

# Some Stable and Unstable Homotopy of Cell Complexes

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# 1. Exponents (at odd primes)

- Instead of answering the very difficult question “what are the homotopy groups of a space?”, we ask for something weaker like “approximately how fast do the homotopy groups grow?”.

- The  $p$ -exponent  $exp_p(G)$  of an abelian group  $G$  is the smallest power  $p^t$  such that

$$p^t \times (p\text{-torsion in } G) = 0.$$

- We measure the growth of homotopy groups of  $X$  by finding bounds for the growth of  $exp_p(\pi_i(X))$ .
- The largest value of  $exp_p(\pi_i(X))$  for  $i > 1$  (if it exists) is called the  $p$ -exponent of  $X$ , and is denoted  $exp_p(X)$ . In this case we can also say  $exp_p(\pi_*(X))$  is bounded.

## 2. Cases of Bounded Exponents

### 2.1. Finite $p$ -local $H$ -spaces

- Building on work of Toda, for odd primes  $p$  Cohen, Moore, and Neisendorfer showed

$$\exp_p(S^{2n+1}) = p^n.$$

- Selick proved the special case

$$\exp_p(S^3) = p.$$

- For every  $n > 0$  the image of the suspension map  $X \rightarrow \Omega^n \Sigma^n X$  on  $\pi_*$  has bounded  $p$ -exponent given  $X$  is a finite connected  $CW$ -complex [Don Stanley].
- A consequence is that every finite connected  $H$ -space (or more generally an  $H$ -space that is the  $p$ -localization of a finite connect  $CW$ -complex) has a bounded  $p$ -exponent [Stanley, Long].

## 2.2. Rationally trivial spaces

- The *Moore space*  $P^n(p^r)$  has  $p$ -exponent  $p^{r+1}$  [Cohen, Moore, and Neisendorfer].
- Any finite wedge of Moore spaces has a bounded  $p$ -exponent.
- A simply connected finite  $CW$ -complex  $X$  is rationally trivial if and only if  $\pi_*^s(X)$  has bounded  $p$ -exponent [Stanley].
- A simply connected finite  $CW$ -complex  $X$  s.t.  $\pi_*(X) \otimes \mathbb{Q}$  is a finite dimensional vector space has a bounded  $p$ -exponent at *almost all* primes  $p$  [McGibbon, Wilkerson].

## 2.3. Conjectured Examples

- Moore conjecture: A simply connected finite  $CW$ -complex  $X$  has  $\pi_*(X) \otimes \mathbb{Q}$  as a finite dimensional vector space if and only if it has bounded  $p$ -exponent at all primes  $p$ .
- Barrats conjecture (weak form): If the identity map on  $\Sigma^2 X$  has order  $p^r$  then  $\exp_p(\Sigma^2 X) = p^{r+1}$ .

### 3. Unbounded Exponents

- Any wedge of spheres does not have a bounded  $p$ -exponent (by Hilton-Milnor Theorem).
- More generally,  $\Sigma X$  is rationally non-trivial implies  $S^m \vee \Sigma X$  does not have a  $p$ -exponent at any prime  $p$  and all  $m > 1$  [Stanley].

note: That  $\Sigma X$  is rationally nontrivial is necessary since  $S^{2k+1} \vee P^n(p^r)$  has  $p$ -exponent  $\max\{p^{r+1}, p^k\}$ .

- Any 2-cell complex  $X = S^m \cup_{\alpha} e^n$  such that the attaching map  $S^{n-1} \xrightarrow{\alpha} S^m$  is of finite order has no bounded  $p$ -exponent when  $n, m \geq 2$  [Selick, Neisendorfer].
- More generally any finite cell complex with torsion free integral homology of dimension  $> 1$  has no bounded exponent at odd primes [Selick].

**Theorem 3.1 (Hilton-Milnor)** *For any spaces  $X$  and  $Y$ , there is a functorial decomposition*

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma(Y \vee \bigvee_{i \geq 1} (X^{(i)} \wedge Y)).$$

□

Iterating the Hilton-Milnor Theorem implies  $\Omega(S^n \vee S^m)$  is the loop space of a certain infinite weak product of spheres of increasing dimension. Then using the known exponents of spheres, an upper bound for the growth of  $\exp_p(\pi_i(S^n \vee S^m))$  can be found.

PROBLEM: For other less trivial spaces, try to obtain bounds for growth of exponents...

MAIN ASSUMPTIONS: From now on assume all spaces are  $CW$ -complexes localized at an odd prime  $p$ . Take some such  $p$ -localized finite cell complex  $X$ , let

$$V = \widetilde{H}_*(X; \mathbb{Z}_p),$$

and let  $M$  be the sum of the degrees of the generators in  $V$ .

Let  $\mathbb{Z}_p[S_k]$  be the group ring of the symmetric group  $S_k$  on  $k$  letters.

- For the  $k$ -fold self smash  $X^{(k)}$ , we have  $\widetilde{H}_*(X^{(k)}; \mathbb{Z}_p) \cong V^{\otimes k}$ .
- $\mathbb{Z}_p[S_k]$  acts on  $V^{\otimes k}$  sending elements to sums of their permutations.
- For each  $\sigma \in \mathbb{Z}_p[S_k]$ , the action of  $\sigma$  on  $V^{\otimes k}$  determines a self map

$$V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}.$$

- When  $X$  is a suspension, one can construct a self-map  $X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)}$  inducing  $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$  on homology.
- If  $\sigma \circ \sigma = \sigma$  (i.e.  $\sigma$  is an idempotent), the mapping telescope  $T(\sigma)$  of the sequence

$$X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)} \xrightarrow{\bar{\sigma}} \dots$$

is a retract of  $X^{(k)}$  with homology isomorphic to the image of  $\sigma$ .

By the Poincare-Birkhoff-Witt Theorem, there is an isomorphism of  $\mathbb{Z}_{(p)}$ -modules

$$T(V) \cong \bigotimes_{i=1}^{\infty} S(L_i(V)),$$

where  $L_n(V)$  is the submodule of length  $n$  Lie brackets in  $V^{\otimes n}$ . Part of this decomposition can be geometrically realized:

**Theorem 3.2 (Jie Wu)** *Let  $X$  be a suspension. There is a decomposition*

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times (\text{Some other space}).$$

for each sequence  $1 < k_1 < k_2 < \dots$  such that

1.  $k_i$  is prime to  $p$ ;
2.  $k_i$  is not a multiple of  $k_j$  whenever  $i \neq j$ .

□

Each space  $L_n(X)$  is a retract of  $X^{(n)}$ , and is isomorphic to  $L_n(V)$  on homology. They are geometrically realized as telescopes  $T(\beta_n)$  using the *Dynkin-Specht-Wever element*

$$\beta_n \in \mathbb{Z}_p[S_n].$$

FURTHER RESTRICTIONS: From now on assume  $X$  is a suspension, and has either only cells of even dimension or cells of odd dimension.

Then the basis of  $V = \widetilde{H}_*(X; \mathbb{Z}_p)$  has generators in degrees corresponding to the dimensions of the cells. Also  $\widetilde{H}_*(X; \mathbb{Z}_{(p)}) \cong V$  and  $\widetilde{H}_*(X^{(i)}; \mathbb{Z}_{(p)}) \cong V^{\otimes i}$  as  $\mathbb{Z}_{(p)}$ -modules.

For the odd case, let  $s_k \in \mathbb{Z}_{(p)}[S_k]$  be

$$s_k = \sum_{\sigma \in S_k} \sigma,$$

and for the even case

$$s_k = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma.$$

In either case it is well known that  $s_k s_k = k! s_k$ .

We obtain the self map  $V^{\otimes \ell} \xrightarrow{s_\ell} V^{\otimes \ell}$  and its geometric realization  $X^{(\ell)} \xrightarrow{\bar{s}_\ell} X^{(\ell)}$ .

**Proposition 3.3** *When  $V$  has  $\ell < p$  generators,*

- (i) *the telescope  $T(s_\ell)$  is a retract of  $X^{(\ell)}$  and*
- (ii)  *$T(s_\ell) = S^M$  where  $M$  is the sum of the degrees for each generator in  $V$ .*

□

**Corollary 3.4** *When  $V$  has  $\ell < p$  generators,*

- (i)  *$\Sigma^M X$  is a retract of  $X^{(\ell+1)}$ ;*
- (ii) *as a submodule of  $\widetilde{H}_*(X^{(\ell+1)}; \mathbb{Z}_p) \cong V^{\otimes \ell+1}$ ,  $\widetilde{H}_*(\Sigma^M X; \mathbb{Z}_p)$  is the image of  $V^{\otimes \ell+1} \xrightarrow{s_\ell^{\otimes \mathbb{1}}} V^{\otimes \ell+1}$ .*

□

Take the self maps

$$V^{\otimes k+1} \xrightarrow{\beta_{k+1}} V^{\otimes k+1}$$

and

$$V^{\otimes k} \xrightarrow{s_k} V^{\otimes k}.$$

**Proposition 3.5** *Suppose a free  $\mathbb{Z}_{(p)}$ -module  $V$  has basis of dimension  $k > 1$  consisting either of only odd degree generators or of even degree generators. Then*

$$(s_k \otimes \mathbb{1}) \circ \beta_{k+1} \circ (s_k \otimes \mathbb{1}) = (k! + (k-1)!)(s_k \otimes \mathbb{1}).$$

□

Therefore the composition

$$\Sigma^M X \longrightarrow X^{(\ell+1)} \longrightarrow L_{\ell+1}(X) \longrightarrow X^{(\ell+1)} \longrightarrow \Sigma^M X$$

is a  $p$ -local homotopy equivalence when  $1 < \ell < p - 1$ .

Notice  $\beta_{k+1} \circ (s_k \otimes \mathbb{1})$  is an idempotent modulo  $p$  when  $k = 2, 3, \dots, p - 2, p$  (excluding  $p - 1$ ).

**Theorem 3.6** *Let  $X$  be a suspension and  $M$  be the sum of degrees of generators in  $V = \widetilde{H}_*(X; \mathbb{Z}_p)$ . When  $V$  has  $1 < \ell < p - 1$  generators, and all generators are either of even or of odd degree, then*

(i)  $\Omega\Sigma^{M+1}X$  is a retract of  $\Omega\Sigma X$ ;

(ii) *Therefore  $\Omega\Sigma^{c_k+1}X$  is a retract of  $\Omega\Sigma X$  for all  $k \geq 0$ , where (recursively)*

$$c_0 = 0$$

and

$$c_k = (\ell + 1)c_{k-1} + M.$$

(iii) *Therefore  $\pi_i^S(\Sigma X)$  is a retract of  $\pi_{i+c_k}(\Sigma X)$  for all  $k$  large enough s.t.  $i < c_k$ .*

□

→ A weaker version of Theorem 3.6 when  $\ell = p$ : Proposition 3.5 implies  $\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1})$  is an idempotent modulo  $p$ . Then the telescope  $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$  splits off from  $L_{\ell+1}(X)$ , and its mod- $p$  homology is isomorphic to that of  $\Sigma^M X$ .

QUESTION: is  $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$  homotopy equivalent to  $\Sigma^M X$  when  $\ell = p$ ?

→ Computational evidence indicates that Proposition 3.5 can be generalised so that

$$(s_\ell^{\otimes k} \otimes 1) \circ \beta_{\ell k+1} \circ (s_\ell^{\otimes k} \otimes 1) = s_\ell^{\otimes k} \otimes 1 \pmod{p},$$

for  $1 < \ell < p - 1$  and all  $k \geq 1$ . So Theorem 3.6 can probably be strengthened considerably.

FURTHER WORK: What happens when  $V$  has *both* even and odd degree generators.

## 4. Exponent Growth (Lower bounds)

An immediate consequence of Theorem 3.6:

**Corollary 4.1** *If  $X$  has  $1 < \ell < p$  cells,*

$$\exp_p(\Sigma X) \geq \exp_p(\Sigma^{M+1} X).$$

□

QUESTION: When is it true that  $\exp_p(Y) \geq \exp_p(\Sigma Y)$ ?

**Theorem 4.2 (Cohen, Neisendorfer)** *Let  $X$  be any  $p$ -localized cell complex consisting of  $\ell < p - 1$  odd cells,  $V = \widetilde{H}_*(X)$ . Then there exists a functorial decomposition*

$$\Omega\Sigma X \simeq A(X) \times \Omega Q(X)$$

*such that  $A(X)$  is a finite  $H$ -space with*

$$H_*(A(X)) \cong S(V)$$

*as primitively generated algebras, and*

$$H_*(\Omega Q(X)) \cong S([L(V), L(V)]) = \bigotimes_{i=2}^{\infty} (S(L_i(V))).$$

□

Given  $X$  consists only of  $1 < \ell < p - 1$  odd cells, Theorem 3.6 and 4.2 imply  $A(\Sigma^{c_k} X)$  is a retract of  $\Omega\Sigma X$  for all  $k \geq 1$  such that  $c_k$  is even. Since

$$T(L_{\ell+1}(V)) \cong \bigotimes_{i=1}^{\infty} S(L_i(L_{\ell+1}(V))) \subseteq S([L(V), L(V)]),$$

this result can be strengthened:

**Theorem 4.3** *If  $V$  consists of  $1 < \ell < p - 1$  odd dimensional generators and  $\ell$  is even:*

$$\Omega\Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{c_i} X) \times (\text{Some other space}).$$

□

If  $N$  is the dimension of the top cell(s) in  $X$ , and  $\bar{X}$  is  $X$  minus an  $N$ -cell, there exist homotopy fibrations

$$A(\Sigma^{c_i} \bar{X}) \longrightarrow A(\Sigma^{c_i} X) \xrightarrow{q} S^{c_i+N}.$$

If the order of the attaching map for the  $N$ -cell that we removed (from  $X$  to get  $\bar{X}$ ) is known (say  $p^t$ ), there exist degree  $p^t$  factorizations

$$S^{c_i+N} \longrightarrow A(\Sigma^{c_i} X) \xrightarrow{q} S^{c_i+N}.$$

In this case, by a result of Theriault we get homotopy fibrations

$$\Omega S^{c_i+N} \times \Omega A(\Sigma^{c_i} \bar{X}) \longrightarrow \Omega A(\Sigma^{c_i} X) \longrightarrow S^{c_i+N}\{p^t\}$$

where  $\exp_p(S^{c_i+N}\{p^t\}) = p^t$ .

## 4.1. The curious case of rank 2

When  $X$  is a suspension with only  $\ell = 2$  even dimensional cells ( $M$  is the sum of their dimensions), from Proposition 3.5

$$L_2(X) = S^M,$$

and

$$L_3(X) = \Sigma^M X.$$

Recall  $c_1 = M$ ,  $c_k = (\ell + 1)c_{k-1} + M = 3c_{k-1} + M$ .

By iteration we have:

**Theorem 4.4** *For any suspended 2-cell complex  $X$  of even cells*

$$\Omega\Sigma X \simeq \prod_{i=1}^{\infty} \Omega(S^{c_i+1}) \times (\text{Some other space}).$$

□

QUESTION: In general, do loop spaces of spheres retract off from  $\Omega\Sigma X$  when  $\ell > 2$ .

## 4.2. Moore Conjecture

Recall simply connected finite  $CW$ -complex is rationally trivial if and only if its stable homotopy has bounded  $p$ -exponent.

Since our space  $\Sigma X$  has either even or odd cells, it is *not* rationally trivial, and so has unbounded  $p$ -exponent on stable homotopy.

Since the stable homotopy of  $\Sigma X$  retracts off from its unstable homotopy (by Theorem 3.6),  $\Sigma X$  has unbounded  $p$ -exponent.

GUESS 1: For any general finite simply connected cell complex  $X$ , there exists an infinite sequence  $Y_1, Y_2, \dots$  of finite dimensional rationally nontrivial spaces such that each  $\Omega Y_i$  is a retract of  $\Omega \Sigma X$  and the mod- $p$  homology of  $Y_{i+1}$  is a suspension of the mod- $p$  homology of  $Y_i$ .

- We saw this happens when when  $X$  has  $\ell < p - 1$  or  $\ell = p$  number of odd or even cells
- When  $X$  has  $\ell = 4$  even cells and  $p \neq 2, 5$ , apparently  $\beta_{\ell k+2} \circ (s_\ell^{\otimes k} \otimes s_2)$  is an idempotent modulo  $p$  for all  $k \geq 1$ . So the guess might be true in this case.
- When  $X$  has a combination of both even and odd cells, then the guess is true when  $\ell = 2$  and  $p > 2$ .

Stanely's result (stable homotopy of  $X$  has finite exponent if and only if  $X$  is rationally trivial) can be restated as: For any infinite sequence of spaces  $Y_1, Y_2, \dots$  such that  $Y_{i+1}$  is a suspension of  $Y_i$ ,

$$\lim_{i \rightarrow \infty} (\exp(Y_i)) = \infty$$

if and only if the rational homology of  $Y_1$  is non-trivial.

The “sufficiency” part of his result might hopefully be strengthened to:

GUESS 2: For any infinite sequence of spaces  $Y_1, Y_2, \dots$  such that the mod- $p$  homology of  $Y_{i+1}$  is a suspension of the mod- $p$  homology of  $Y_i$ , the rational homology of each  $Y_i$  is non-trivial  $\Rightarrow \lim_{i \rightarrow \infty} (\exp(Y_i)) = \infty$

**5. THANKS!**