Bounded Cohomology

Mladen Bestvina

Definition of $H_b^n(X)$

Let X be a topological space. We have singular (co)chain complexes:

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(with coefficients $\mathbb{R}$).
Let $C^n_b(X) = \{c \in C^n(X) | \sup_{\sigma:\Delta^n \to X} |c(\sigma)| < \infty\}$
and we have the bounded cochain complex

$$0 \rightarrow C^0_b(X) \rightarrow C^1_b(X) \rightarrow \cdots$$

whose cohomology is bounded cohomology

$$H^n_b(X)$$

Basic properties of $H_b^n(X)$

- canonical map $H_b^n(X) \to H^n(X)$,
- $H_b^1(X) = 0$ for any X,
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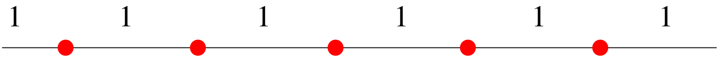
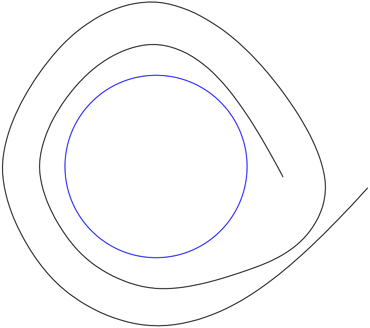
Caution: We cannot use simplicial chain complex in place of the singular complex. E.g. for $X = \mathbb{R}$ the cochain that assigns 1 to each edge is an essential bounded cocycle.



Examples

Example

$X = S^1$. Then $H_b^1(S^1) \to H^1(S^1) = \mathbb{R}$ is 0. Say $c \in C_b^1(S^1)$ is a cocycle. Let $\sigma_n : [0,1] \to S^1$ be $t \mapsto e^{2\pi i n}$, so σ_n/n represents the fundamental class. Thus $c(\sigma_n/n) \to 0$ and so $c(\sigma_1) = 0$.



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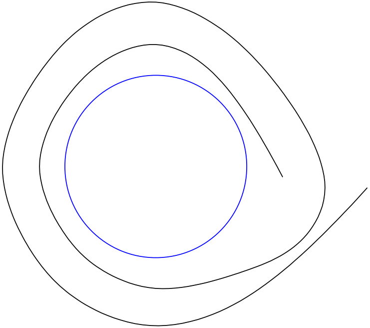


Example

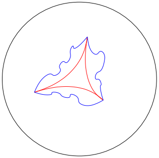
$X = T$, a torus. Then $H^2_b(T) \to H^2(T) = \mathbb{R}$ is 0. The argument is similar; the key is that there is a degree n map $T \to T$.

Example

X a closed oriented hyperbolic surface. Let $c \in C_b^2(X)$ be defined by

$$c(\sigma) = \text{signed area of } \hat{\sigma}$$

where $\hat{\sigma}$ is the straightened singular simplex; it agrees with σ on the vertices, but sends edges to geodesics.



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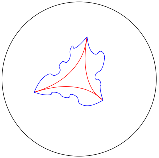
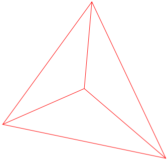
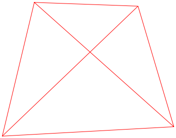


- c is a cocycle since the signed area of the boundary of a tetrahedron is 0,
- c is bounded since the area of any geodesic triangle is $< \pi$,





  - c evaluates to $Area(X) = (2g-2)\pi$ on the fundamental class. So $H_b^2(X) \to H^2(X)$ is onto.

Gromov norm in homology

For $x \in H_n(X;\mathbb{R})$ define

$$||x|| = \inf\{\sum |a_i| \mid x = [a_1\sigma_1 + a_2\sigma_2 + \cdots + a_k\sigma_k]\}$$

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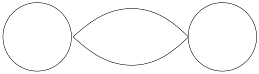
There is a non-degenerate pairing

$$Im[H_b^n(X) \to H^n(X)] \times H_n(X)/(\text{classes of norm } 0) \to \mathbb{R}$$

Bounded cohomology of groups

Let G be a discrete group. There is a contractible free G-complex Ω with vertices G and a k-simplex is an ordered $(k+1)$-tuple of vertices.



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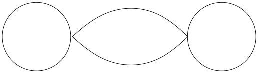

Recall that the group cohomology $H^n(G)$ of G is the cohomology of the simplicial cochain complex of Ω/G, or more succinctly, of

$$0 \to F(G,\mathbb{R})^G \to F(G^2,\mathbb{R})^G \to F(G^3,\mathbb{R})^G \to \cdots$$

where $F(G^n,\mathbb{R})^G$ are G-invariant functions on G^n.

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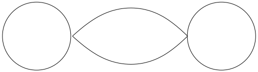

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Bounded group cohomology $H_b^n(G)$ of G is the bounded version:

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Amenable means that one can associate average to any bounded function $f : G \to \mathbb{R}$ which is

- linear, $Av(af + bg) = aAv(f) + bAv(g)$,
- monotone, $f \geq 0 \Rightarrow Av(f) \geq 0$,
- $G - invariant$, $Av(fL_g) = Av(f)$ for $L_g : G \to G$ left translation by g, and
- $Av(1) = 1$.

e.g. solvable groups are amenable.

Proof.

If $p : X \to Y$ is a k-sheeted covering map there is the transfer map

$$\tau : C^n(X) \to C^n(Y)$$

given by averaging:

$$\tau(c)(\sigma) = \frac{1}{k}\sum_{i=1}^{k} c(\tilde{\sigma}_i)$$

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Transfer works with amenable covers for bounded cohomology (i.e. covers with amenable deck group). Applying to $p : \widetilde{K(G,1)} \to K(G,1)$ we see that

$$p^* : H_b^n(K(G,1)) \to 0$$

is injective. □

What does bounded cohomology measure?
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The field consists of two halves, studying image and the kernel of

$$H_b^n(G) \to H^n(G)$$

Kernel results

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Image results

- (Lafont-Schmidt 2006) If G is the fundamental group of a closed n-dimensional locally symmetric space of noncompact type, then $H_b^n(G) \to H^n(G) = \mathbb{R}$ is onto.

Quasi-homomorphisms $G \longrightarrow \mathbb{R}$ and $H_b^2(G)$

Definition

A quasi-homomorphism on a group G is a function $\phi : G \longrightarrow \mathbb{R}$ such that

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Let

$$\widetilde{QH}(G) = QH(G)/(Hom(G,\mathbb{R}) + \ell^{\infty}(G))$$

Quasi-homomorphisms $G \to \mathbb{R}$ and $H_b^2(G)$

Proposition

$\widetilde{QH}(G) = Ker[H_b^2(G) \to H^2(G)]$.

Proof.

A function $\phi : G \to \mathbb{R}$ can be viewed as a 1-cochain in $C^1(\Omega/G)$ that to the 1-cell e_g assigns $\phi(g)$. Thus $\delta(\phi)$ is a 2-cocycle that to a 2-cell $E_{ab=c}$ assigns $\phi(a) + \phi(b) - \phi(c)$ and so $\delta(\phi)$ is bounded when ϕ is a quasi-homomorphism. This gives

$$QH(G) \to Ker[H_b^2(G) \to H^2(G)]$$

It is an exercise in recalling definitions that this is onto and the kernel is $Hom(G, \mathbb{R}) + \ell^\infty(G)$. □

Integer coefficients

The statement that $H_b^2(\mathbb{Z};\mathbb{R}) = 0$ implies that $\widetilde{QH}(\mathbb{Z}) = 0$ i.e. every quasi-homomorphism $\phi : \mathbb{Z} \to \mathbb{R}$ is within bounded distance from $n \mapsto sn$ for some $s \in \mathbb{R}$. This leads to:

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Proof.

$$H_b^2(\mathbb{Z};\mathbb{Z}) = Ker[H_b^2(\mathbb{Z};\mathbb{Z}) \to H^2(\mathbb{Z};\mathbb{Z})] = \widetilde{QH}(\mathbb{Z};\mathbb{Z}) = Hom(\mathbb{Z};\mathbb{R})/Hom(\mathbb{Z};\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$$

□

Example

$\phi(n) = [n/2]$

Rotation number and Euler number

When G acts on S^1 by orientation preserving homeomorphisms there is the associated Euler class $e \in H^2(G; \mathbb{Z})$.

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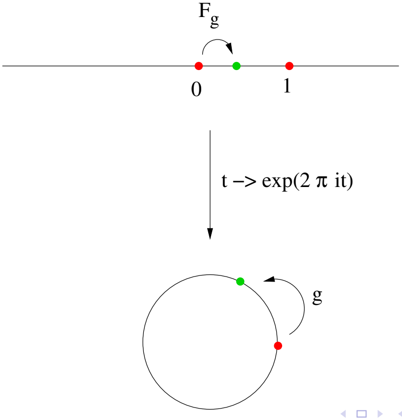
- e is the obstruction to lifting the action to $\mathbb{R} \to S^1$.
- Explicit cocycle: for $g \in G$ choose the lift $F_g : \mathbb{R} \to \mathbb{R}$ with $F_g(0) \in [0,1)$. Then let $E_{ab=c} \mapsto F_a F_b F_c^{-1}$ (translation by 0 or 1).



Rotation number and Euler number

- (Ghys) This cocycle is bounded so we have a lift $\tilde{e} \in H_b^2(G; \mathbb{Z})$.
- (Ghys) When $G = \mathbb{Z}$ the bounded Euler class is the rotation number in $H_b^2(\mathbb{Z}; \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$.

Rotation number and Euler number

Theorem (Ghys 1984)

- *Two conjugate actions of G on S^1 have the same bounded Euler class.*
- *If two actions of G with dense orbits have the same bounded Euler class, then they are conjugate.*

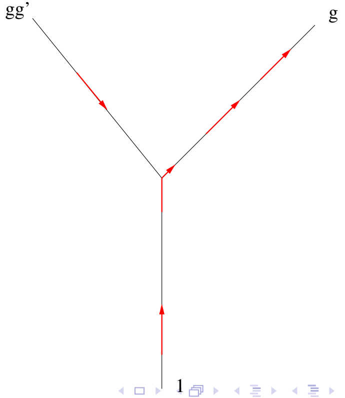
The Brooks construction, 1981

$G = F_2 =< a, b >$. Fix an element of G, say $w = ab$. Define $\phi_w : F_2 \to \mathbb{Z}$ for reduced words g by:

$$\phi_w(g) = \text{maximal \# of nonoverlapping copies of } w - \text{maximal \# of nonoverlapping copies of } w^{-1}$$

E.g.
$\phi_{ab}(babbaBABab) = 2 - 1 = 1.$

ϕ_w is a quasi-homomorphism: $|\phi(gg') - \phi(g) - \phi(g')| \leq 6$. All non-straddling copies of w have canceling contributions.



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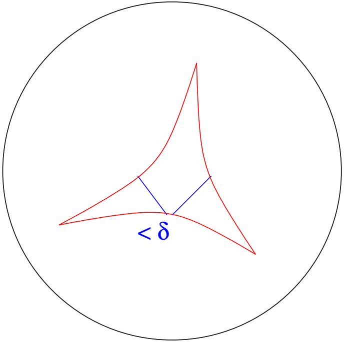
In fact, $\dim \widetilde{QH}(F_2) = \infty$ by choosing a suitable sequence of w's.

The Brooks construction, generalizations

Efforts to generalize Brooks' construction to other groups:
$\dim \widetilde{QH}(G) = \infty$ for

- G Gromov hyperbolic (Epstein-Fujiwara 1997). This means that the Cayley graph of G is Gromov hyperbolic, i.e. geodesic triangles are uniformly thin: for some $\delta > 0$ and all geodesic triangles ABC we have $\overline{AB} \subset N_\delta(\overline{AC} \cup \overline{BC})$.

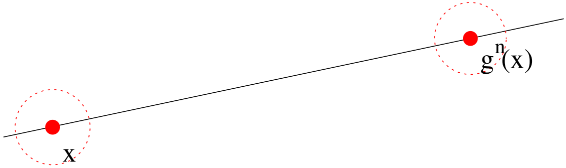


The Brooks construction, generalizations

- G has a Weakly Properly Discontinuous (WPD) action on a Gromov hyperbolic space X (B-Fujiwara 2002):
  - G is not virtually cyclic,
  - there are hyperbolic elements (positive translation length), and
  - for every hyperbolic element g, every $x \in X$, and every $D > 0$ there is $n > 0$ so that

$$\{h \in G \mid d(x, h(x)) < D, d(g^n(x), hg^n(x)) < D\}$$

is finite.



The Brooks construction, generalizations

Main Example: Mapping class group acting on the curve complex.

Corollary

- *(Masur-Kaimanovich 1996, Farb-Masur 1998) A lattice in a higher rank symmetric space does not embed in any mapping class group.*
- *The Cayley graph of a (non-virtually abelian) mapping class group contains arbitrarily large balls consisting entirely of pseudo-Anosov homeomorphisms.*

The Brooks construction, generalizations

- G has a nonelementary WPD action on a CAT(0) space with rank 1 elements (B-Fujiwara, 2008).

E.g. many right-angled Artin groups and Coxeter groups are in this category.

New developments

- (Monod-Remy 2006) There are groups G where $\widetilde{QH}(G)$ is nonzero but finite dimensional.
  Idea: Start with G where $\widetilde{QH}(G) = 0$ i.e. $H_b^2(G) \to H^2(G)$ is injective, but there is a class $0 \neq e \in H^2(G)$ in the image. Let $\tilde{G}$ be the central extension with Euler class e. Then $H^2(\tilde{G}) = H^2(G)/e$ but $H_b^2(\tilde{G}) = H_b^2(G)$.

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Definition (Manning)

A geodesic metric space X satisfies the bottleneck property if there is $\Delta \geq 0$ such that for any two points $p, q \in X$ there is a midpoint $r \in X$ so that any path from p to q intersects $B(r, \Delta)$.

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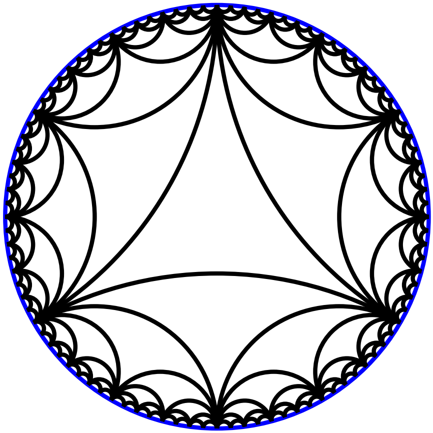
Theorem (Manning)

A geodesic metric space X is a quasi-tree if and only if it satisfies the bottleneck property.

The Farey graph



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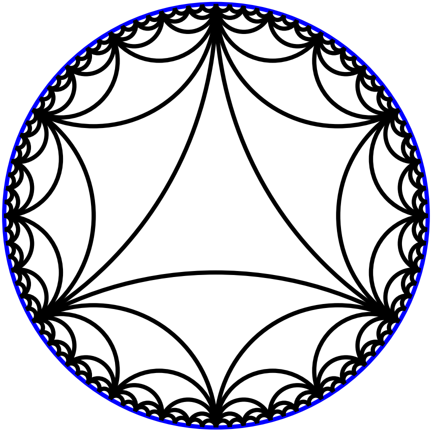

Exercise

The Farey graph is a quasi-tree.

New developments, continued

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- (B-Bromberg-Fujiwara 2009) If G is a group for which generalized Brooks construction is in place then G has many WPD actions on quasi-trees.
  - This gives an alternative way of showing that these groups have $\dim \widetilde{QH}(G) = \infty$.
  - There are many hyperbolic groups that satisfy Kazhdan's Property (T). These groups don't have non-trivial actions on trees, but have many interesting actions on quasi-trees.

Conjecture

Let G be finitely generated. Then

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G admits a quotient that has a WPD action on a quasi-tree

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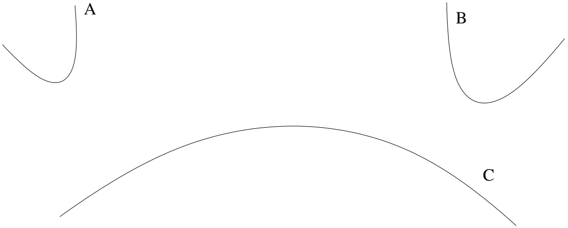
Contrast with the Stallings' theorem:
If a finitely generated group G has infinitely many ends (i.e. $\dim H^1(G; \mathbb{Z}G) = \infty$) then G acts nontrivially on a tree with finite edge stabilizers.
Current status: About $\frac{3}{4}$ of the conjecture is proved.

Analogy

Let G be a discrete group of isometries of $\mathbb{H}^n$ and $\mathbf{Y}$ the set of axes of loxodromic elements in finitely many conjugacy classes of G.



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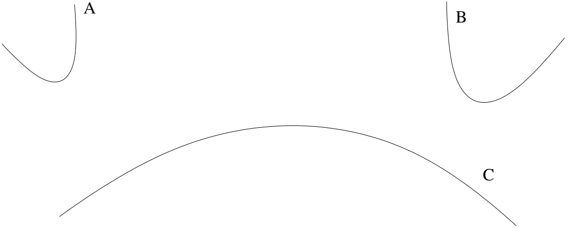

Features:

- $A, Y \in \mathbf{Y}$ implies the (nearest point) projection $\pi_Y(A)$ is bounded.

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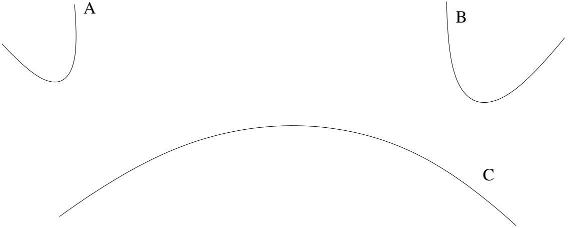

Features:

- $A, Y \in \mathbf{Y}$ implies the (nearest point) projection $\pi_Y(A)$ is bounded.
- Define
$$d_Y(A, B) = \mathrm{diam}(\pi_Y(A \cup B))$$
Then at most one of 3 numbers $d_Y(A, B), d_A(B, Y), d_B(A, Y)$ can be big.

Axioms

Let $\mathbf{Y}$ be a set and for every $Y \in \mathbf{Y}$ let $d_Y : \mathbf{Y} \setminus \{Y\} \times \mathbf{Y} \setminus \{Y\} \to [0, \infty)$ be a "distance function" satisfying

- $d_Y(A, B) = d_Y(B, A)$,
- $d_Y(A, C) \leq d_Y(A, B) + d_Y(B, C)$,
- there is $\xi > 0$ such that $\min\{d_C(A, B), d_B(A, C)\} < \xi$, and
- there is $K_0 > 0$ such that for all A, B

$$\{C \in \mathbf{Y} | d_C(A, B) > K_0\}$$

  is finite.

Construction of the quasi-tree

Definition

Fix $K >> K_0, \xi$. $T(\mathbf{Y})$ is the graph whose vertex set is $\mathbf{Y}$ and A, B are joined by an edge if $d_Y(A, B) < K$ for all Y.

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Theorem (BBF)

$T(\mathbf{Y})$ is a quasi-tree.

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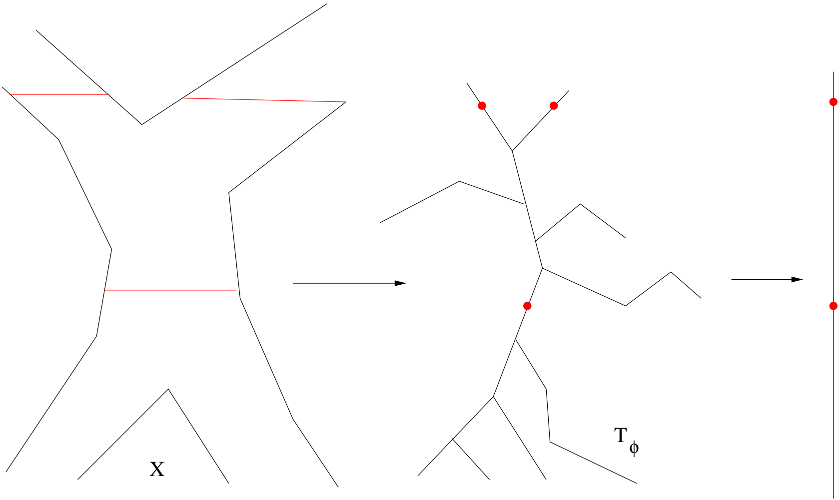
Observe: If $\phi : G \to \mathbb{R}$ is a homomorphism, then we can construct an equivariant map $X \to \mathbb{R}$ and the associated monotone-light factorization $X \to T_\phi \to \mathbb{R}$ produces a G-tree T_ϕ.



New developments – Manning's work

Manning (2005): Any quasi-homomorphism $\phi : G \to \mathbb{R}$ gives rise to a natural action of G on a quasi-tree T_ϕ.

Definition

ϕ a bushy quasi-homomorphism if there are rays $r_1^+, r_2^+ : [0, \infty) \to T_\phi$ converging to distinct ends of T_ϕ whose images in $\mathbb{R}$ go to $+\infty$, and similarly there are rays $r_1^-, r_2^- : [0, \infty) \to T_\phi$ converging to distinct ends of T_ϕ whose images in $\mathbb{R}$ go to $-\infty$.

Theorem (Manning 2005)

If G has a bushy quasi-homomorphism then $\dim H_b^2(G) = \infty$.

Conjectures

Let G be finitely generated. Then

$$\dim \widetilde{QH}(G) = \infty \iff$$

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Contrast with the Stallings theorem:

Theorem

Let G be finitely generated. Then

$$H^1(G; \mathbb{Z}G) \neq 0 \iff$$

G has a nontrivial action on a tree with finite edge stabilizers