On Symmetry of Flat Manifolds

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Outline

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Outline

Flat Manifolds and Bieberbach Groups

- X – compact, connected, flat Riemannian manifold (flat manifold for short).
- $\Gamma = \pi_1(X)$ – fundamental group of X – Bieberbach group.
- X is isometric to $\mathbb{R}^n/\Gamma$.
- Γ determines X up to affine equivalence.

Abstract Definition of Bieberbach Groups

Definition

Bieberbach group is a torsion-free group defined by a short exact sequence

$$0 \longrightarrow M \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

- G – finite group (holonomy group of Γ).
- M – faithful G-lattice, i.e. faithful and free $\mathbb{Z}G$-module, finitely generated as an abelian group.
- Element $\alpha \in \mathrm{H}^2(G, M)$ corresponding to the above extension is special, i.e. $\mathrm{res}^G_H \alpha \neq 0$ for every non-trivial subgroup H of G.

Group of Affinities

- $\mathrm{Aff}(X)$ – group of affine self equivalences of X.
  - $\mathrm{Aff}(X)$ is a Lie group.
- $\mathrm{Aff}_0(X)$ – identity component of $\mathrm{Aff}(X)$.
  - $\mathrm{Aff}_0(X)$ is a torus.
  - Dimension of $\mathrm{Aff}_0(X)$ equals $b_1(X)$ – the first Betti number of X ($b_1(X) = \mathrm{rk}\ \mathrm{H}^0(G, M)$).

Theorem (Charlap, Vasquez 1973)

$$\mathrm{Aff}(X)/\mathrm{Aff}_0(X) \cong \mathrm{Out}(\Gamma)$$

Corollary

$\mathrm{Aff}(X)$ *is finite iff* $b_1(X) = 0$ *and* $\mathrm{Out}(\Gamma)$ *is finite.*

Problem

Problem (Szczepański 2006)

Which finite groups occur as outer automorphism groups of Bieberbach groups with trivial center.

Theorem (Belolipetsky, Lubotzky 2005)

For every $n \geq 2$ and every finite group G there exist infinitely many compact n-dimensional hyperbolic manifolds M with $\mathrm{Isom}(M) \cong G$.

Calculating $\mathrm{Out}(\Gamma)$

Theorem (Charlap, Vasquez 1973)

$\mathrm{Out}(\Gamma)$ *fits into short exact sequence*

$$0 \longrightarrow \mathrm{H}^1(G, M) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow N_\alpha/G \longrightarrow 1.$$

- N_α – stabilizer of $\alpha \in \mathrm{H}^2(G, M)$ under the action of $N_{\mathrm{Aut}(M)}(G)$ defined by

$$n * a(g_1, g_2) = n \cdot a(n^{-1} g_1 n, n^{-1} g_2 n).$$

Finite Groups of Affinities of Flat Manifolds

- (Szczepański, Hiss 1997)
  - C_2 – two flat manifolds.
  - $C_2 \times (C_2 \wr F)$, where $F \subset S_{2k+1}$ is cyclic group generated by the cycle $(1, 2, \ldots, 2k+1), k \geq 2$.
- (Waldmüller 2003)
  - A flat manifold with no symmetries.
- (Lutowski, PhD)
  - $C_2^k, k \geq 2$.

Outline

Lattice Basis

Definition

Bieberbach group is a torsion-free group defined by a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

- $G \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ – integral representation of G.
- G acts on $\mathbb{Z}^n$ by matrix multiplication.
- Element $\alpha \in \mathrm{H}^2(G, \mathbb{Z}^n)$ corresponding to the above extension is special.

Special Element of $\mathrm{H}^2(G, M)$

- $\alpha \in \mathrm{H}^2(G, \mathbb{Z}^n)$ is special iff $\mathrm{res}^G_H \alpha \neq 0$ for every $1 \neq H < G$.
- By the transitivity of restriction – enough to check subgroups of prime order.
- By the action '$*$' of normalizer – enough to check conjugacy classes of such groups.
- Since $\mathrm{H}^2(G, \mathbb{Z}^n)$ is hard to compute, we use an isomorphic group $\mathrm{H}^1(G, \mathbb{Q}^n/\mathbb{Z}^n)$.

Holonomy Group

- $G = M_{11}$ – Mathieu group on 11 letters.
- $|G| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$.
- G has a presentation

$$G = \langle a, b | a^2, b^4, (ab)^{11}, (ab^2)^6, ababab^{-1}abab^2ab^{-1}abab^{-1}ab^{-1} \rangle.$$

- Representatives of conjugacy classes of G:
  - Order 2: $\langle a \rangle$,
  - Order 3: $\langle (ab^2)^2 \rangle$,
  - Order 5: Sylow subgroups,
  - Order 11: Sylow subgroups.

The Lattice – Definition

- The lattice is given by integral representation of G.
- M_1, M_3, M_4 – representation from Waldmüller's example of degree 20,44,45 respectively.
- M_3 – sublattice of index 3 of Waldmüller's lattice of degree 32, given by the orbit of the vector

$$\underbrace{(2, 1, \ldots, 1)}_{32}.$$

- $M := M_1 \oplus \ldots \oplus M_4$.

The Lattice – Properties

i	Degree	$\mathbb{C}$-irr	$\mathrm{H}^1(G, M_i)$	$\mathrm{H}^2(G, M_i)$	$\lvert\langle\alpha_i\rangle\rvert$	$\lvert H_i\rvert$
1	20	No	0	C_6	6	3
2	32	No	C_3	C_5	5	5
3	44	Yes	0	C_6	6	2
4	45	Yes	0	C_{11}	11	11

- $\mathrm{H}^1(G, M) = \bigoplus_{i=1}^{4} \mathrm{H}^1(G, M_i) = C_3$.

Torsion-Free Extension

Proposition

Extension Γ of M by G defined by $\alpha := \alpha_1 \oplus \ldots \oplus \alpha_4 \in \mathrm{H}^2(G, M)$ is torsion-free.

Next Step in Calculating $\mathrm{Out}(\Gamma)$

- Recall short exact sequence

$$0 \longrightarrow \mathrm{H}^1(G, M) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow N_\alpha/G \longrightarrow 1.$$

- $\mathrm{H}^1(G, M) = C_3$.
- Next step: calculate N_α.

Calculation of Stabilizer

Step 1: Centralizer

- $C_{\mathrm{Aut}(M)}(G) = C_{\mathrm{Aut}(M_1)}(G) \times \ldots \times C_{\mathrm{Aut}(M_4)}(G)$.
- M_3, M_4 – absolutely irreducible, thus $C_{\mathrm{Aut}(M_i)}(G) = \langle -1 \rangle, k = 3, 4$.
- For $k = 1, 2 : C_{\mathrm{Aut}(M_k)}(G) = U(\mathrm{End}_{\mathbb{Z}G}(M_k))$. We have:
  - $\mathrm{End}_{\mathbb{Z}G}(M_1) \cong \mathbb{Z}[\sqrt{-2}]$,
  - $\mathrm{End}_{\mathbb{Z}G}(M_2) \cong \mathbb{Z}[\frac{3\sqrt{-11}-1}{2}] \subset \mathbb{Z}[\sqrt{-11}, \frac{1}{2}]$.
- $U(\mathrm{End}_{\mathbb{Z}G}(M_k)) = \langle -1 \rangle, k = 1, 2$.

Corollary

$$C_{\mathrm{Aut}(M)}(G)_\alpha = 1.$$

Calculation of Stabilizer

Step 2: Normalizer

- Since $\mathrm{Out}(G) = 1$, we have

$$N_{\mathrm{Aut}(M)}(G) = G \cdot C_{\mathrm{Aut}(M)}(G).$$

Corollary

$$N_{\mathrm{Aut}(M)}(G)_\alpha = G.$$

Flat Manifold with Odd-Order Group of Symmetries

Theorem

If X is a manifold with fundamental group Γ, then $\mathrm{Aff}(X) \cong C_3$.

Further properties of Γ

- $\mathrm{Aut}(\Gamma)$ is a Bieberbach group.
- $\mathrm{Out}(\mathrm{Aut}(\Gamma)) = 1$.
- $$\exists_{\Gamma' \triangleleft \Gamma} \, [\Gamma \colon \Gamma'] = 3 \wedge \mathrm{Out}(\Gamma') \cong C_3^2.$$

Outline

Direct Factors of Centerless Groups

Definition

Group is *directly indecomposable*, if it cannot be expressed as a direct product of its nontrivial subgroups.

Theorem (Golowin, 1939)

If a group G has a trivial center, then any two decompositions to a direct product of subgroups

$$G = \prod_{\alpha} H_\alpha = \prod_{\beta} F_\beta$$

have common subdecomposition.

Structure of Automorphisms of Γ^n

- Γ – directly indecomposable centerless Bieberbach group.
- $n \in \mathbb{N}$.
- $\Gamma^n := \underbrace{\Gamma \times \ldots \times \Gamma}_{n}$.
- $\Gamma_i := \{e\}^{i-1} \times \Gamma \times \{e\}^{n-i}$, $i = 1, \ldots, n$.

Lemma

$$\forall_{\varphi \in \mathrm{Aut}(\Gamma^n)} \exists_{\sigma \in S_n} \forall_{1 \leqslant i \leqslant n} \; \varphi(\Gamma_i) = \Gamma_{\sigma(i)}$$

Automorphism and Outer Automorphism Group

Corollary

Let Γ *be a directly indecomposable Bieberbach group with a trivial center and* $n \in \mathbb{N}$. *Then*

$$\operatorname{Aut}(\Gamma^n) = \operatorname{Aut}(\Gamma) \wr S_n,$$

hence

$$\operatorname{Out}(\Gamma^n) = \operatorname{Out}(\Gamma) \wr S_n.$$

Subgroups of Outer Automorphism Groups

- By Waldmüller's example: S_n occurs as an outer automorphism group of a Bieberbach group with a trivial center.

Corollary

Let G be a finite group. There exists a flat manifold X, with $b_1(X) = 0$ and monomorphism

$$i\colon G \to \mathrm{Aff}(X),$$

such that $[\mathrm{Aff}(X) \colon i(G)]$ is finite.

Generalization of the Lemma

Theorem

Let Γ_i, $i = 1, \ldots, k$, *be mutually nonisomorphic directly indecomposable Bieberbach groups with trivial center. Let* $n_i \in \mathbb{N}$, $i = 1, \ldots, k$. *Then*

$$\mathrm{Out}(\Gamma_1^{n_1} \times \ldots \times \Gamma_k^{n_k}) \cong \mathrm{Out}(\Gamma_1) \wr S_{n_1} \times \ldots \times \mathrm{Out}(\Gamma_k) \wr S_{n_k}.$$

Summary

- There exists a Bieberbach group with a trivial center and odd-order, non-trivial outer automorphism group.
- (Outer) automorphism group of a Bieberbach group depends on its direct component.
- Every finite group can be realized as a subgroup of finite index in outer automorphism group of a Bieberbach group with trivial center.

Thank you!