

Verification of database-driven systems via amalgamation

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ABSTRACT

We describe a general framework for static verification of systems that base their decisions upon queries to databases. The database is specified using constraints, typically a schema, and is not modified during a run of the system. The system is equipped with a finite number of registers for storing intermediate information from the database and the specification consists of a transition table described using quantifier-free formulas that can query either the database or the registers.

Our main result concerns systems querying XML databases – modeled as data trees – using quantifier-free formulas with predicates such as the descendant axis or comparison of data values. In this scenario we show an EXPSpace algorithm for deciding reachability.

Our technique is based on the notion of amalgamation and is quite general. For instance it also applies to relational databases (with an optimal PSPACE algorithm).

We also show that minor extensions of the model lead to undecidability.

Categories and Subject Descriptors

F.4.1 [Mathematical Logic and Formal Languages]:
Mathematical Logic

General Terms

Logic, automata

Keywords

Database-driven Systems, Register Automata, Amalgamation, Fraïssé classes

1. INTRODUCTION

In this paper we describe a general framework for static verification of *database-driven* systems. Such a system bases its decisions upon queries to databases. Typical examples

are web services, web applications, or data-centric business processes. These systems can be complex and error prone. Computer-aided static analysis can improve their robustness and correctness.

In order to perform static analysis, the behavior of the database-driven system is specified in a suitable formalism; the desired properties of its executions are also specified in a suitable formalism. The computer then automatically checks whether all runs of the system verify the expected properties.

As advocated in [10], classical software verification techniques have serious limitations when applied to such systems – the main reason is that they abstract away data values, resulting in serious loss of semantics for both the system and the properties being verified.

For this reason, several specific formalisms have been designed allowing meaningful specification of relational database-driven systems. See for instance [8, 6, 4, 5, 9]. As demonstrated in [9], for these scenarios, the system can be described using a register automaton whose transition rules are quantifier-free first-order formulas querying the database and the registers. The correctness criterion for executions of the system is specified using a language mixing queries to the database and temporal behavior that can easily be translated into the same register automata model. Altogether the static analysis problem boils down to testing the existence of a database such that the register automaton has an accepting run driven by that database.

In this paper we develop general techniques for testing reachability of such automata models. These techniques encompass the examples cited above concerning relational databases, but also apply to XML databases, and – in general – to any kind of structures having “good” model properties.

Following [9], we specify database-driven systems using transition rules controlling their workflow. Each such rule may be based on the result of quantifier-free queries to the database. The database is not fixed and may vary from run to run. It is however restricted to range over a certain class of databases typically specified using a schema and possibly several other constraints. Moreover the system has only read access to the database and the database does not change during a run.

To give an idea of the setting we are dealing with, let us describe a toy example of a database-driven system \mathcal{S} that fits into our framework. The system \mathcal{S} is equipped with one register capable of storing nodes of XML documents. We specify the transitions of \mathcal{S} as follows:

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The node stored in the register after the transition is a descendant of the node stored in the register before the transition and the attribute \mathbf{a} of both nodes (before and after) contains the same data value.

Furthermore, we specify that in the initial configuration of the system, the register stores the root of the tree, and in an accepting configuration, it must store some leaf of the tree. Note that the transitions of the system do not modify the database.

We are interested in the following question: is there some XML document t such that the described system has an accepting run driven by t ? We may ask more detailed questions: is there some XML document t satisfying a certain XML schema such that the described system has an accepting run driven by t ?

In general our goal is to give an algorithm for the following problem, parametrized by a class \mathcal{C} of databases.

- **Input.** A database-driven system.
- **Output.** Does the system have some finite accepting run driven by some database in \mathcal{C} ?

We show that if the class \mathcal{C} of databases satisfies a certain model-theoretic assumption – namely, it is a computable *Fraïssé class* – then there exists an algorithm for the described problem.

Our main technical result shows that many natural classes of databases are Fraïssé. Examples include: all databases over a given relational schema, three-colorable graphs (more generally, any property of databases expressed as a Constraint Satisfaction Problem), XML documents viewed as data trees satisfying a given XML schema (more generally, any property of trees recognized by a tree automaton).

The most interesting and most difficult result is the XML case. In this scenario the database is an XML document that must verify a certain XML schema. The system can query the XML document using the descendant axis, the document order and the closest common ancestor relation. It can also test equality or inequality between attribute values. Our generic technique shows that in this setting the above problem is decidable in EXPSpace.

In the setting of relational databases, we derive from our generic technique an optimal PSPACE decision procedure.

We also show how extending slightly the expressive power of these systems quickly leads to undecidability. For instance, in the XML setting, allowing the system to use the sibling axis or the child axis in its queries leads to undecidability.

Comparison with previous work. The model described in this paper generalizes the previous existing models of automata introduced for relational database-driven systems [8, 6, 4, 5, 9]. In particular, the domain can be linearly ordered and the specification of the database-driven system may use this order within the quantifier-free formulas. In this paper we notice that the key is the Fraïssé property, which holds for linear orders, and show that the XML setting is also Fraïssé.

In terms of results, this paper considers only finite runs and generalizes all the previous known results concerning the existence of finite runs. As shown in several of the

above cited papers, the existence of infinite runs lead to additional challenges which are not solved by the Fraïssé property alone. However, in all the practical cases mentioned here the existence of infinite runs can be reduced to the existence of finite runs using a Ramsey argument as described in [9].

2. DATABASE-DRIVEN SYSTEMS

We model databases as finite structures over finite schemas containing relation and function symbols. We use standard terminology from model theory (see [7] for a reference); we briefly recall the relevant notions below.

Basic notions.

A *schema* Σ is a finite set of relation symbols and function symbols, each with a given arity (0-ary function symbols are *constant* symbols). A *model*, or *structure*, \mathbb{A} over a schema Σ is a set $\text{dom}(\mathbb{A})$ – the *domain* of \mathbb{A} – together with an interpretation $s^{\mathbb{A}}$ for each symbol $s \in \Sigma$ as a relation or function over the domain of an appropriate arity, as described by the schema. A structure is said to be *finite* if its domain is finite. A *database* is a finite structure over a given schema.

By *substructure* we always mean in this paper an *induced substructure*, i.e. a restriction of the initial structure to a subset of its domain, which is closed under the function symbols from the schema.

A *homomorphism* from a structure \mathbb{A} to a structure \mathbb{B} , is a mapping $h : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B})$ that preserves the relations and functions from Σ , i.e. $(a_1, \dots, a_k) \in R^{\mathbb{A}}$ implies $h(a_1, \dots, a_k) \in R^{\mathbb{B}}$, and $h(f^{\mathbb{A}}(a_1, \dots, a_k)) = f^{\mathbb{B}}(h(a_1, \dots, a_k))$, for all tuples a_1, \dots, a_k of elements of $\text{dom}(\mathbb{A})$ and function/relation symbols of arity k . An *isomorphism* is a bijective homomorphism whose inverse mapping is also a homomorphism. An *automorphism* is an isomorphism from \mathbb{A} to itself. Finally an *embedding* is a mapping h that is an isomorphism onto the substructure induced by the image of h .

We assume familiarity with first-order logic. We write $\mathbb{A} \models_{\text{val}} \varphi$ to express the fact that a first-order formula φ holds in the structure \mathbb{A} with the valuation val for its free variables.

Database-driven systems.

A *database-driven system* over a schema Σ is described by the following components.

- A finite set of *control states* $Q = \{p, q, \dots\}$
- A finite set of *registers* $X = \{x, y, \dots\}$
- A subset of *initial states* $I \subseteq Q$
- A subset of *accepting states* $F \subseteq Q$
- Finitely many *transition rules* of the form:

$$p \xrightarrow{\delta} q$$

where p, q are control states and δ is a quantifier-free first-order formula over the schema Σ with free variables in the set $X \times \{\text{new}, \text{old}\}$. The formula δ is called the *guard* of the transition and relates the values of the registers before and after the transition.

Fix a database-driven system as described above. A *configuration* is a triple $(\mathbb{D}, q, \text{val})$, where:

- \mathbb{D} is a database over the schema Σ ;
- q is a control state;
- $\text{val} : X \rightarrow \mathbb{D}$ is a valuation, which maps the registers to elements in the domain of \mathbb{D} .

We say that there is a *transition* between configurations $(\mathbb{D}_{\text{old}}, q_{\text{old}}, \text{val}_{\text{old}})$ and $(\mathbb{D}_{\text{new}}, q_{\text{new}}, \text{val}_{\text{new}})$, if

- $\mathbb{D}_{\text{old}} = \mathbb{D}_{\text{new}}$ (*transitions do not modify the database*)
- There is a transition rule $q_{\text{old}} \xrightarrow{\delta} q_{\text{new}}$ such that

$$\mathbb{D}_{\text{old}} \models_{\text{val}} \delta$$

where $\text{val}(x, i) = \text{val}_i(x)$ for $x \in X, i \in \{\text{old}, \text{new}\}$.

A *run* of the system is a sequence of configurations that begins in a configuration with an initial state, and where two consecutive configurations are connected by a transition. In this paper, we are interested in finite runs. A run is *accepting* if the control state in its last configuration is accepting. It follows from the definition that for each run, there is some database \mathbb{D} that is shared by all configurations in the run. We say that the run is *driven by* \mathbb{D} . Note that different runs of the same system may be driven by different databases.

Example 1. Consider directed graphs, where some of the nodes are colored red, and the remaining nodes are white. This corresponds to a schema with one binary edge predicate E and one unary predicate red .

We describe a database-driven system whose accepting runs trace odd-length cycles of red nodes. The system has the following components.

- The control states are $\{\text{start}, q_0, q_1, \text{end}\}$. The initial state is start and the accepting state is end .
- The registers are x, y .
- There is a transition rule $q_0 \xrightarrow{\delta} q_1$, where the guard δ is $(x_{\text{old}} = x_{\text{new}}) \wedge E(y_{\text{old}}, y_{\text{new}}) \wedge \text{red}(y_{\text{new}})$.

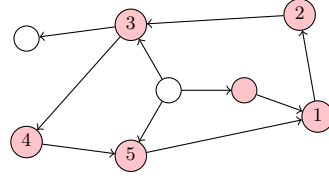
There is also a transition rule $q_1 \xrightarrow{\delta} q_0$, with the same guard. This means that the system alternates between the states q_0 and q_1 , each time moving the content of register y along an edge to some red node (the content of register x stays in place).

There is a transition rule $\text{start} \xrightarrow{\alpha} q_0$, where the guard α is

$$x_{\text{old}} = x_{\text{new}} = y_{\text{old}} = y_{\text{new}}$$

There is also a transition $q_1 \xrightarrow{\alpha} \text{end}$, with the same guard. This means that, in order to exit the initial state, both registers need to point to the same vertex; likewise, in order to enter the accepting state, both registers need to point to the same vertex.

Here is an example run of the system. The run is driven by the database that is the graph \mathbb{G} depicted below. The nodes of the graph are colored red or white; the numbers are not part of the database, they are used to identify the nodes.



An accepting run of the system, driven by \mathbb{G} is:

$$(\mathbb{G}, \text{start}, [1, 1]) \rightarrow (\mathbb{G}, q_0, [1, 1]) \rightarrow (\mathbb{G}, q_1, [1, 2]) \rightarrow (\mathbb{G}, q_0, [1, 3]) \rightarrow (\mathbb{G}, q_1, [1, 4]) \rightarrow (\mathbb{G}, q_0, [1, 5]) \rightarrow (\mathbb{G}, q_1, [1, 1]) \rightarrow (\mathbb{G}, \text{end}, [1, 1]),$$

where $[i, j]$ denotes the valuation that maps x to the node marked with i and y to the node marked with j .

In general, the described system has an accepting run driven by a graph \mathbb{G} if and only if there is a cycle in \mathbb{G} of odd length, consisting only of red nodes.

The emptiness problem. In this paper, we study the following decision problem, which is parametrized by a class \mathcal{C} of databases over a common schema Σ , and called *emptiness of database-driven systems over \mathcal{C}* .

- **Input.** A database-driven system over Σ .
- **Output.** Does the system have some finite accepting run driven by some database in \mathcal{C} ?

Actually, in some of our results also a finite description of the class \mathcal{C} will be given on input. The following observation shows that the problem is PSPACE-hard for almost any choice of parameter \mathcal{C} .

LEMMA 1. *The emptiness problem for database-driven systems over \mathcal{C} is PSPACE-hard if \mathcal{C} contains at least one database with at least two elements.*

Existential guards. Before describing our results in more detail, we point out that replacing quantifier-free formulas by existential formulas in the guards when specifying the system does not affect the expressive power nor the decidability results, as quantified variables can be simulated by using extra registers and nondeterminism.

FACT 2. *For every database-driven system with existential guards one can compute in linear time a database-driven system with quantifier-free guards accepting the same runs driven by the same databases.*

However, as we shall show later on, further extensions of the guards, such as boolean combinations of existential formulas, break decidability.

3. DECIDABILITY RESULTS

In this section, we present the main results of the paper, which show that emptiness of database-driven systems is decidable over certain classes of databases.

3.1 XML documents and regular tree languages

This class is motivated by XML databases. We work with vertex-labeled, unranked and sibling-ordered trees. We use the standard terminology for trees: root, leaf, descendant, ancestor, child, parent, sibling. The *next sibling* of a node x is the first (and therefore unique) sibling after x in document order, which might not exist if x is a rightmost sibling. The *following sibling* is the transitive (but not reflexive) closure of the next sibling relation. Likewise, each node has at most one *previous sibling*, but possibly many *preceding siblings*. We use the standard notion of regular languages for unranked trees. The automaton model is presented in Section 5.3.

It is easy to see that in the presence of a successor relation database-driven systems can simulate counters and are therefore undecidable. See also Section 6.1. For this reason we disallow in our model the use of the child, parent, next sibling and previous sibling relations and only allow relations such as ancestor, descendant, following and preceding sibling. As a matter of fact we can also include the document order and the closest common ancestor function that maps x, y to the node that is a descendant of all common ancestors of both x and y .

We model a tree t as a database, denoted by $\text{Treedb}(t)$, whose domain is the nodes of the tree, and which is equipped with the following predicates and functions:

- A unary predicate for every possible node label (there are finitely many labels);
- Binary predicates for document order (denoted by \leq_{doc}) and descendant order (denoted by \preceq_v);
- A binary function for closest common ancestor, which is denoted by $x \wedge y$. Observe that the descendant relation is defined in terms of this function by a quantifier-free formula:

$$x \preceq_v y \quad \text{iff} \quad x = x \wedge y$$

If the set of node labels is A , then the schema above is denoted by $\text{TreeSchema}(A)$.

Our main result on trees is that emptiness of database-driven systems is decidable over any regular tree language, even when the description of the regular language is also part of the input.

THEOREM 3. *The following problem is decidable:*

- **Input.** *A tree automaton defining a language L of trees labeled by an alphabet A , and a database-driven system over $\text{TreeSchema}(A)$.*
- **Output.** *Is there a tree $t \in L$ and an accepting run of the system driven by $\text{Treedb}(t)$?*

For a fixed tree automaton, the problem is PSPACE-complete. When both the tree automaton and the system are given on input, the problem is in EXPSpace.

The database-driven systems are only allowed to access the trees through quantifier-free formulas that use the predicates included in $\text{TreeSchema}(A)$. By Fact 2, we could also allow the systems to use existential formulas defined in terms of the predicates in $\text{TreeSchema}(A)$. Some navigation predicates for trees, such as child, next sibling, or even simply

sibling are not definable this way. We will later show (Section 6) that adding any one of the above three predicates leads to undecidability.

The proof of Theorem 3 will be given in Sections 4 and 5.

Adding data values. We show in Section 4.4 a composition method implying that our results extend to databases storing data values allowing equality tests. In particular, the result of Theorem 3 remains valid if each node of the tree also carries a data value in \mathbb{N} and the query can test these values using equalities or inequalities. The complexity bound is not affected by this extension.

3.2 Homomorphisms

In this section, we consider schemas with relations only, and no functions. Suppose that \mathbb{G} and \mathbb{H} are two databases over the same schema. Suppose that \mathbb{H} is some database. By $\mathcal{HOM}(\mathbb{H})$ we denote the class of all databases over the schema of \mathbb{H} that map homomorphically to \mathbb{H} . In other words, $\mathbb{G} \in \mathcal{HOM}(\mathbb{H})$ if and only if there is a homomorphism $f : \mathbb{G} \rightarrow \mathbb{H}$.

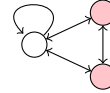
The database \mathbb{H} is called the *template* for the class $\mathcal{HOM}(\mathbb{H})$. Examples of $\mathcal{HOM}(\mathbb{H})$ include n -colorable graphs for every n (when \mathbb{H} is a n -clique).

We show that if a class of databases can be defined as $\mathcal{HOM}(\mathbb{H})$ for some \mathbb{H} , then it admits an algorithm for emptiness of database-driven systems. As for Theorem 3, we actually prove a stronger result where the template is also part of the input.

THEOREM 4. *The following problem is PSPACE-complete.*

- **Input.** *A template database \mathbb{H} and a database-driven system over the schema of \mathbb{H}*
- **Output.** *Does the system have an accepting run driven by some database in $\mathcal{HOM}(\mathbb{H})$?*

Example 2. Let \mathbb{H} be the graph below, with nodes colored red or white



Then a graph \mathbb{G} maps homomorphically to \mathbb{H} if and only if there is no red cycle of odd length in \mathbb{G} . On the other hand, the system from Example 1 has a \mathbb{G} -driven run if and only if there is some red cycle of odd length in \mathbb{G} . Therefore, there is no database $\mathbb{G} \in \mathcal{HOM}(\mathbb{H})$ such that the system has an accepting run driven by \mathbb{G} .

The proof of Theorem 4 will be given in Section 4.

Adding data values. As for the previous case, the result of Theorem 4 remains valid if each node of the tree also carries a data value in \mathbb{N} and the query can test these values using equalities or inequalities.

4. THE METHOD

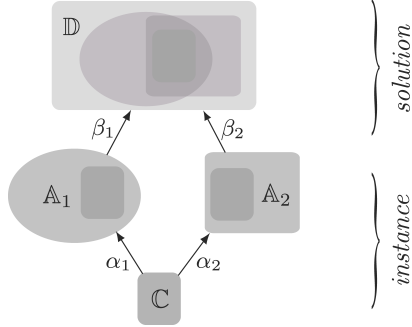
Both decidability results stated in the previous section, namely Theorems 3 and 4, are proved using the same method. The method is presented in this section. The general idea is

to add some more predicates or functions to the databases, so that the resulting class of databases has good closure properties, of which the most important is closure under amalgamation.

4.1 Fraïssé classes and amalgamation

An *instance of amalgamation* consists of two embeddings of the same database \mathbb{C} into two other databases:

$$\alpha_1 : \mathbb{C} \rightarrow \mathbb{A}_1 \quad \alpha_2 : \mathbb{C} \rightarrow \mathbb{A}_2.$$



A *solution* to the instance is a database \mathbb{D} , together with embeddings

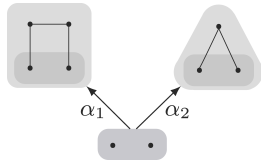
$$\beta_1 : \mathbb{A}_1 \rightarrow \mathbb{D} \quad \beta_2 : \mathbb{A}_2 \rightarrow \mathbb{D}.$$

such that the diagram above commutes, i.e. $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

Following [7, Chapter 6, Section 6.1] a *Fraïssé class* is a class \mathcal{C} of databases over a common schema such that:

- \mathcal{C} is *closed under embeddings*: if \mathbb{C} is a database in \mathcal{C} and \mathbb{D} is any database over the same schema that embeds into \mathbb{C} , then $\mathbb{D} \in \mathcal{C}$;
- \mathcal{C} is *closed under amalgamation*: every instance of amalgamation, where the databases $\mathbb{A}_1, \mathbb{A}_2, \mathbb{C}$ all belong to \mathcal{C} , has a solution \mathbb{D} that belongs to \mathcal{C} ; and
- \mathcal{C} has the *joint embedding property*: every two databases from \mathcal{C} can be embedded into a single database from \mathcal{C} .

Example 3. Consider a schema with one binary relation. The reader can verify that Fraïssé classes over this schema include: all finite linear orders, all finite directed graphs, and all equivalence relations over finite sets. The class of forests (understood as directed graphs) is not closed under amalgamation: the instance depicted below does not have a solution which is a forest.



We will show that, under weak assumptions, emptiness for database-driven systems is decidable over Fraïssé classes. The weak assumptions are that membership in the Fraïssé class is decidable, and that if the schema contains function symbols, then sets of elements of bounded size cannot generate databases of unbounded size via images of functions.

Let \mathbb{C} be a database and S a subset of its domain. We say that S *generates* \mathbb{C} if there is no proper substructure $\mathbb{C}' \subsetneq \mathbb{C}$ that contains S . A database is called *n-generated* if its domain has a subset of size n that generates it. The *blowup function* of a Fraïssé class \mathcal{C} is the function

$$\text{blowup}_{\mathcal{C}} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$$

that maps $n \in \mathbb{N}$ to the least upper bound on the size of n -generated databases in \mathcal{C} . Note that in the absence of function symbols in the schema, any n -generated database has size n and therefore $\text{blowup}_{\mathcal{C}}(n) = n$.

THEOREM 5. *Let \mathcal{C} be a Fraïssé class with membership in PSPACE. Emptiness of database-driven systems is decidable for \mathcal{C} in space*

$$\log(n) \cdot \text{poly}(\text{blowup}_{\mathcal{C}}(2k))$$

where n is the number of control states and k is the number of registers in the database-driven system.

PROOF. We describe a nondeterministic algorithm for the emptiness problem. A configuration $(\mathbb{A}, q, \text{val})$ of the system is called *small* if the database \mathbb{A} belongs to \mathcal{C} and is generated by the contents of the registers as described by val . Note that the size of \mathbb{A} is bounded by $\text{blowup}_{\mathcal{C}}(k)$ and by our assumption on membership in \mathcal{C} , testing $\mathbb{A} \in \mathcal{C}$ requires space polynomial in $\text{blowup}_{\mathcal{C}}(k)$. Consider two small configurations

$$(\mathbb{A}_{\text{old}}, q_{\text{old}}, \text{val}_{\text{old}}) \quad (\mathbb{A}_{\text{new}}, q_{\text{new}}, \text{val}_{\text{new}}).$$

We say that there is a *sub-transition* between them if there is a database $\mathbb{A} \in \mathcal{C}$ and two embeddings

$$f_{\text{old}} : \mathbb{A}_{\text{old}} \rightarrow \mathbb{A} \quad f_{\text{new}} : \mathbb{A}_{\text{new}} \rightarrow \mathbb{A}$$

such that

$$(\mathbb{A}, q_{\text{old}}, f_{\text{old}} \circ \text{val}_{\text{old}}) \quad (\mathbb{A}, q_{\text{new}}, f_{\text{new}} \circ \text{val}_{\text{new}})$$

is a transition of the system. Checking if there is a sub-transition requires space polynomial in $\text{blowup}_{\mathcal{C}}(2k)$ because the class is closed under embeddings and we can therefore assume that the database \mathbb{A} is generated by the images of the valuations $f_{\text{old}} \circ \text{val}_{\text{old}}$ and $f_{\text{new}} \circ \text{val}_{\text{new}}$. This leads to the following nondeterministic algorithm.

1. Nondeterministically guess a small configuration where the state is initial.
2. If the state of the current configuration is final, then terminate and accept. Otherwise, nondeterministically guess a new small configuration accessible from the previous one by a sub-transition. Repeat step 2.

The space consumption of the algorithm is as required by the theorem. The proof of its correctness is based on the fact that the class \mathcal{C} is closed under embeddings and under amalgamation. In particular, from a run of our algorithm we construct a run of the system by amalgamating the databases that appear in the small configurations.

Completeness. Consider an accepting run of the system, driven by some database $\mathbb{A} \in \mathcal{C}$:

$$(\mathbb{A}, q_0, \text{val}_0) (\mathbb{A}, q_1, \text{val}_1) \cdots (\mathbb{A}, q_n, \text{val}_n)$$

Let \mathbb{B}_i be the substructure of \mathbb{A} generated by the content of the registers at step i . By closure under embedding, $\mathbb{B}_i \in \mathcal{C}$

for all i . Moreover \mathbb{A} witnesses the fact that there is a sub-transition from $(\mathbb{B}_i, q_i, \text{val}_i)$ to $(\mathbb{B}_{i+1}, q_{i+1}, \text{val}_{i+1})$. Hence our algorithm has an accepting run.

Soundness. Assume that our algorithm has a run:

$$(\mathbb{B}_0, q_0, \text{val}_0)(\mathbb{B}_1, q_1, \text{val}_1) \cdots (\mathbb{B}_n, q_n, \text{val}_n)$$

By induction on n we exhibit a database \mathbb{B} and embeddings $f_i : \mathbb{B}_i \rightarrow \mathbb{B}$ such that

$$(\mathbb{B}, q_0, f_0 \circ \text{val}_0)(\mathbb{B}, q_1, f_1 \circ \text{val}_1) \cdots (\mathbb{B}, q_n, f_n \circ \text{val}_n)$$

is a valid run of the database-driven system. For $n = 0$ we take $\mathbb{B} = \mathbb{B}_0$ and f_0 the identity.

Assume we have \mathbb{B} for the run until step $(n-1)$ and consider $(\mathbb{B}_n, q_n, \text{val}_n)$. By definition there is a database \mathbb{A} and embeddings $g_{n-1} : \mathbb{B}_{n-1} \rightarrow \mathbb{A}$ and $g_n : \mathbb{B}_n \rightarrow \mathbb{A}$ such that there is a transition from $(\mathbb{A}, q_{n-1}, g_{n-1} \circ \text{val}_{n-1})$ to $(\mathbb{A}, q_n, g_n \circ \text{val}_n)$. But we also have an embedding $f_{n-1} : \mathbb{B}_{n-1} \rightarrow \mathbb{B}$. By closure under amalgamation, we get a database \mathbb{A}' and embeddings $f : \mathbb{A} \rightarrow \mathbb{A}'$ and $g : \mathbb{B} \rightarrow \mathbb{A}'$ with good commuting properties. It is now easy to verify that \mathbb{A}' is the database we were looking for with embeddings $g \circ f_i$ for $i < n$ and $f \circ g_n$ for $i = n$.

This proves the correctness of the algorithm. \square

Observe that the algorithm does not use the joint embedding property. There are two (related) reasons why we use the joint embedding property: first, it is part of the classical definition of a Fraïssé class; second it is necessary for the Fraenkel-Mostowski approach described in the Section 4.5.

4.2 Semi-Fraïssé classes

For some of the classes we are interested in, Theorem 5 will not work, because the classes are not closed under amalgamation.

Example 4. Consider graphs that are 2-colorable or, equivalently, have no odd-length cycle. This class is $\mathcal{HOM}(\mathbb{H})$ where \mathbb{H} is a 2-clique, over the schema Σ consisting of one binary relation.

This class is not closed under amalgamation. Indeed, there is an odd-length cycle in every solution of the instance of amalgamation depicted in Example 3.

A solution to the problem is to consider not 2-colorable graphs, but *2-colored* graphs, i.e. graphs with a 2-coloring. This corresponds to considering an extended schema Γ , with the original binary edge relation, and two unary predicates denoting the colors. The 2-clique \mathbb{H} lifts to a canonical graph $\tilde{\mathbb{H}}$ over this schema, where each node gets a different color. The class $\mathcal{HOM}(\tilde{\mathbb{H}})$ is now closed under amalgamation, and is Fraïssé.

In the example above, we added some structure to the databases in order to recover amalgamation. We formalize this strategy below. Let \mathbb{G} be a database over a schema Γ and let Σ be a subset of the schema Γ . The Σ -projection $\Sigma(\mathbb{G})$ of \mathbb{G} is the same as \mathbb{G} , only the interpretation is restricted to the smaller schema. The Σ -projection of a class \mathcal{C} of databases over Γ , denoted by $\Sigma(\mathcal{C})$, is defined pointwise.

The proof of the following lemma is fairly simple as quantifier-free formulas are invariant under extending the domain or the schema. For a class of databases \mathcal{C} , by $\text{substructures}(\mathcal{C})$ we denote the smallest class of databases containing \mathcal{C} that is closed under embeddings.

LEMMA 6. *Let \mathcal{C} be a class of finite databases over a schema Σ . Suppose that \mathcal{D} is a Fraïssé class over a schema $\Gamma \supseteq \Sigma$, such that*

$$\mathcal{C} \subseteq \Sigma(\mathcal{D}) \subseteq \text{substructures}(\mathcal{C}).$$

Then emptiness of database-driven systems over \mathcal{C} is decidable with the same complexity bounds as over \mathcal{D} .

A class \mathcal{C} for which there exists a Fraïssé class \mathcal{D} that satisfies the assumptions of the above lemma will be called a *semi-Fraïssé* class. In Example 4, the class $\mathcal{HOM}(\mathbb{H})$ is semi-Fraïssé, as witnessed by $\mathcal{HOM}(\tilde{\mathbb{H}})$.

4.3 HOMs are Semi-Fraïssé

As a simple application of the method, we prove Theorem 4. By Theorem 5 and Lemma 6 it is a consequence of the following lemma:

LEMMA 7. *If \mathbb{H} is a finite database, then $\mathcal{HOM}(\mathbb{H})$ is a semi-Fraïssé class.*

PROOF. The schema Γ is the schema Σ extended by a family $\{h\}_{h \in \mathbb{H}}$ of unary predicates, one for each element of the domain of \mathbb{H} . We may view the database \mathbb{H} as a database $\tilde{\mathbb{H}}$ over the extended schema Γ , where a node $h \in \mathbb{H}$ gets the color h . It is easy to see that $\mathcal{HOM}(\mathbb{H})$ is the Σ -projection of $\mathcal{HOM}(\tilde{\mathbb{H}})$. To complete the lemma, we prove that $\mathcal{HOM}(\tilde{\mathbb{H}})$ is Fraïssé.

We only show here amalgamation, the other two properties being trivial. Consider an instance $\mathbb{A}_1, \mathbb{A}_2, \mathbb{C}$ of amalgamation. The desired structure \mathbb{D} is simply constructed from the disjoint union of \mathbb{A}_1 and \mathbb{A}_2 by identifying the images of \mathbb{C} . It remains to show that $\mathbb{D} \in \mathcal{HOM}(\tilde{\mathbb{H}})$. This is witnessed by the mapping sending each node of \mathbb{D} to its color. The reader can verify that this mapping is a homomorphism. \square

Note that the schema of $\tilde{\mathbb{H}}$ in the proof of Lemma 7 contains no function symbols, hence we have $\text{blowup}_{\mathcal{HOM}(\tilde{\mathbb{H}})}(n) = n$, and the complexity is PSPACE as desired.

4.4 Data values

In this section we show how Theorems 3 and 4 extend to databases whose nodes are additionally equipped with data values, and where the transition systems may test equality and inequality of data values. The method is again very general – the data values themselves may carry some structure; we only require that the data values come from a homogeneous relational structure. After introducing some preliminary notions, we describe this general setting.

Homogeneous structures. The notion of homogeneity comes from model theory. An infinite structure \mathbb{F} over a schema Σ is called *homogeneous* if every isomorphism $f : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ between two finite substructures $\mathbb{F}_1, \mathbb{F}_2$ of \mathbb{F} can be extended to an automorphism \tilde{f} of \mathbb{F} .

Homogeneous structures abound; important examples include:

- The set of natural numbers, with the equality relation denoted \sim , denoted $\langle \mathbb{N}, \sim \rangle$;
- The rational numbers, with the linear ordering, denoted $\langle \mathbb{Q}, < \rangle$.

A theorem of Fraïssé (see [7, Chapter 6] and also Section 4.5) says that we can associate to every Fraïssé class an infinite countable structure, called the *Fraïssé limit* of the *Fraïssé class* which is a homogeneous structure.

Data values. Fix a homogeneous structure \mathbb{F} , whose elements will model *data values*. We assume that the schema of \mathbb{F} is purely relational, i.e. does not contain function symbols. For instance \mathbb{F} could be the structure $\langle \mathbb{N}, \sim \rangle$ or $\langle \mathbb{Q}, < \rangle$.

Consider a finite database \mathbb{A} over a schema Σ . Let $\lambda : \mathbb{A} \rightarrow \mathbb{F}$ be any labeling of the nodes of \mathbb{A} by elements of \mathbb{F} . We denote by $\mathbb{A} \otimes \lambda$ the (finite) database extending \mathbb{A} by symbols from the schema of \mathbb{F} , which are interpreted in $\mathbb{A} \otimes \lambda$ via the mapping λ : if R is a relation symbol in \mathbb{F} , then

$$(\mathbb{A} \otimes \lambda) \models R(x_1, \dots, x_k) \iff \mathbb{F} \models R(\lambda(x_1), \dots, \lambda(x_k)).$$

The schema of $\mathbb{A} \otimes \lambda$ is therefore the union of the schema of \mathbb{A} and the schema of \mathbb{F} . The database $\mathbb{A} \otimes \lambda$ can be seen as a database whose nodes are additionally labeled by data values, and the database contains relation symbols from \mathbb{F} allowing to compare the data values. If \mathcal{C} is a class of finite databases, then by $\mathcal{C} \otimes \mathbb{F}$ we denote the class of databases of the form $\mathbb{A} \otimes \lambda$, where $\mathbb{A} \in \mathcal{C}$ and λ is a mapping from \mathbb{A} to \mathbb{F} . By $\mathcal{C} \odot \mathbb{F}$ we denote subset of $\mathcal{C} \otimes \mathbb{F}$ consisting of those databases $\mathbb{A} \otimes \lambda$, where the mapping λ is *injective*, i.e. each node gets a different data value¹.

Example 5. Let t be a finite tree and let $\text{Treedb}(t)$ be the corresponding database. Let $\lambda : t \rightarrow \mathbb{N}$ be a labeling of the nodes of t by natural numbers. Then the database $\text{Treedb}(t) \otimes \lambda$ can be seen as a tree equipped with data values (or *attributes*); two nodes x, y store the same attribute if $x \sim y$.

Example 6. Let \mathbb{G} be a finite graph and let $\lambda : \mathbb{G} \rightarrow \mathbb{N}$ be an injective labeling of \mathbb{G} by natural numbers. Then the database $\mathbb{G} \otimes \lambda$ can be seen as a graph whose nodes are natural numbers: because λ is injective, we can identify a node x with the number $\lambda(x)$. If \mathcal{G} denotes the class of all graphs, then the structures in $\mathcal{G} \odot \langle \mathbb{N}, \sim \rangle$ can be interpreted as graphs on natural numbers. Similarly, the structures in $\mathcal{G} \odot \langle \mathbb{Q}, < \rangle$ can be interpreted as graphs on rational numbers; in particular, their nodes are linearly ordered.

Using the theorem of Fraïssé and a construction for combining two Fraïssé classes into one class, we can obtain the following proposition whose proof is omitted here.

PROPOSITION 1. *Let \mathbb{F} be a purely relational homogeneous structure, such that deciding whether a finite database embeds into \mathbb{F} can be done in PSPACE. Then, for any Fraïssé class \mathcal{C} (over any schema), the classes $\mathcal{C} \otimes \mathbb{F}$ and $\mathcal{C} \odot \mathbb{F}$ are Fraïssé classes, with the same blowup function as \mathcal{C} .*

As a consequence of Proposition 1 and Lemma 7 we get the following extension of Theorem 4.

COROLLARY 8. *The following problem is decidable in PSPACE:*

- **Input.** A relational database \mathbb{H} a database-driven system over the union of the schemas of \mathbb{H} and \mathbb{F} .

¹We consider the two variants \otimes and \odot because in relational databases, we want every value to be unique – to avoid redundancy – while in XML databases, attributes are used for identifying distinct nodes. See Examples 5 and 6.

- **Output.** Is there a database $(\mathbb{A} \otimes \lambda) \in \mathcal{HOM}(\mathbb{H}) \odot \mathbb{F}$ and an accepting run of the system driven by \mathbb{A} ?

Special cases of this result – without the condition $\mathbb{A} \in \mathcal{HOM}(\mathbb{H})$ – have been proved earlier for $\mathbb{F} = \langle \mathbb{N}, \sim \rangle$ in [5] and $\mathbb{F} = \langle \mathbb{Q}, < \rangle$ in [4].

Our abstract machinery applies also to database-driven systems, where the databases are trees with data values.

THEOREM 9. *The following problem is decidable:*

- **Input.** A tree automaton defining a language L of trees labeled by an alphabet A , and a database-driven system over the schema $\text{TreeSchema}(A) \cup \{\sim\}$.
- **Output.** Is there a tree $t \in L$, a labeling λ of t by elements of \mathbb{N} , and an accepting run of the system driven by $\text{Treedb}(t) \otimes \lambda$?

For a fixed tree automaton, the problem is PSPACE-complete. When both the tree automaton and the system are input, the problem is in EXPSpace.

Remark 1. Theorem 9 works for any countable homogeneous structure \mathbb{F} such that testing whether a given finite database \mathbb{A} embeds into \mathbb{F} can be done in PSPACE. For instance $\langle \mathbb{N}, \sim \rangle$ could be replaced by $\langle \mathbb{Q}, < \rangle$. Actually it only matters that $\text{substructures}(\mathbb{F})$ is a Fraïssé class. In particular, as

$$\text{substructures}(\langle \mathbb{N}, < \rangle) = \text{substructures}(\langle \mathbb{Q}, < \rangle),$$

the result also hold for $\mathbb{F} = \langle \mathbb{N}, < \rangle$. Similarly, by considering semi-Fraïssé classes instead of Fraïssé, the result also works with $\langle \mathbb{N}, < \rangle$ augmented with constants, thus capturing the setting of [9].

4.5 Fraenkel-Mostowski sets

In this section, we comment on a bigger picture that contains Theorem 5, but also implies other results, such as emptiness of database-driven systems with pushdowns. To simplify the discussion, we only focus on decidability and not on complexity. The bigger picture is called nominal sets, or Fraenkel-Mostowski sets.

The Fraïssé limit. We begin by observing that instead of talking about a class of finite databases, we can talk about a single limit structure (which is usually infinite so it should not be called a database). The theorem of Fraïssé says that if \mathcal{C} is a Fraïssé class, then there exists a single countable (but usually infinite) homogeneous structure \mathbb{F} – the *Fraïssé limit* of \mathcal{C} – such that the databases in \mathcal{C} are exactly the finitely generated substructures of \mathbb{F} . Fraïssé limits have many good model-theoretic properties, for instance they are ω -categorical.

What is the connection with database-driven systems? A run of a database-driven system is finite, and therefore visits only finitely many elements of a database with its registers. It follows that a database-driven system has a run driven by some finite database in \mathcal{C} if and only if it has a run driven by the Fraïssé limit of \mathcal{C} . Therefore, instead of studying emptiness over a class \mathcal{C} , we could study emptiness of a system that uses registers to store elements of the Fraïssé limit.

Fraenkel-Mostowski sets and their automata. Automata that store values from a Fraïssé limit in their registers have already been studied in [2], as part of a more general framework called *Fraenkel-Mostowski sets*. From the results in [2] it follows that emptiness for such automata is decidable, which implies the decidability result in Theorem 5. (We included a proof of Theorem 5 to make this paper self-contained, and also to get the precise complexity.) Apart from finite automata with registers, the Fraenkel-Mostowski framework contains other computational devices with decidable emptiness, which can then be used to get decidability results for extensions of database-driven systems. The results concerning these devices include:

- Emptiness is decidable for pushdown automata, which are allowed to store elements of a Fraïssé limit both in their state and on the pushdown, see [2]. This implies decidable emptiness for a natural pushdown extension of database-driven systems.
- Emptiness is decidable for tree automata, where a configuration can have more than one successor configuration. This implies decidable emptiness for a natural branching extension of database-driven systems.
- Under additional assumptions, which hold for regular tree languages but not for equivalence relations and HOMs, even certain alternating automata have decidable emptiness [1]. This implies decidable emptiness for a certain alternating extension of database-driven systems.

We would like to point out that the first two results (pushdown automata and tree automata) can be easily obtained without using the abstract framework of Fraenkel-Mostowski sets (this is no longer true for alternating automata, where the proof is quite involved and follows the lines of [3]). We believe, however, that seeing database-driven systems as a special case of automata in Fraenkel-Mostowski sets gives a uniform explanation for the decidability results. We plan to give a more detailed discussion of the Fraenkel-Mostowski connection, including a precise definition of the extended database-driven models, in the full version of this paper.

5. REGULAR TREE LANGUAGES

In this section, we prove Theorem 3, which is the main technical result of the paper. The theorem says that emptiness is decidable for database-driven systems over regular tree languages. Since the proof is quite technical, we begin by illustrating the main ideas in the case of words.

5.1 Regular Word Languages

Like in the case of trees, to a word w over an alphabet A , we associate a database $\text{Worddb}(w)$, where the domain is the positions of the word, there are unary predicates $\{a(x)\}_{a \in A}$ for the labels, and a binary predicate $x < y$ for the natural order on word positions. Call $\text{WordSchema}(A)$ the schema of this database.

THEOREM 10. *The following problem is PSPACE-complete.*

- **Input.** A regular word language $L \subseteq A^*$, given by an NFA, and a database-driven system over the schema $\text{WordSchema}(A)$.
- **Output.** Is there a word $w \in L$ and an accepting run of the system driven by $\text{Worddb}(w)$?

The rest of Section 5.1 is devoted to showing the above theorem. Fix a regular word language $L \subseteq A^*$. Let Q be the states of an NFA that recognizes L . Define

$$\text{Worddb}(L) = \{\text{Worddb}(w) : w \in L\}.$$

For a class of databases \mathcal{C} , let \mathcal{C}^* denote the closure of \mathcal{C} under disjoint unions. The point of studying \mathcal{C}^* is that it is guaranteed to have the joint embedding property. In the specific case of $\mathcal{C} = \text{Worddb}(L)$, the disjoint union is defined so that positions from different words are incomparable with respect to $<$.

We will prove that $\text{Worddb}(L)^*$ is a semi-Fraïssé class, and therefore has decidable emptiness for database-driven systems. The following lemma reduces emptiness from $\text{Worddb}(L)$ to $\text{Worddb}(L)^*$.

LEMMA 11. *For every database-driven system \mathcal{S} , there is a database-driven system \mathcal{S}^* , such that emptiness of \mathcal{S} over $\text{Worddb}(L)$ is equivalent to emptiness of \mathcal{S}^* over $\text{Worddb}(L)^*$.*

PROOF SKETCH. Define the system \mathcal{S}^* as extending \mathcal{S} with a new register. The idea is that this new register stores some position of the word. The new register does not change contents during the whole run, and all other registers are required to be comparable with the new register in the order $<$. Apart from this, the other registers behave as in the system \mathcal{S} . Even when driven by a disjoint union of words, the registers will only use positions from one of the words. \square

From this point, our aim is to prove that $\text{Worddb}(L)^*$ is a semi-Fraïssé class. In particular, we do not consider database-driven systems any more.

Fix an automaton \mathcal{A} recognizing the language L . We assume the automaton does not contain useless states: every state in the automaton is reachable from some initial state, and that from every state an accepting state can be reached. We also assume that for each state q of the automaton, there is a unique letter a that can be read in that state, i.e. a unique letter such that the automaton contains transitions of the form $p \xrightarrow{a} q$ (the state q is not unique, of course). This assumption can be enforced by splitting each state into one copy for each letter of the input alphabet. Denote by \rightarrow the one-step reachability relation on states in the automaton, i.e. $p \rightarrow q$ holds if there is a transition $p \xrightarrow{a} q$. Let \rightarrow^+ be the transitive closure of this relation, i.e. reachability via nonempty words. We will be interested in strongly connected components of this relation, which we call *components*. We adopt the convention that if a state q is not reachable from itself, then it is also in a component, which contains only the state q . Thanks to this convention, the components form a partition of the states of the automaton. We denote components by Γ .

We define a *pre-run* of the automaton to be an input word, together with a labeling of positions by states, where position x gets the state after reading it. (In particular, the first state of the run, before reading any position, does not appear in the labeling.) A pre-run, call it ρ , is interpreted as a database, denoted by $\text{Rundb}(\rho)$ as follows:

- There are the original predicates $\{a(x)\}_{a \in A}$ and $x < y$ for the input word. In other words, $\text{Rundb}(\rho)$ extends the database $\text{Worddb}(w)$, where w is the input word in the run.
- There are unary state predicates $\{q(x)\}_{q \in Q}$ for the states in the run.

- For each component Γ of the automaton, there is a unary function $leftmost_\Gamma(x)$ that maps a position x to the leftmost position before x that has a state in component Γ . If there is no such appearance, then $leftmost_\Gamma(x)$ is x . Likewise, we have a $rightmost_\Gamma(x)$ unary function². We use the name *pointers* for the *leftmost* and *rightmost* functions.

Define \mathcal{C} to be the closure under substructures of

$$\{\text{Rundb}(\rho) : \rho \text{ is a run.}\}.$$

PROPOSITION 2. \mathcal{C} is closed under amalgamation.

Before showing the proposition, we show how it implies Theorem 10.

PROOF OF THEOREM 10. The theorem will follow from the items below thanks to Lemmas 6 and 11.

1. It is not difficult to see that the projection assumption (when projecting a pre-run to its input word) in Lemma 6 is satisfied:

$$\begin{aligned} \text{Worddb}(L)^* &\subseteq \text{WordSchema}(\mathcal{C})^* \subseteq \\ &\subseteq \text{substructures}(\text{Worddb}(L)^*) \end{aligned}$$

2. \mathcal{C}^* is a Fraïssé class. The class \mathcal{C}^* is closed under substructures by definition. It is also closed under isomorphism. The joint embedding property is easy because we can simply take disjoint unions³. Closure of a class under amalgamation is preserved by the operation $\mathcal{C} \mapsto \mathcal{C}^*$, and therefore \mathcal{C}^* is closed under amalgamation by Proposition 2. In conclusion, \mathcal{C}^* is a Fraïssé class, and so $\text{Worddb}(L)^*$ is a semi-Fraïssé class.
3. The blowup of \mathcal{C}^* is small. There are at most $|Q|$ components in the automaton. Since we have two unary functions per component, the blowup function for \mathcal{C}^* is at most $n \mapsto 2|Q| \cdot n$. This gives the PSPACE complexity bound.

□

We now resume the proof of Proposition 2. We will use the following characterization of \mathcal{C} .

LEMMA 12. Let ρ be a pre-run, where the states are q_1, \dots, q_n listed from left to right. Then $\text{Rundb}(\rho) \in \mathcal{C}$ if and only if

$$q_1 \rightarrow^+ q_2 \rightarrow^+ \dots \rightarrow^+ q_n$$

Instead of proving that \mathcal{C} is closed under amalgamation, we prove that it is closed under *inclusion amalgamation*. An instance of inclusion amalgamation consists of two databases \mathbb{A} and \mathbb{B} that are *consistent*, i.e. the functions and predicates are defined the same way on the elements that appear in both domains. A solution of inclusion amalgamation is a database \mathbb{C} that contains both \mathbb{A} and \mathbb{B} as substructures.

LEMMA 13. Let \mathcal{C} be a class of structures closed under isomorphism. Then \mathcal{C} is closed under amalgamation if and only if it is closed under inclusion amalgamation.

²It would seem more natural to define $leftmost_\Gamma$ and $rightmost_\Gamma$ as nullary functions, i.e. constants. We choose unary functions for two reasons: to make the tree case more similar, and to make disjoint unions of runs easier.

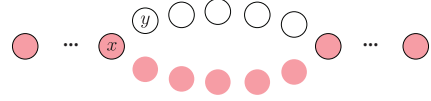
³The joint embedding property is the reason why we work with \mathcal{C}^* and $\text{Worddb}(L)^*$ instead of \mathcal{C} and $\text{Worddb}(L)$.

PROOF OF PROPOSITION 2. The proof is more wordy than it needs to be, because we want it to have the same structure as the proof for the more complicated case of trees.

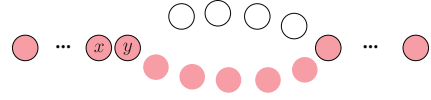
Consider an instance of inclusion amalgamation in the class \mathcal{C} , i.e. two pre-runs ρ_1 and ρ_2 such that the databases $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho_2)$ are consistent and in \mathcal{C} . Define $\text{Rundb}(\rho)$ to be the common part, i.e. the intersection of the two databases, which is a substructure of both, so it also belongs to \mathcal{C} .

We need to show a database in \mathcal{C} that contains both $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho_2)$ as substructures. The proof is by induction on the number of elements in ρ_1 not in ρ . In the induction base $\text{Rundb}(\rho_1)$ is a substructure of $\text{Rundb}(\rho_2)$, and we already have a solution to amalgamation.

For the induction step, suppose that y is in the domain of ρ_1 , but not in ρ . Choose y so that its preceding position in ρ_1 , call it x , is already in ρ . The situation is illustrated in the following picture:



In the picture, the positions of ρ_1 are colored, the positions of ρ_2 have a black border, the positions of ρ are colored and have a black border. To advance the induction, we add y to ρ_2 , as in the following picture:



Define a pre-run ρ'_2 , by adding position y (with its state and input label) to ρ_2 right after x , with the same state as y . We claim that

1. $\text{Rundb}(\rho'_2) \in \mathcal{C}$;
2. $\text{Rundb}(\rho_2) \subseteq \text{Rundb}(\rho'_2)$;
3. $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho'_2)$ are consistent.

If we prove the claims above, then we are done. This is because $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho'_2)$ are an instance of inclusion amalgamation with a smaller induction parameter. The induction assumption says that some database in \mathcal{C} contains both $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho'_2)$, and therefore it also contains $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho_2)$.

Let Γ be the component of the state in y . Consider the values of the pointers $leftmost_\Gamma$ that are assigned to the position x in the three databases $\text{Rundb}(\rho)$, $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho_2)$. Because the databases are consistent, these are all the same position, i.e.

$$leftmost_\Gamma^\rho(x) = leftmost_\Gamma^{\rho_1}(x) = leftmost_\Gamma^{\rho_2}(x)$$

Call the position above y_{left} , it is before y in ρ_1 . Likewise, we define a position y_{right} .

To prove $\text{Rundb}(\rho'_2) \in \mathcal{C}$, we use Lemma 12. By this lemma, in the run ρ_2 , all positions between y_{left} and y_{right} have states in component Γ . The position y is added between these positions, so it does not violate the condition from Lemma 12.

To prove $\text{Rundb}(\rho_2) \subseteq \text{Rundb}(\rho'_2)$, we only need to show that the functions in $\text{Rundb}(\rho'_2)$ are defined the same way

for the positions from $\text{Rundb}(\rho_2)$. This is not difficult to see, because in $\text{Rundb}(\rho'_2)$ there is one new position, which is both followed and preceded by states in the same component. The same argument shows that $\text{Rundb}(\rho_1)$ and $\text{Rundb}(\rho'_2)$ are consistent. \square

5.2 Proof Strategy for Regular Tree Languages

We now resume the proof of Theorem 3, which says that emptiness is decidable for database-driven systems over regular tree languages. We use the same proof strategy as for words. For a tree language L , define

$$\text{Treedb}(L) = \{\text{Treedb}(t) : t \in L\}.$$

Like in the case of words, we will have a class \mathcal{C} that represents substructures of runs.

5.3 Tree automata and their components

We begin by presenting the model of tree automata that we use. Out of the many equivalent models of automata on unranked trees, we choose a model where the runs are easier to pump.

A *tree automaton* consists of:

- An input alphabet A . The automaton is used to accept or reject trees labeled by A .
- A set of states Q . A *run* of the automaton over an input tree is a labeling of the tree nodes by states from Q , subject to some local consistency requirements described below.
- As in the word case, we assume that for each state q , there is a unique input letter a that can be used in that state.
- The automaton has distinguished subsets of: *leaf states*, which are the only states allowed for leaves, *root states*, which are the only states allowed for the root, and *rightmost states*, which are the only states allowed for rightmost children.
- A binary first-child relation $\rightarrow_{\text{firstchild}}$ on states. In a run, if a node has state q and its leftmost child has state p , then $p \rightarrow_{\text{firstchild}} q$ holds.
- A binary next-sibling relation $\rightarrow_{\text{nextsibling}}$ on states. In a run, if a node has state q and its next sibling has state p , then $p \rightarrow_{\text{nextsibling}} q$ holds.

A tree is accepted by the automaton if it admits some run (we do not distinguish between runs and accepting runs). Let us fix for the rest of Section 5 a tree automaton as described above, which recognizes a tree language L .

Components. Define two binary relations on states of the automaton: a relation

$$\rightarrow_h \stackrel{\text{def}}{=} \rightarrow_{\text{nextsibling}}^+$$

that corresponds to following sibling, and a relation

$$\rightarrow_v \stackrel{\text{def}}{=} (\rightarrow_{\text{firstchild}} \circ \rightarrow_{\text{nextsibling}}^*)^+$$

that corresponds to descendant. We use the name *descendant component* for strongly connected components of the relation \rightarrow_v , and the name *horizontal component* for strongly connected components of the relation \rightarrow_h . Again, we adopt the convention that when a state is not reachable from itself

by \rightarrow_v , then it still forms a (singleton) descendant component, likewise for horizontal component. We distinguish two kinds of descendant components:

- A descendant component Γ is called *branching* if in some run, some node with state in Γ has two children with states in Γ .
- A descendant component Γ is called *linear* otherwise. This means that in every run, every node with state in Γ has at most one child with a state in Γ .

For a descendant component, define a set of states $\text{left}(\Gamma)$ as follows. Suppose that in some run, a node x has two descendants y and z , such that z is before y in document order, and is not on the path from x to y . If the states in x and y are in Γ , then we put the state in z into the set $\text{left}(\Gamma)$. The set $\text{right}(\Gamma)$ is defined similarly.

5.4 The class \mathcal{C}

We are now ready to define the class \mathcal{C} , which contains databases representing runs.

The pointers. Define a *pre-run* to be any tree where each node is labeled by a letter $a \in A$, as well as a state $q \in Q$ such that a is the unique letter that can be read in state q . A pre-run need not satisfy the consistency conditions in the definition of a tree automaton run. A node x in a pre-run is called *component maximal* if none of its children have a state in the same descendant component. For a pre-run ρ , we define $\text{Rundb}(\rho)$ to be the following database.

- We have the standard database $\text{Treedb}(t)$ for the input tree t : the node labels, the descendant order, the document order, and the closest common ancestor function.
- For each state q , there is a unary function $\text{leftmost}_q(x)$ defined as follows. If x is a component maximal node, then $\text{leftmost}_q(x)$ maps x to the leftmost child with a state in q . If x is not component maximal, or it has no children with state q , then the function is “undefined”, which is encoded by $\text{leftmost}_q(x) = x$.
- In the same way, we define a function $\text{rightmost}_q(x)$, but for the rightmost child with a state in Γ .
- For each descendant component Γ , there is a unary function $\text{ancestormost}_\Gamma(x)$, whose value is the last node on the path from x to the root that has label q . If there is no such node, then the function is “undefined”, which is encoded by $\text{ancestormost}_\Gamma(x) = x$.
- Suppose that x is a node whose state is in a linear descendant component Γ . Then $\text{descendantmost}(x)$ maps x to the unique descendant of x that has a state in Γ , and has no children with states in Γ . If the state of x is not in a linear descendant component, then the function is “undefined”, which is encoded by $\text{down}(x) = x$.

The class \mathcal{C} . Define \mathcal{C} to be the closure under substructures of the class

$$\{\text{Rundb}(\rho) : \rho \text{ is a run of the automaton}\}.$$

In other words, a database belongs to \mathcal{C} if it can be extracted from a run (not a pre-run) so that nodes are extracted together with the values of their pointers. Following the same

proof as in Theorem 10, to prove Theorem 3, it will be sufficient to prove the following results, whose proofs are omitted here.

LEMMA 14. *The blowup function for C^* is $n \mapsto c \cdot n$, where the constant c is exponential in the state space Q .*

PROPOSITION 3. *C is closed under amalgamation.*

6. UNDECIDABLE MODELS

We consider in this section several ways of extending the model. In most cases, the extensions lead to undecidability. For instance, if the trees are additionally equipped with the child relation or the sibling relation, then the emptiness problem becomes undecidable, even for a fixed tree language. A more interesting question is what happens if we extend the expressive power of the logics used for describing the transitions of the system. These extensions also quickly lead to undecidability.

6.1 Child and sibling axes

Adding axes such as *next sibling* or *child* leads to undecidability. The reason is that already for unary words with the successor relation on positions, we get undecidability. More precisely, a unary word w can be viewed as a structure whose domain is the set $1, 2, \dots, |w|$ of positions of w , and $\text{succ}(x, y)$ holds if $y - x = 1$.

FACT 15. *Let L be any infinite set of words over the unary alphabet, viewed as structures over the schema consisting of the binary symbol succ . Then, the following problem is undecidable.*

- **Input.** *A database-driven system over the schema consisting of succ .*
- **Output.** *Is there a word $w \in L$ and an accepting run of the system driven by the word w ?*

PROOF SKETCH. Using one register, the system can simulate a counter of a counter machine: a transition can increment or decrement the counter using the relation succ . There are no zero tests, but this can be simulated by keeping one register z that is never changed (using $z_{\text{old}} = z_{\text{new}}$ as a conjunct in all rules); then a zero test of the counter is simulated by the formula $x = z$. Since the halting problem is undecidable for two-counter machines, the fact follows. \square

It follows immediately from Fact 15 that in the presence of the *child* or *next sibling* axis it is undecidable whether a database-driven has an accepting run.

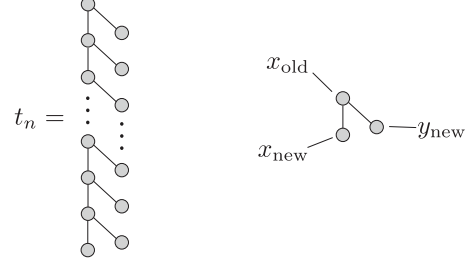
The sibling axis.

We show that even extending the set of predicates by the sibling relation also leads to undecidability. Formally, we model a tree t as a database with two predicates: the closest common ancestor \wedge and the transitive, symmetric and irreflexive binary *sibling* relation (the document order nor the unary predicates are needed for this undecidability result).

FACT 16. *There exists a regular tree language L over a unary alphabet, such that the following problem is undecidable:*

- **Input.** *A database-driven system over the schema consisting of \wedge and *sibling*.*
- **Output.** *Is there a tree $t \in L$ and an accepting run of the system driven by the tree t ?*

PROOF SKETCH. The language L is defined as the set of trees of the form t_n , where $n \in \mathbb{N}$ and t_n is the tree depicted in the left-hand side of the figure below, of height n .



The reduction is again from counter machines. We show that a system can simulate a counter using a register x .

To simulate incrementation of the counter, the machine uses an auxiliary register y (see right-hand side of the figure above), and follows a transition whose guard is the following formula:

$$(x_{\text{old}} = (x_{\text{new}} \wedge y_{\text{new}})) \quad \wedge \quad \text{sibling}(x_{\text{new}}, y_{\text{new}})$$

These conditions guarantee that x_{new} is a child of x_{old} (they do not guarantee that x_{new} is the left child, as is the case in the figure, but this is not necessary).

Decrementation of a counter is obtained by swapping “old” with “new” in the guard. As in the proof of Fact 15, using additional counters, one can simulate zero tests. This way, a database-driven system using the predicates \wedge and the successor relation can simulate a counter machine. \square

Remark 2. We don’t know whether emptiness is decidable for database-driven systems over the schema consisting of the sibling relation, the document order and the vertical order, but not the closest common ancestor.

6.2 Rules that are not existential

A legitimate question is whether one can extend the expressivity of our model by extending the power of the formulas defining the transitions of the system, for instance by allowing first-order formulas, while preserving the decidability of the emptiness problem.

We have seen in Fact 2 that systems with transitions guarded by existential formulas can be simulated by systems where the transitions are guarded by quantifier-free formulas. However, already allowing boolean combinations of existential formulas quickly leads to undecidability. For instance, in the tree case, this is a consequence of Fact 15. Indeed, using boolean combinations of existential formulas, one can define the *child* axis:

$$\text{child}(x, y) \iff x \preceq_v y \quad \wedge \quad \neg \exists z : x \prec_v z \prec_v y$$

6.3 Data tree patterns

We also considered the setting where the queries are data tree patterns. A data tree pattern selects data values within a tree, depending on the existence of nodes whose positions verify the tree pattern (we use an injective semantics for tree

patterns, where each node of the tree pattern must match a different node of the tree). With our terminology, a tree pattern is a special case of an existential formula; the hope being that systems using boolean combination of tree patterns would be decidable.

To make this setting fit into our formalism, we assume the system is over $\text{DataTreeSchema}(A)$ and has rules guarded by boolean combinations of formulas of the following form (called tree pattern formulas):

$$\delta(\bar{x}_{\text{new}}, \bar{x}_{\text{old}}) : \exists^{\neq} v_1, v_2, \dots, v_l \quad \phi(v_1, \dots, v_l),$$

where the notation \exists^{\neq} implies that the nodes v_1, \dots, v_l are pairwise distinct (to reflect the injective semantics of tree patterns), and ϕ is a conjunction of conjuncts of the following three possible forms:

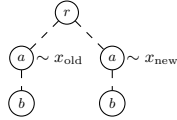
$$v_i \sim x_j, \quad v_i \preceq_v v_j, \quad \lambda(v_i).$$

Note that the restriction on ϕ implies that a formula δ cannot (directly) tell whether two registers x and y of the system point to the same node; it can only test whether they have the same data value.

Example 7. Consider trees labeled by $\{a, b, r\}$. The following tree pattern:

$$\begin{aligned} & \exists^{\neq} v, l_a, l_b, r_a, r_b. \left(a(l_a) \wedge b(l_b) \wedge a(r_a) \wedge b(r_b) \wedge r(v) \right) \wedge \\ & (v \preceq_v l_a \preceq_v l_b) \wedge (v \preceq_v r_a \preceq_v r_b) \wedge (u_a \sim x_{\text{old}}) \wedge (u_a \sim x_{\text{new}}) \end{aligned}$$

can be graphically represented as follows (dashed lines represent descendant relationships):



THEOREM 17. *There is a language of A -labeled trees L such that the following decision problem is undecidable.*

- **Input.** A database-driven system over the schema $\{\preceq_v, \sim, \{a\}_{a \in A}\}$ where transitions are boolean combinations of tree pattern queries.
- **Output.** Is there a tree $t \in L$ and an accepting run of the system driven by the tree t ?

Having described the applications and limitations of our framework, we end this paper.

7. REFERENCES

- [1] Mikołaj Bojańczyk, Laurent Braud, Bartek Klin, and Slawomir Lasota. Towards nominal computation. In *Symp. on Principles of Programming Languages (POPL)*, pages 401–412, 2012.
- [2] Mikołaj Bojańczyk, Bartek Klin, and Slawomir Lasota. Automata with group actions. In *Symp. on Logic in Computer Science (LICS)*, pages 355–364, 2011.
- [3] Stéphane Demri and Ranko Lazic. LTL with the freeze quantifier and register automata. In *Symp. on Logic in Computer Science (LICS)*, pages 17–26, 2006.
- [4] Alin Deutsch, Richard Hull, Fabio Patrizi, and Victor Vianu. Automatic verification of data-centric business processes. In *Intl. Conf. on Database Theory (ICDT)*, 2009.
- [5] Alin Deutsch, Liying Sui, and Victor Vianu. Specification and verification of data-driven web applications. *J. Comput. Syst. Sci.*, 73(3):442–474, 2007.
- [6] Alin Deutsch, Liying Sui, Victor Vianu, and Dayou Zhou. A system for specification and verification of interactive, data-driven web applications. In *Intl. Conf. on Management of Data (SIGMOD)*, 2006.
- [7] W. Hodges. *A shorter model theory*. Cambridge University Press, 1997.
- [8] Sheila A. McIlraith, Tran Cao Son, and Honglei Zeng. Semantic web services. *IEEE Intelligent Systems*, 16(2):46–53, 2001.
- [9] Luc Segoufin and Szymon Toruńczyk. Automata based verification over linearly ordered data domains. In *Intl. Symp. on Theoretical Aspects of Computer Science (STACS)*, 2011.
- [10] Victor Vianu. Automatic verification of database-driven systems: a new frontier. In *Intl. Conf. on Database Theory (ICDT)*, pages 1–13, 2009.