This exercise talks about the so-called *Joker Game*. The concept was introduced in [KS15], see `http://www.mimuw.edu.pl/~mskrzypczak/docs/#`, Conference Papers, "*On Determinisation of Good-For-Games Automata*", Section E.2 of Appendix.

Consider a non-deterministic co-Büchi automaton $\mathcal{A} = (A, Q, q_{\mathbf{I}}, \Delta, R)$ where A is a finite alphabet, Q is a finite set of states, $q_{\mathbf{I}}$ is an initial state, $\Delta \subseteq Q \times A \times Q$ is a transition relation, and $R \subseteq \Delta$ is a set of *rejecting* transitions. A sequence of states $\rho \in Q^\omega$ is a *run* of $\mathcal{A}$ over $\alpha \in A^\omega$ if $\rho(0) = q_{\mathbf{I}}$ and for every n we have $(\rho(n), \alpha(n), \rho(n+1)) \in \Delta$. A run ρ over a word α is *accepting* if only finitely many times $(\rho(n), \alpha(n), \rho(n+1)) \in R$. The set of words over which $\mathcal{A}$ has an accepting run is called the *language recognised by* $\mathcal{A}$ and denoted $\mathrm{L}(\mathcal{A})$. For technical reasons we assume that for every $q \in Q$ and $a \in A$ there is at least one $q' \in Q$ such that $(q, a, q') \in \Delta$.

Definition 0.1. *An automaton $\mathcal{A}$ is* Good-For-Games *if there exists a function $\sigma\colon A^* \to Q$ such that for every $\alpha \in \mathrm{L}(\mathcal{A})$ the sequence*

$$\big(\sigma(\epsilon),\ \sigma(\alpha(0)),\ \sigma(\alpha(0)\alpha(1)),\ \sigma(\alpha(0)\alpha(1)\alpha(2)),\ \ldots\big)$$

is an accepting run of $\mathcal{A}$ over α.

Fix a non-deterministic co-Büchi automaton $\mathcal{A}$.

Definition 0.2. *The Joker Game on $\mathcal{A}$ (denoted $\mathcal{G}^{\text{Joker}}$) is defined on the set of positions $Q \times Q$. The initial position is $(q_{\mathbf{I}}, q_{\mathbf{I}})$. The game is played in rounds $n = 0, 1, \ldots$, in a round n starting in a position (p_n, q_n) the following actions are performed:*

- $\forall$ *chooses a letter $a_n \in A$,*
- $\exists$ *chooses a transition $p_n \xrightarrow{a_n} p_{n+1}$ of $\mathcal{A}$,*
- $\forall$ *either:*
  - *chooses a transition $q_n \xrightarrow{a_n} q_{n+1}$ of $\mathcal{A}$,*
  - *or plays* JOKER *and chooses a transition $p_n \xrightarrow{a_n} q_{n+1}$ of $\mathcal{A}$.*

After such a round the game moves to the position (p_{n+1}, q_{n+1}).

Now, the priority of an edge corresponding to a round as above is either:

- 2 *if $\forall$ played* JOKER*,*
- *otherwise 2 if the transition $q_n \xrightarrow{a_n} q_{n+1}$ is rejecting in $\mathcal{A}$,*
- *otherwise 1 if the transition $p_n \xrightarrow{a_n} p_{n+1}$ is rejecting in $\mathcal{A}$,*
- *otherwise 0.*

Therefore, $\mathcal{G}^{\text{Joker}}$ is in fact a finite parity game with priorities $\{0, 1, 2\}$. $\exists$ wins an infinite play of $\mathcal{G}^{\text{Joker}}$ if the highest priority seen infinitely often is even.

An infinite play of the above game produces: an ω-word $\alpha = a_0 a_1 \ldots$, a run $\rho = p_0 p_1 \ldots$ of $\mathcal{A}$, and a *pseudo-run* $\tau = q_0 q_1 \ldots$ — each time $\forall$ plays JOKER, the successive state q_{n+1} may not be accessible from q_n via a transition of $\mathcal{A}$. However, since the acceptance condition is prefix-independent, if $\forall$ played only finitely many times JOKER then it makes sense to ask whether the pseudo-run τ is accepting over α.

Note that there are the following possibilities for the limes superior of the priorities of edges during this play:

- 0 and both ρ and τ are accepting over α,
- 1 and the pseudo-run τ is accepting over α but ρ is not,
- 2 and either $\forall$ played infinitely many times JOKER or τ is not accepting over α.

Therefore, we obtain the following fact.

Fact 0.3. $\exists$ *wins a play as above if and only if either:*

- $\forall$ *played* JOKER *infinitely many times,*
- τ *is not accepting over* α*, or*
- ρ *is accepting over* α*.*

Fact 0.4. *If a given automaton* $\mathcal{A}$ *is GFG then* $\exists$ *wins* $\mathcal{G}^{\text{Joker}}$*.*

Proof. It is enough to use the function σ from the definition of GFG as a strategy of $\exists$. ■

The exercise is to prove or disprove the following conjecture.

Conjecture 0.5 (Kuperberg 2014)**.** *If* $\exists$ *wins* $\mathcal{G}^{\text{Joker}}$ *then* $\mathcal{A}$ *is GFG.*

It is quite easy to see that if we start with a general non-deterministic parity automaton and consider $\mathcal{G}^{\text{Joker}}$ defined as above with the winning condition for $\exists$ defined as in Fact 0.3, then Conjecture 0.5 does not hold.

References

[KS15] Denis Kuperberg and Michał Skrzypczak. On determinisation of good-for-games automata. In *ICALP (2)*, pages 299–310, 2015.