T}^{k-1})^{(k-2)}$ denoted the $(k-2)$-skeleton of $\mathbb{T}^{k-1}$.

**Theorem 11.** **(Fox)** *Let $X$ be a path connected space. Then:*

(1) *If $[-,-] : \pi_k(X) \times \pi_l(X) \to \pi_{k+l-1}(X)$ is the Whitehead product, $\alpha \in \pi_k(X)$, $\beta \in \pi_l(X)$ and $k + l - 1 \leq n$ then the imbeddings $\pi_k(X) \to \tau_n(X)$ and $\pi_l(X) \to \tau_n(X)$ can be so chosen that*

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1};$$

(2) *the Fox group $\tau_n(X)$ is algebraically determined by the homotopy groups $\pi_1(X)$, ..., $\pi_n(X)$ and those Whitehead products $[-,-] : \pi_k(X) \times \pi_l(X) \to \pi_{k+l-1}(X)$ for which $k + l - 1 \leq n$.*

Next, given a space $X$, we define the *Gottlieb-Fox groups* to be the evaluation subgroups

$$G\tau_n := G\tau_n(X, x_0) := \mathrm{Im}(ev_* : \tau_n(\mathsf{Map}(X, X), \mathsf{id}_X) \to \tau_n(X, x_0))$$

of the torus homotopy groups $\tau_n$ for $n \geq 1$.

Next, let $G$ denote a finite group acting on a compactly generated Hausdorff path connected space $X$ with a basepoint. The associated pair $(X, G)$ is called in the literature a *transformation* *group*.

**F. Rhodes, F., Proc. London Math. Soc. 16 (1966)**, introduced the notion of the fundamental group $\sigma(X, x_0, G)$ of the pair $(X, G)$, where $x_0$ is a basepoint in $X$.

A typical element in $\sigma(X, x_0, G)$ is the homotopy class $[\alpha; g]$ consisting of a path $\alpha$ in $X$ and a group element $g \in G$ such that $\alpha(0) = x_0, \alpha(1) = gx_0$. The multiplication is given by

$$[\alpha_1; g_1] * [\alpha_2; g_2] := [\alpha_1 + g_1\alpha_2; g_1g_2].$$

It is easy to see that the groups $\pi_1(X, x_0)$, $\sigma(X, x_0, G)$ and $G$ fit into the following short exact sequence

$$1 \to \pi_1(X, x_0) \to \sigma(X, x_0, G) \to G \to 1.$$

Then, **F. Rhodes, F., Canad. J. Math. 21 (1969)**, defined higher groups $\sigma_n(X, x_0, G)$ of the pair $(X, G)$ for $n \geq 1$ which is an extension of $\tau_n(X, x_0)$ by $G$ so that

$$1 \to \tau_n(X, x_0) \to \sigma_n(X, x_0, G) \to G \to 1$$

is exact.

Let $C_n = I \times \mathbb{T}^{n-1}$. We say that a map $f : C_n \to X$ is of *order* $g \in G$ provided $f(0, t_2, \ldots, t_n) = x_0$ and $f(1, t_2, \ldots, t_n) = g(x_0)$ for $(t_2, \ldots, t_n) \in \mathbb{T}^{n-1}$. Denote by $[f; g]$ the homotopy class of a map $f : C_n \to X$ of order $g$ and by $\sigma_n(X, x_0, G)$ the set of all such homotopy classes.

We define an operation $*$ similar to the one on $\sigma(X, x_0, G)$ on the set $\sigma_n(X, x_0, G)$, i.e.,

$$[f'; g'] * [f; g] := [f' + g'f; g'g].$$

Next, we consider the evaluation subgroups of the Rhodes groups $\sigma_n$ for $n \geq 1$.

Given a $G$-space $X$, define the pointwise action of $G$ on the space $\mathrm{Map}(X, X)$, i.e., $(gf)(x) := gf(x)$ for $g \in G$, $f \in \mathrm{Map}(X, X)$ and $x \in X$.

The evaluation subgroup

$$G\sigma_n := G\sigma_n(X, x_0, G) := \mathrm{Im}(ev_* : \sigma_n(X^X, \mathrm{id}_X, G) \to \sigma_n(X, x_0, G))$$

of $\sigma_n$ is called the $n$-th *Gottlieb-Rhodes group* of a $G$-space $X$.

The group $G\sigma_n$ was already defined by M. Woo and Y. Yoon.

To relate the Gottlieb-Rhodes groups with the Gottlieb-Fox groups, we consider the homomorphism $p_n : G\sigma_n \longrightarrow G$ given by $[f;g] \mapsto g$ for $[f;g] \in G\sigma_n$.

**Theorem 12.** *The following sequence*

$$1 \longrightarrow G\tau_n \longrightarrow G\sigma_n \xrightarrow{p_n} G_0 \longrightarrow 1 \tag{7}$$

*is exact. Here, $G_0$ is the subgroup of $G$ consisting of elements $g$ considered as homeomorphisms of $X$ which are freely homotopic to* $\mathrm{id}_X$.

Finally, we have three evaluation subgroups: the classical Gottlieb groups $G_n(X) = G_n(X, x_0)$, the Gottlieb-Fox groups $G\tau_n(X) = G\tau_n(X, x_0)$ and the Gottlieb-Rhodes groups

$$G\sigma_n(X, G) = G\sigma_n(X, x_0, G),$$

where $(X, G)$ is a transformation group.

At the end, for any pointed spaces $X$ and $V$, we can define the *generalized Gottlieb group*

$$G(\Sigma V, X) = \mathrm{Im}\,(ev_*[\Sigma V(X^X, \mathrm{id}_X)] \to [\Sigma V, (X, x_0)])$$

and the $V$*-Fox group*

$$\tau_V(X) = [(V \sqcup *), X].$$

Given a space $W$, the group $[\Sigma V, X]$ can be regarded as a subgroup of $\tau_{V\times W}(X)$ via the projection $V \times W \to V$.

**Proposition 13.** *The generalized Gottlieb group* $G(\Sigma V, X)$*, regarded as a subgroup of* $\tau_{V\times W}(X)$ *is central in* $\tau_{V\times W}(X)$*. In particular, it is central in* $[\Sigma V, X]$*.*

Now, given $\alpha \in [\Sigma V, X]$ and $\beta \in [\Sigma W, X]$, consider the generalized Whitehead product (for details, see **M. Arkowitz, Pacific J. Math. 12 (1962), 7-23)**)

$$\alpha \circ \beta : \Sigma(V \wedge W) \to X.$$

Then, the composite

$$\Sigma(V \times W \sqcup *) \to \Sigma(V \times W) \to \Sigma(V \wedge W) \overset{\alpha\circ\beta}{\to} X$$

determines an element in the $V \times W$-Fox group $\tau_{V\times W}(X)$.

**Theorem 14.** (1) *Given* $\alpha \in [\Sigma V, X]$ *and* $\beta \in [\Sigma W, X]$, *the image of the generalized Whitehead product* $\alpha \circ \beta$ *in* $\tau_{V\times W}(X)$ *is the commutator* $[\alpha, \beta]$ *of* $\alpha$ *and* $\beta$.

(2) *If* $\alpha \in G(\Sigma V, X)$ *then the generalized Whitehead product* $\alpha \circ \beta = 0$.