Loops on polyhedral products: geometric models and homology

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Overview

$$\left.\begin{array}{l} K \text{ — a simplicial complex with } m \text{ vertices} \\ m \text{ based topological spaces } \mathbf{X} = (X_1, \ldots, X_m) \end{array}\right\} \mapsto$$

$$\mapsto \text{ polyhedral product } X_1 \vee \cdots \vee X_m \subset \mathbf{X}^K \subset X_1 \times \cdots \times X_m.$$

Particular cases appeared in works of Porter, Lemaire, Segal, in full generality defined by Anick (1982).
New life: Davis-Januszkiewicz, Buchstaber-Panov, Denham-Suciu, Bahri-Bendersky-Cohen-Gitler, Felix-Tanre etc...

This talk is about $\Omega\mathbf{X}^K$, $H_*\Omega\mathbf{X}^K$.

Overview

- Homotopy and homology of loop spaces
  - geometric models for loops spaces;
  - diagonal subspace arrangements;
  - stable homotopy splittings of loop spaces;
  - homology spittings for loop spaces;
  - higher operations in loop space homology.
- Commutative algebra
  - infinite resolutions for monomial rings;
  - Koszulness, Golodness etc;
  - Poincare series of monomial rings.
- Toric topology
  - Geometric models for the loop spaces of (quasi)-toric manifolds;
  - Loop space homology of (quasi)-toric manifolds and moment-angle complexes.

Polyhedral product: definition

Notation

K is a simplicial complex on the vertex set $[m] = \{1, 2, \ldots, m\}$
$\mathbf{X} = (X_1, \ldots, X_m)$ is a sequence of based topological spaces.

Definition

A polyhedral product, or *K-product*, is the subspace $\mathbf{X}^K \subset X_1 \times \cdots \times X_m$ defined by

$$(x_1, \ldots, x_m) \in \mathbf{X}^K \Leftrightarrow$$

for any $\tau \notin K$ there exists $i \in \tau$ such that x_i is a base-point of X_i.

Examples

Examples

1. if $K = \Delta[m]$ is the full $(m-1)$-dimensional simplex $\Rightarrow \mathbf{X}^K = X_1 \times \cdots \times X_m$;
2. if K is the set of disjoint vertices $\Rightarrow \mathbf{X}^K = X_1 \vee \cdots \vee X_m$;
3. if $K = \partial\Delta[m]$ is the boundary of the simplex $\Rightarrow \mathbf{X}^K$ is the fat wedge of $\mathbf{X}$;
4. if $K = \operatorname{skel}_i \Delta[m]$ — the full i-dimensional skeleton of the simplex, then $\mathbf{X}^K$ is known as a generalized fat wedge $T_i(\mathbf{X})$.

Problem

Notation

Ω is the functor of based loops;
$H_*(A; k)$ denotes the homology of a space A with coefficients in a field k.
If A is a loop space, then $H_*(A; k)$ has a natural structure of algebra with so called Pontryagin product.

Problem

Find the homotopy type of the loop space $\Omega\mathbf{X}^K$ having the loop spaces $\Omega\mathbf{X}$.
Calculate the loop homology algebra $H_*(\Omega\mathbf{X}^K; k)$ having the algebras $H_*(\Omega\mathbf{X}; k)$.

Simple examples

Homotopy

- $\Omega(X_1 \times \cdots \times X_m) \cong \Omega X_1 \times \cdots \times \Omega X_m$;
- $\Omega(X_1 \vee \cdots \vee X_m) \simeq \Omega X_1 * \cdots * \Omega X_m$, where $*$ is the free product of topological monoids.

Homology

- $H_*(\Omega(X_1 \times \cdots \times X_m); k) \simeq H_*(\Omega X_1; k) \otimes \cdots \otimes H_*(\Omega X_m; k)$;
- $H_*(\Omega(X_1 \vee \cdots \vee X_m); k) \simeq H_*(\Omega X_1; k) \sqcup \cdots \sqcup H_*(\Omega X_m; k)$. Here $\sqcup$ is the free product of connected graded algebras.

'Zero' approximation

We have m monomorphisms $H_*(\Omega X_i; k) \hookrightarrow H_*(\Omega \mathbf{X}^K; k)$. Define $\mathcal{F}_0 H_*(\Omega \mathbf{X}^K; k)$ as the subalgebra in $H_*(\Omega \mathbf{X}^K; k)$ generated by all the images of those monomorphisms.

Proposition

$$\mathcal{F}_0 H_*(\Omega \mathbf{X}^K; k) \cong \sqcup_{i=1}^m H_*(\Omega X_i; k)/\sim,$$

with

$$[x, y] = 0$$

for $x \in H_*(X_i; k)$, $y \in H_*(X_j; k)$ *when* $\{i, j\} \in K$.

Remark: it depends only on 1-skeleton of K.

Flag K

When is the zero approximation exact?
Define the class of simplicial complexes which are determined by their 1-skeleton.

Definition

- $\tau \subset [m]$ is called a *missing face* for K if $\partial\tau \subset K$ but $\tau \notin K$.
- A simplicial complex K is called *flag* if any missing face has dimension 1.

Theorem 1 (D.)

$\mathcal{F}_0 H_*(\Omega \mathbf{X}^K; k) \hookrightarrow H_*(\Omega \mathbf{X}^K; k)$ is an isomorphism if and only if K is flag.

Labelled configuration spaces

Classical results for $\Omega^n\Sigma^n Y$, Y — connected

- Milgram-May-Segal model:

$$\Omega^n\Sigma^n Y \simeq C(\mathbb{R}^n, Y) := \sqcup F(\mathbb{R}^n, k) \times_{\Sigma_k} Y^k / \sim;$$

- Snaith splitting:

$$C(\mathbb{R}^n, Y) \cong \vee_{k\in\mathbb{N}} F(\mathbb{R}^n, k)_+ \wedge_{\Sigma_k} Y^{\wedge k};$$

- homology calculations;
- Browder operations.

Labelled configuration spaces with collisions

Idea

Use the theory of labelled configuration spaces with collisions.

We construct $\mathbf{C}_K = \sqcup_{I \in \mathbb{N}^m} \mathbf{C}_K(I)$ where $\mathbf{C}_K(I)$ has equivalent descriptions as

1. configuration space of particles with labels and collisions;
2. complements of diagonal subspace arrangements (+ sometimes inequalities). E.g.

$$\mathbf{C}_K(1,\dots,1) = \mathbb{R}^m - \{(t_1,\dots,t_m) \mid \\ \mid t_{j_1} = \cdots = t_{j_n} \text{ for some } \{j_1,\dots,j_n\} \notin K)\}.$$

The case of suspensions

It works very well when each X_i is a suspension: $\mathbf{X} = \Sigma\mathbf{Y}$.

Proposition (Snaith-type stable splitting, D.)

$$\Omega(\Sigma\mathbf{Y})^K \simeq_s \bigvee_{I=(i_1,\dots,i_m)\in\mathbb{N}^m} \mathbf{C}_K(I)_+ \wedge Y_1^{\wedge i_1} \wedge \cdots \wedge Y_m^{\wedge i_m}.$$

This implies the homology splitting:

$$H_*(\Omega(\Sigma\mathbf{Y})^K; k) \simeq_s \oplus_{I\in\mathbb{N}^m} H_*(\mathbf{C}_K(I); k) \otimes \tilde{H}_*(\mathbf{Y}; k)^{\otimes I}.$$

$H_*(\mathbf{C}_K; k)$ — the 'algebra of operations'.

General case

The idea still works in case of general $\mathbf{X}$!

Modified idea

Add collisions using the monoid structure on ΩX_i for $i \in [m]$.

Proposition (Geometric model for the loop space)

$$\Omega \mathbf{X}^K \simeq \sqcup \mathbf{C}_K(I) \times (\Omega \mathbf{X})^I / \sim$$

Price:

- no stable splitting anymore;
- more equivalence relations.

Loop homology splitting in general case

Theorem 2 (D.)

If $X_1, \dots, X_m$ are 1-connected and k is a field, then the following algebra isomorphism holds

$$H_*(\Omega \mathbf{X}^K) \cong H_*(\mathbf{C}_K) \otimes_{\mathbf{Ass}} \tilde{H}_*(\Omega \mathbf{X}) :=$$

$$= \bigoplus_{I \in \mathbb{N}^m} H_*(\mathbf{C}_K(I)) \otimes \tilde{H}_*(\Omega \mathbf{X})^{\otimes I} / \sim$$

where the equivalence relation are determined by action of the certain "doubling" operations on $H_*(\mathbf{C}_K)$:

$$\mathbf{C}_K(I) \to \mathbf{C}_K(I + e_j)$$

and by the Pontryagin product on $H_*(\Omega X_i)$.

Back to flag complexes

Proof of Theorem 1.

$$\mathcal{F}_0 H_*(\Omega \mathbf{X}^K; k) \cong H_*(\Omega \mathbf{X}^K; k) \Leftrightarrow H_*(\mathbf{C}_K; k) \cong H_0(\mathbf{C}_K; k) \Leftrightarrow$$
$$\text{the arrangements consists only of hyperplanes} \Leftrightarrow K \text{ is flag.}$$

□

In other words, in case of flag complexes all the higher operations $H_{\geq 1}(\mathbf{C}_K; k)$ vanish.

Question

For a non-flag K any minimal subspace of codimension ≥ 2 gives a non-trivial higher operation. What are these operations?

Higher commutators in loop space homology (Williams)

Defining system

Let $\alpha \in H_*(\Omega Y)$, $i \in [m]$ and $\beta_i \in C_*(\Omega Y)$ represent those classes. A defining system for the higher commutator product $\{\alpha_1, \ldots, \alpha_m\}$ is a family of chains $\beta_J \in C_*(\Omega Y)$ indexed by nonempty proper subsets of $[m]$, which satisfies the following conditions

$$d\beta_J = \sum_{S_1 \sqcup S_2 = J} [\beta_{S_1}, \beta_{S_2}],$$

where $[\cdot, \cdot]$ denotes the graded commutator.

Definition

A higher commutator product $\{\alpha_1, \ldots, \alpha_m\}$ is the homology class of the chain

$$\sum_{S_1 \sqcup S_2 = [m]} [\beta_{S_1}, \beta_{S_2}].$$

Loops on a fat wedge

Let $K = \partial\Delta[m]$ for $m \geq 3$. Then $\mathbf{C}_K(1,\ldots,1) \simeq S^{m-2}$.

Proposition

For $K = \partial\Delta[m]$ the monomorphism

$$\tilde{H}_{m-2}(\mathbf{C}_K(1,\ldots,1);k) \otimes \tilde{H}_*(\Omega X_1;k) \otimes \cdots \otimes \tilde{H}_*(\Omega X_m;k) \hookrightarrow H_*(\Omega \mathbf{X}^K;k)$$

realizes a higher commutator product.

The similar construction works for any missing face of K.

Next approximation: single higher products

Let $\mathcal{F}_1 H_*(\Omega \mathbf{X}^K; k)$ be the subalgebra in $H_*(\Omega \mathbf{X}^K; k)$ generated by

- the subalgebras $H_*(\Omega X_i; k)$, $i \in [m]$;
- the higher products of elements from $\tilde{H}_*(\Omega \mathbf{X}; k)$ taken for each missing face.

Question

For which simplicial complexes K this approximation is exact?

We do not know the general answer. Certain sufficient conditions will be given later.

Iterated higher products

An example when the approximation $\mathcal{F}_1$ is not exact.

Example

K — a simplicial complex on [5] which missing faces are $\{1,2,5\},\{3,4,5\}$.
The algebra $H_*(\mathbf{C}_K;k)$ is not generated by 0- and 1-dimensional classes.

Proof

$$\mathbf{C}_K(1,1,1,1,1) \simeq T^2 = S^1 \times S^1,$$

so there is a nontrivial class in $H_2(\mathbf{C}_K)$. This class is not a product of 0- and 1-dimensional classes due to natural multi-degree reasons. □

Actually, this operation is $\{\{x_1,x_2,x_5\},x_3,x_4\}$ for $x_i \in H_*(\Omega X_i;k)$.
Conclusion: the iterated higher commutators can be needed.

All higher operations

Question

Do all single and iterated higher commutator products generate all the higher operations?

Answer: we do not know.

The positive answer for a simplicial complex K implies nice corollaries:

1. all the relations in Theorem 1 takes form of Leibnitz rule;
2. any homology class in the complements of diagonal subspace arrangements can be realized as embedded product of spheres;

Generalized Leibnitz rule

2 types of relations for generators:

1. from equivalence relations of Theorem 1;
2. the relations in $H_*(\mathbf{C}_K;k)$.

One of the advantages of using single and iterated higher products: classical form of those relations.

Generalized Leibnitz rule

If an element of $H_*(\mathbf{C}_K;k)$ can be written as a higher commutator $\{c_1,\dots,c_n\}$ then the equivalence relations of Theorem 1 can be written as

$$\{c_1,\dots,c_j\cdot c_j',\dots,c_s\}=\{c_1,\dots,c_j,\dots,c_s\}c_j'\pm c_j\cdot\{c_1,\dots,c_j',\dots,c_s\}.$$

Presentation of the algebra $H_*(\Omega \mathbf{X}^K; k)$: summary

Generators

- elements of $H_*(\Omega X_i)$, $i = 1, \dots, m$.
- the single higher commutators of elements from different $H_*(\Omega X_i)$;
- iterated higher commutators;
- anything else?

Relations

- relations among the operations from $H_*(\mathbf{C}_K; k)$;
- generalized Leibnitz rule for higher operations.
- relations of Theorem 2 for 'anything else'..

Connection with problems in commutative algebra

Definition

Exterior Stanley-Reisner ring:

$$\Lambda(K) = \Lambda[v_1, \ldots, v_m]/I_{SR},$$

where the ideal I_{SR} is generated by all the monomials $v_{j_1} \ldots v_{j_n}$ for $\{j_1, \ldots, j_n\} \notin K$.

Theorem (D., cf. Peeva-Reiner-Welker)

After certain regrading of elements the following algebra isomorphism holds

$$H_*(\mathbf{C}_K; k) \cong \mathrm{Ext}_{\Lambda(K)}(k; k).$$

Remark: Peeva, Reiner, Welker proved the analogous cohomology statement for the complement of the diagonal arrangement $D_K = \mathbf{C}_K(1, \ldots, 1)$.

Again back to flag complexes: Koszulness

Definition

The algebra A is called Koszul if $\mathrm{Ext}_A(k;k)$ is generated by $\mathrm{Ext}^1_A(k;k)$.

Froberg proved that any polynomial algebra with *quadratic monomial relations* is Koszul. This statement together with Theorem 2 give a new proof of Theorem 1:

$$K \text{ is flag } \Leftrightarrow \text{ the algebra } \Lambda(K) \text{ is Koszul} \Leftrightarrow$$
$$\mathrm{Ext}_{\Lambda(K)}(k;k) \cong k[u_1,\dots,u_m]/([u_i,u_j]=0 \text{ if } \{i,j\}\in K).$$

The earlier proofs of certain particular cases of Theorem 1 used this machinery: Papadima-Suciu (the spaces X_i's are spheres), Panov-Ray ($X_i=\mathbb{C}P^\infty$).

Class $\mathcal{C}$ over k

We define a class of simplicial complexes - the class $\mathcal{C}$ over k imposing some homology condition on their 'copure skeletons'. E.g. all shifted complexes (which are defined combinatorially) are in the class $\mathcal{C}$ over any field k.

Test example

$K = \mathrm{skel}_i\,\Delta[m]$ for $i \geq 1$, $m \geq 4$ is in class $\mathcal{C}$ over any field. It is not flag!

Full answer for complexes from $\mathcal{C}$

Theorem

If K is in $\mathcal{C}$ over k, the algebra $H_*(\Omega\mathbf{X}^K;k)$ has the following algebra presentation.

Generators:

(a) $[x]$ *for any class* $x \in \tilde{H}(\Omega X_i)$ *of the same degree:* $\deg[x] = \deg x$.

(b) *the higher commutator products*

$$\{x_{j_1},\dots,x_{j_s}\}$$

for each missing face $\tau = \{j_1,\dots,j_s\}$ *in* K *with* $s \geq 3$, *and for any set of* $x_{j_s} \in \tilde{H}(\Omega X_{j_s})$ *(with the shift of degree by* $(s-2)$*).*

Relations (1) *for any* $x_i, x_{i'} \in H_*(\Omega X_i)$, $[x_i]\cdot[x_{i'}] = [x_i x_{i'}]$;

(2) *commutative relations;*

(3) *generalized Leibniz rule;*

(4) *generalized Jacobi rule.*

Relations

Commutative relations:

$$[x_i] \cdot [x_j] - (-1)^{\vee}[x_j] \cdot [x_i] = 0$$

for $x_i \in H_*(\Omega X_i; k)$, $x_j \in H_*(\Omega X_j; k)$ when $\{i, j\} \in K$.

Generalized Leibnitz rule:

$\{x_{i_1}, \ldots, x_{i_l} x'_{i_l}, \ldots, x_{i_s}\} =$
$\{x_{i_1}, \ldots, x_{i_l}, \ldots, x_{i_s}\} \cdot [x'_{i_l}] + (-1)^{\vee}[x_{i_l}] \cdot \{x_{i_1}, \ldots, x'_{i_l}, \ldots, x_{i_s}\}$

Generalized Jacobi rule:

$$\sum_{j:\tau-\{i_j\}\notin K} (-1)^{\vee}[\{x_{i_1}, \ldots, \hat{x_{i_j}}, \ldots, x_{i_s}\}, [x_{i_j}]] = 0$$

for any $\tau = \{i_1, \ldots, i_s\} \subset [m]$ with $\partial\tau \not\subset K$ but $\mathrm{skel}_{s-3}\,\tau \subset K$.

(Quasi)-toric manifolds

Informal definition

(Quasi-)toric manifolds are smooth $2n$-dimensional manifolds with smooth action of T^n such that the orbit space is diffeomorphic to a simple polytope P^n.

n-dim simple polytopes $\leftrightarrow$ $(n-1)$-dim simplicial complexes
$P^n \mapsto\ K$ combinatorially dual to ∂P^n.

Combinatorial data

A quasitoric manifold is determined by the following data: a polytope P^n with numbering of its facets: $1, \ldots, m$, and an epimorphism $\lambda : T^m \to T^n$.

Example: projective spaces

Example

$M^{2n} = \mathbf{C}P^n$ with the action

$$(t_1, \dots, t_n)(z_0 : \dots : z_n) = (z_0 : t_1 z_1 : \dots : t_n z_n),$$

where $t_i \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
The orbit polytope is $P^n = \Delta[n+1]$.
$\lambda : T^{n+1} \to T^n$

Connection with K-products

Theorem (Buchstaber-Panov)

Borel construction:

$$M^{2n} \times_{T^n} ET^n \simeq (\mathbb{C}P^\infty)^K.$$

We get the fibration

$$M^{2n} \to (\mathbb{C}P^\infty)^K \to (\mathbb{C}P^\infty)^n,$$

which splits after looping:

$$\Omega(\mathbb{C}P^\infty)^K \simeq \Omega M^{2n} \times T^n.$$

This splitting doen't respect the multiplication of loops!

Loops on toric manifolds (with Nige Ray)

Homotopy modify the geometric model for $(\mathbb{C}P^\infty)^K$ to a monoid:

$$\mathcal{L}_K = \sqcup_{I \in \mathbb{N}^m} \tilde{C}_K(I) \times (S^1)^I / \sim$$

We define the space $\mathcal{G}$ of very special geodesics on M^{2n} trasversal to T^n-orbits in such a way that $\mathcal{L}_K$ is homeomorphic to the piecewise geodesics on M^{2n} starting in the fixed point and ending in the same orbit. Consider the composition of homomorphisms:

$$\varphi : \mathcal{L}_K \to T^m \xrightarrow{\lambda} T^n.$$

Theorem (D., Nige Ray)

Kerφ *provides a geometric model for* ΩM^{2n}. *It is the space of piecewise geodesics with pieces from* $\mathcal{G}$ *and which start and end at the fixed point.*