Bounded depth data trees

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Abstract. A data tree is a tree where each node has a label from a finite set, and a data value from a possibly infinite set. We consider data trees whose depth is bounded beforehand. By developing an appropriate automaton model, we show that under this assumption various formalisms, including a two variable first-order logic and a subset of XPath, have decidable emptiness problems.

1 Introduction

A data tree is a tree where each node has a label from a finite set, and a data value from a possibly infinite set. We consider trees where there is a fixed bound on the depth of nodes. For recognizing properties of such trees, we define an automaton model that traverses the trees in a depth-first manner. We show that the emptiness problem for the automata is decidable, by a reduction to reachability for priority multicounter automata, a powerful model for recognizing word languages [32]. The automaton model is used to show decidability of the satisfiability problem for a two-variable first-order logic, and also for a fragment of XPath. In the logic and XPath, we allow a rich vocabulary of navigational predicates, such as document order, thus extending the work from [6], where only successor axes were allowed.

The main application area for this paper is static analysis tasks for XML databases. We would like to develop tools that automatically answer questions such as: does property a of XML documents always imply property b?; or: is property a vacuously true?

A very successful approach to static analysis has been to use tree automata.An XML document is modeled as a tree, where the labels of the tree correspond to tag names in the document. Many formalisms for XML can be represented as tree automata, possibly extended with additional features, (see, e.g., [29, 23]). Using this representation, a large body of techniques for tree automata can be applied to solving static analysis tasks.

A drawback of the tree automaton approach is that it considers only the tag names, and ignores other content stored in the document. For instance, one cannot express key constraints such as: "every two nodes have different values

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stored in their `unique_key` attribute". Such constraints are clearly important for databases, and can be expressed in, say, XPath. One way of extending the tree automata approach beyond mere tag names is to consider data trees. In a data tree, each node has a label from a finite set, and a data value from a possibly infinite set. The data values are used to model the content of the document. Recently, there has been flurry of research on models with data, including data words [19, 31, 13, 14, 5, 22, 3], and data trees [8, 1, 4, 16, 6].

The typical tool for solving logics over trees without data is a finite-state automaton. When data is added, the appropriate automaton almost always involves counting: ranging from automata with semilinear constraints on runs [6], through vector-addition systems [5], and on to faulty counter machines [14] and lossy channel machines [22]. Complexities are often high: non-primitive recursive, e.g. [14] and some results in [22], or as hard as reachability in vector-addition systems [5] (a decidable problem, but not known to be primitive recursive [26, 21]).

Due to the above, logics for objects with data are usually quite weak. For data trees, the present cutting edge is a fragment of first-order logic, where only two variables are used, and only the child and next-sibling axes are allowed for testing spatial relationship [6]. This logic has decidable emptiness, but most extensions are undecidable: adding a third variable, adding a second data value, adding order on data values. One question left open in [6] was whether the logic remains decidable if we add their transitive closures (i.e. descendant and following sibling)? The outlook is not optimistic, since the extended problem subsumes reachability for tree vector-addition systems, a difficult open problem [12].

In this paper, we try to deal with the extended axes. We show that if a fixed bound on the depth is imposed, the logic from [6] remains decidable after the descendant and following sibling axes (and even document order) are added to the signature. (The following sibling axis is more interesting than the descendant axis in bounded depth trees.) In terms of XPath, we extend the fragment from [6], by allowing all navigational axes in Core XPath ([17]) in path expressions, and considerably stronger predicate expressions, where the data values of two relative paths can be compared, as long as the paths belong to the same subtree.

Another motivation to consider bounded depth trees is that the lower bounds in [6] are somewhat artificial, using constructions alien to actual XML documents. Indeed, many XML data bases are wide, but not very deep (see, e.g.,[9]). Therefore, considering trees of arbitrary depth, which turns out to be a major technical difficulty, need not reflect problems in the real world. It is, however, sometimes crucial to compare elements on a horizontal axis (which nodes are later/earlier in the document), something that cannot be done by the logic in [6].

What do we gain by bounding the depth? The main idea is that a bounded depth tree actually bears more resemblance to a word than a tree. If a bounded depth tree is written down as a string (the way XML documents are stored in text files), a finite string automaton can recover the tree structure by using its finite control to keep track of the path leading to the current node. This simple observation is the essence of our approach. However, it is not immediately clear

how the string automaton should deal with data values. We discover that the appropriate model is an extension of multicounter automata, where a limited form of zero test is allowed [32].

The paper is structured as follows. In Section 2 we define bounded depth data trees, and some notions for discussing them. Section 3 contains the main contributions of the paper. Here, we present our automaton model, describe its basic properties, and prove that the corresponding emptiness problem is decidable. Section 4 describes a fragment of first-order logic, which, thanks to the automaton model, can be shown to have a decidable satisfiability problem. Section 5 describes applications for XPath. Due to space limitations, many proofs have been omitted, and will appear in the full version of the paper.

2 Definitions

To simplify technicalities, we do not actually consider data trees, but data forests. Informally, a data forest is an ordered sequence of data trees. Formally, a *data forest* is a partial function

$$t : \mathbb{N}^+ \to \Sigma \times \Delta$$

with nonempty finite domain. The set Σ is called the *alphabet* and is required to be finite, while the set Δ is called the *data domain*, and may be infinite (usually, we use the natural numbers for Δ). The *nodes* of the forest are elements of the domain of t. The first coordinate of $t(x)$ is called *the label* of the node x, while the second coordinate of $t(x)$ is called the *data value* of x. Furthermore, the set of nodes must be closed under parents and previous siblings:

- The *parent* of a node $a_1 \cdots a_n a_{n+1} \in \mathbb{N}^*$ is the node $a_1 \cdots a_n$.
- The *previous sibling* of a node $a_1 \cdots a_n \in \mathbb{N}^*$ is the node $a_1 \cdots a_{n-1}(a_n - 1)$. (A node with $a_n = 0$ has no previous sibling.)

Preceding siblings are defined by taking the transitive closure the previous sibling. The opposite of previous/preceding siblings are next/following siblings. A node has at most one previous/next sibling, but possibly many preceding/following siblings. A *root* in a forest is any node without a parent; there may be many roots. The *depth* of a node $a_1 \cdots a_n$ is the number n; in particular each root has depth 1. The opposite of parent is *child*. The transitive closure of the child relation is the *descendant* relation, similarly *ancestors* are defined for parents. A *leaf* is a node without children.

A *depth k data forest* is one where all leaves have depth k. We could also consider forests where leaves have depth at most k; however the more general type can be easily encoded in the special one by adding dummy nodes. When considering depth k data forests, we assume without loss of generality that the label set Σ is partitioned into k disjoint sets $\Sigma_1, \ldots, \Sigma_k$ such that nodes of depth i are only allowed to use labels from Σ_i. This assumption can be easily ensured by expanding the alphabet.

A *class* of a data forest is a maximal set of nodes with the same data value.

Let t be a forest. The *depth-first-search traversal* (DFS traversal) of t is a sequence $v_1, \ldots, v_n$ of nodes of t satisfying:

- Each non-leaf node appears twice, and each leaf appears once.
- If $i < n$ and v_i appears for the first time, i.e. $v_i \notin \{v_1, \ldots, v_{i-1}\}$, then v_{i+1} is the leftmost child of v_i, except if v_i is a leaf, in which case v_{i+1} is the next sibling of v_i, or, if v_i is a rightmost child, the parent of v_i.
- If $i < n$ and v_i is seen for the second time, i.e. $v_i \in \{v_1, \ldots, v_{i-1}\}$, then v_{i+1} is the next sibling of v_i, or the parent of v_i if v_i is a rightmost child.

There is only one DFS traversal, and it must begin with the leftmost root and end with the rightmost root. Later on, it will be convenient that non-leaf nodes are visited twice. If we remove repetitions from the DFS traversal (by deleting second occurrences), we get the *document ordering* on nodes of a forest.

3 Automata

This section contains the main contribution of the paper. In Section 3.1, we define an automaton model for bounded depth data forests. After showing some properties that can be recognized by our automata in Section 3.2, we show in Section 3.3 that the automata have decidable emptiness. The decidability proof is by reduction to reachability in an extended model of multicounter automata (Petri nets). Therefore, we have no primitive recursive upper bound for the complexity; lower bounds are also open.

The automaton model we define can be seen as an extension of the class memory automata for words from [3] to bounded depth forests. These are, in turn, a variant of the data automata from [5]. The basic idea is to use one class memory automaton per depth level in the forest.

3.1 Class memory automata for forests of bounded depth

A depth k forest class memory automaton (k-FCMA) is defined as follows. It has $k+1$ state spaces: $Q, Q_1, \cdots, Q_k$. Each has an initial and a final subset:

$$I, F \subseteq Q \quad I_1, F_1 \subseteq Q_1, \quad \cdots \quad , I_k, F_k \subseteq Q_k \ .$$

The idea is that the states Q_i will be used to examine data values of nodes at depth at least i, while the states in Q are used to examine properties that do not involve data.

The automaton runs on an input depth k forest by visiting its nodes in the DFS sequence (in particular, non-leaf nodes are visited twice). At every moment of its run, it keeps its current state $q \in Q$ – called the *global state* – as well as k *class memory functions* of the form

$$f_1 : \Delta \to Q_1 \qquad \cdots \qquad f_k : \Delta \to Q_k \ .$$

Therefore, a configuration of the automaton consists of: a node v of the forest t, the global state q and the class memory functions $f_1, \ldots, f_k$. (Thanks to

the class memory functions, the automaton is a type of infinite-state system, which contributes to the high complexity of emptiness. Each configuration can be finitely represented, since the class memory functions have finite non-initial support.) At the beginning of the run, v is the leftmost root, q is set to be a designated *initial state* $q_I \in Q$, while all the class memory functions $f_1, \ldots, f_k$ assign initial states to all data values $d \in \Delta$. (If there are many initial states, this produces nondeterminism.)

A single step of the automaton works as follows. Assume that the automaton is in a node v of depth i with data value d. Depending on the global state, the values of $f_1(d), \ldots, f_i(d)$, and the label of v, the automaton picks a new global state and new values of $f_1(d), \ldots, f_i(d)$. It then advances to the next node in the DFS traversal. Therefore, the transition function is a set of rules from

$$\bigcup_{i=1,\ldots,k} Q \times Q_1 \times \cdots \times Q_i \times \Sigma_i \times Q \times Q_1 \times \cdots \times Q_i$$

Note that since Σ is partitioned into sets $\Sigma_1, \ldots, \Sigma_k$, the label of a node determines its depth. In particular, the automaton knows if it is descending into a successor, moving to the right sibling, or ascending into the parent.

Furthermore, when the automaton has just read for the second time a rightmost sibling v at depth $i \in \{1, \ldots, k\}$ (or for the first time, if v is a leaf), it does some further processing on the class memory function f_i, which we call a *check-reset*. (The check-reset is done after the transition corresponding to the second visit in v has been applied.) First, the automaton checks if the class memory function f_i is *accepting*, i.e. all data values are assigned either initial or accepting states. If this is not the case, the run is aborted and cannot be continued. If this check succeeds, the class memory function f_i is reset, by assigning the initial state (nondeterministically, if there is more than one) to all data values.

The automaton accepts the data forest if, after completing the DFS traversal, it has an accepting global state (and the last-check reset has been successful). Note however, that before this happens, a large number of memory check-resets must be successfully carried out.

Example 1. Consider the following property of depth k forests: each data value occurs at most once. To recognize this property, the automaton only uses the states Q_1 (all other state spaces Q and $Q_2, \ldots, Q_k$ contain one state q, which is both initial and final, and is never modified). There are two states in Q_1: an initial state *new* and a final state *old*. The transition function advances *new* to *old*, while *old* has no outgoing transitions. In other words, there is only one transition for each letter $a \in \Sigma$:

$$(q, new, q, \cdots, q, a, q, old, q \cdots, q).$$

3.2 Some properties of FCMA

In this section we present some properties of bounded depth data forests that can be recognized by FCMA. Apart from being useful later on, the results in this section are meant to give a feeling for what FCMA can do.

Fact. FCMA are closed under union and intersection.

When the depth of a data forest is limited to 1, the forest is a *data word*, as considered in [5]. Furthermore, *data automata*, the automaton model introduced in [5] to recognize properties of data words, coincides with the restriction of FCMA to depth 1. Lemma 1 below can be used to transfer results about data words to data forests.

Lemma 1. *Let $\mathcal{A}$ be a data automaton. The following properties of data forests are recognized by FCMA:*

- *For every node v, the children of v, when listed from left to right, form a data word accepted by $\mathcal{A}$.*
- *For every node v, the descendants of v, when listed in document order, form a data word accepted by $\mathcal{A}$.*

Sometimes it is convenient to see how the data value of a node is related to the data values of its neighborhood. The *profile* of a node is information about which nodes among its ancestors, previous and next siblings have the same data value. Once the depth k of forests is fixed, there are finitely many possible profiles. The following lemma shows that these can be tested by an automaton:

Lemma 2. *For each possible profile p, there is an FCMA that recognizes the language: "a node has label a if and only if it has profile p".*

We will also need to use FCMAs to recognize languages of the form: "for every class, a given property holds". Here we present a general result of this type. Note that it is not clear what we mean when saying that a class satisfies some property, since it is not clear how the nodes of a class should be organized once they are taken out of the data forest. Here we use one such definition, which we call a *take-out*. Let t be a forest and V a set of nodes in this forest. The nodes of the take-out are nodes of V, along with their ancestors. The labels in the take-out are inherited from t, except we add a special marker to distinguish if a node is from V, or just an ancestor of a node from V. The take-out is a forest without data, where leaves may have different depths.

Lemma 3. *Let L be a regular forest language (without data). An FCMA can test if the take-out of every class belongs to L.*

3.3 Decidable emptiness for the automata

In this section, we will show that emptiness is decidable for k-FCMA. The proof is by reduction to emptiness of priority multicounter automata. Note that universality is undecidable even for 1-FCMA, as it is already undecidable for data automata over words.

Priority multicounter automata A priority multicounter automaton is an automaton over words (without data) that has a number of counters, which can be incremented, decremented and tested for zero. (Multicounter automata with zero tests correspond to Petri nets with inhibitor arcs.) To keep the model decidable, the zero tests are restricted. This is where the priorities come in.

More formally, a *priority multicounter automaton* has a set C of counters, a state space Q and an input alphabet Σ. Furthermore, the counters come with a distinguished chain of subsets: $C_1 \subseteq \cdots \subseteq C_m \subseteq C$.

The automaton reads a word $w \in \Sigma^*$ from left to right, possibly using ϵ-transitions. At each point in its run, the automaton has a current state $q \in Q$ and a non-negative counter assignment $c \in \mathbb{N}^C$. At the beginning, a designated initial state is used, and all the counters are empty.

In a transition, the automaton reads a letter – possibly ϵ – from the word. Depending on this letter the automaton changes its state, and performes a *counter operations*, that is, it increases a counter, decrements a counter, or checks that all counters in C_i, for some i, are empty.

The above operations can fail: if a decrement is done on an empty counter; or if a zero test fails. When the counter operation fails, the transition fails and the run is aborted. The automaton accepts if at the end of the word it has reached a designated accepting state. The following difficult result has been shown in [32]:

Theorem 1. *Emptiness is decidable for priority multicounter automata.*

Note that priority multicounter automata are an extension of multicounter automata (where the zero tests are not allowed). In particular, no primitive recursive emptiness algorithm is known.

Reduction to priority multicounter automata We now show that emptiness for FCMA can be reduced to emptiness of priority multicounter automata. In particular, thanks to Theorem 1, emptiness is decidable for FCMA.

Let t be a depth k forest, and let $v_1, \ldots, v_n$ be its DFS traversal. Let $\text{trav}(t)$ be the word over Σ containing the labels of $v_1, \ldots, v_n$. Since $\text{trav}(t)$ does not use the data values, it is irrelevant if t is a data forest or a non-data forest.

Theorem 2. *Emptiness is decidable for k-FCMA, for all $k \in \mathbb{N}$. Furthermore, for each k-FCMA $\mathcal{A}$, the set $\{\text{trav}(t) : t \text{ is accepted by } \mathcal{A}\}$ is accepted by an (effectively obtained) priority multicounter automaton.*

By Theorem 1, the first clause of the theorem follows from the second one. This section is therefore devoted to simulating a k-FCMA with a priority multicounter automaton.

We fix a k-FCMA $\mathcal{A}$. We assume that in every transition

$$(q, q_1, \ldots, q_i, a, r, r_1, \ldots, r_i) \ ,$$

none of the states $r_1, \ldots, r_i$ are initial; and if some q_j is initial, then so are $q_{j+1}, \ldots, q_i$. Any k-FCMA can be effectively transformed into one satisfying the above assumptions.

The priority multicounter automaton that recognizes the traversals is defined as follows. It is a conjunction of two automata. The first one checks that the depths indicated by the labels are consistent with a DFS traversal, i.e. the input word belongs to $\{\text{trav}(t) : t \text{ is a depth } k \text{ forest}\}$. Since the latter is a regular word language, we do not even need to use counters.

The real work is done by the second automaton, which we call $\mathcal{B}$. To simplify presentation, we use a slightly extended notion of transition. We will later comment on how the extended notion can be realized by a standard priority multicounter automaton. The control states of $\mathcal{B}$ are the global states Q of $\mathcal{A}$. It has a counter for each of the states in $Q_1, \ldots, Q_k$ used in the class memory functions (we assume these state spaces are disjoint).

When the simulating automaton $\mathcal{B}$ is in state $q \in Q$, and the input letter is $a \in \Sigma_i$ (with $i = 0, \ldots, k$) the automaton performs the following actions:

1. As preprocessing for the transition, $\mathcal{B}$ may nondeterministically choose to increment any counter corresponding to an initial state.
2. In the next step, $\mathcal{B}$ nondeterministically picks a transition
$$(q, q_1, \ldots, q_i, a, r, r_1, \ldots, r_i)$$
of the simulated k-FCMA $\mathcal{A}$. It decrements counters $q_1, \ldots, q_i$, and then increments the counters $r_1, \ldots, r_i$.
3. In the third step, $\mathcal{B}$ sets its finite control to the the state r from the transition chosen in step 2.
4. The last step corresponds to the check-reset and is carried out if the next label is going to be from Σ_{i-1} (this corresponds to a rightmost successor node appearing for the second time in the DFS, or for the first time, if the node is a leaf). In this case, the automaton $\mathcal{B}$ tests that all counters in
$$Q_i \setminus (F_i \cup I_i) \tag{1}$$
are empty, and then empties all the counters in Q_i.

We call such a sequence of actions a *macrotransition*. A macrotransition can be carried out by a multicounter automaton with zero checks, by using ϵ-transitions and additional control states. Perhaps the most delicate point is the last step in the macrotransition. First of all, the automaton needs to know the next label. Here, we can nondeterministically guess the next label in advance; this nondeterministic guess is then validated in the next step. (The degenerate case of $j = 0$ is handled by using ϵ-transitions.)

At first glance, the automaton is not a priority multicounter automaton, since the zero checks in (1) are done for disjoint counters. But this can easily be fixed, by imposing a chain discipline on the zero checks. Indeed, when the automaton is doing the zero check in (1), we know that in the previous moves it has emptied the counters $Q_{i+1}, \ldots, Q_k$. Therefore, it could equivalently zero check the counters
$$Q_i \setminus (F_i \cup I_i) \quad \cup \quad Q_{i+1} \quad \cup \quad \cdots \quad \cup \quad Q_k \ .$$

Furthermore, the emptying of the counters in Q_i, which is done after (1), can be simulated by a sequence of nondeterministic decrements on Q_i and then a zero check on $Q_i \cup \cdots \cup Q_k$. The automaton $\mathcal{B}$ accepts if it reaches an accepting global state after processing all the nodes. It is fairly clear that if $\mathcal{A}$ accepts t, then $\mathcal{B}$ accepts $\mathrm{trav}(t)$. Theorem 2 then follows once we show the converse:

Lemma 4. *If t is a depth k forest whose DFS traversal is accepted by $\mathcal{B}$, then t can be labeled with data values so that the resulting data forest is accepted by $\mathcal{A}$.*

Proof
Consider an accepting run of $\mathcal{B}$, with macrotransitions $m_1, \ldots, m_n$. Let $v_1, \ldots, v_n$ be the DFS traversal of the forest t. These nodes correspond to the macrotransitions $m_1, \ldots, m_n$. Recall that each macrotransition corresponds (in step 2) to a transition of the automaton $\mathcal{A}$. Let then $\delta_1, \ldots, \delta_n$ be the sequence of transitions of $\mathcal{A}$ that corresponds to $m_1, \ldots, m_n$.

We will assign data values to nodes of the forest t, so that the result s is accepted by $\mathcal{A}$, using the run $\delta_1, \ldots, \delta_n$. This is done progressively for $v_1, \ldots, v_n$, so that at each intermediate step $j = 0, \ldots, n$ the following invariant is satisfied.

Assume that data values have been assigned to nodes $v_1, \ldots, v_j$. The sequence $\delta_1, \ldots, \delta_j$ is a partial run of $\mathcal{A}$ on t (that has read nodes $v_1, \ldots, v_j$) such that:

> For each class memory function $f_i \in \{f_1, \ldots, f_k\}$, and each non-initial state $q \in Q_i$, the counter q contains the number of data values d with $f_i(d) = q$.

This invariant can be easily shown by induction on j. □

4 A two-variable logic for bounded depth data forests

In this section, we define a first-order logic that can express properties of data forests. Formulas of this logic can be effectively compiled into FCMA; in particular this logic has decidable satisfiability thanks to Theorem 2. Variables quantify over nodes. Only two variables, x and y, are allowed. Furthermore, data values can only be compared for equality, via a predicate $x \sim y$. On the other hand, we allow a large body of navigational predicates.

For a fixed depth k, we define the logic FO^2_k as the two-variable fragment of FO, with the following predicates (some parameterized by $i = 1, \ldots, k$):

$d_i(x)$	x has depth i
$a(x)$	x has label a (here a is a label from Σ)
$x \downarrow_i y$	y is a descendant of x and $\mathit{depth}(y) - \mathit{depth}(x) = i$
$x \downarrow_+ y$	y is a descendant of x
$x + 1 = y$	x is the left sibling of y
$x \prec y$	x comes before y in the document ordering (the ordering produced by a pre-order traversal)
$x < y$	x and y are siblings, and x is to the left of y
$t_i(x, y)$	$x \preceq y$ and the nodes x, y share the same depth i ancestor but not the same depth $i + 1$ ancestor

$t_0(x,y)$	x, y do not have a common ancestor
$x \sim y$	x and y have the same data value
$x \oplus y$	y is the *class successor* of x, that is, $x \sim y$, x comes before y in the document ordering, and there is no z between x and y (in the document ordering) which has the same data value.

The semantic of the logic is defined as usual. For instance, the following is a long way of saying that all nodes have the same data value:

$$\forall x \forall y (x \downarrow_+ y \Rightarrow x \sim y) \wedge (x + 1 = y \Rightarrow x \sim y) \ .$$

The predicates d_i, $\downarrow_+, \downarrow_i$ and $<$ are syntactic sugar, and can be removed from the signature without loss of expressivity. For instance, $d_i(x)$ is the same as $t_i(x,x)$. In similar ways, $\downarrow_i$, $\downarrow_+$ can be defined in terms of t_i, and $<$ can be defined in terms of t_i and $\prec$. Since we only have two variables, $x+1=y$ cannot be defined in terms of $<$, and $x \oplus y$ cannot be defined in terms of $\prec$ and $\sim$.

In the following example, we show that thanks to the bounded depth assumption, two-variable formulas can express properties that seemingly require three variables.

Example 2. We write a formula $\varphi(x)$, which holds in a node x that has two distinct descendants $y \sim z$. The natural formula would be

$$\varphi(x) \quad = \quad \exists y \exists z \ (y \neq z \ \wedge \ y \sim z \ \wedge \ x \downarrow_+ y \ \wedge \ x \downarrow_+ z) \ .$$

The problem is that this formula uses three variables. We will show that for depth k forests, φ can be written with only two variables. The idea is to do a disjunction over the finitely many possible depths i of the node x:

$$\varphi(x) = \bigvee_i d_i(x) \wedge \exists y (x \downarrow_+ y \wedge \exists x (x \neq y \wedge x \sim y \wedge \bigvee_{j \geq i} t_j(x,y))) \ .$$

In the above formula, the second existential quantifier $\exists x$ actually corresponds to the node z. We do not need to verify if the new node x is a descendant of the "real" node x in the free variable; this is a consequence of $t_i(x,y)$.

Theorem 3. *Every language definable in FO^2_k can be recognized by a k-FCMA.*

Corollary 1. *Satisfiability is decidable for FO^2_k.*

5 XPath

We now apply our results to show decidability of some static analysis tasks for XML. Our approach closely mirrors that in [6]. To avoid repetition, we only explain which expressive power can be *added* to the fragment *LocalDataXPath* from [6], while preserving decidability over bounded depth trees. Since we have the predicates $\downarrow_+$ and $<$ in our logic, we can, unlike [6], capture the XPath axes `descendant` (`ancestor`) and `following` (`preceding`). Also, we can allow

attribute comparisons in which *both* sides of the (in-)equality are *relative*, as long as they stay *within the subtree* rooted at the node to which they are relative.

As in [6], decidability is shown by encoding XPath into two-variable logic. We first give an example that illustrates how the bounded depth can be used to encode XPath expressions that could not be handled in [6]. This example is similar to Example 2 in the Section 4.

Example 3. Consider the XPath expression

$$\texttt{child} :: a/\texttt{child} :: b/@B_1 = \texttt{child} :: c/\texttt{next} - \texttt{sibling} :: d/@B_2.$$

It is not allowed in LocalDataXPath, since both sides of the equality are relative paths. For bounded depth trees, however, we can encode it into the logic from Section 4, using a technique similar to that of Example 2.

We now define *BDXPath* (Bounded Depth XPath). It is defined the same way as LocalDataXPath from [6]; except that BDXPath can use all the navigational axes in Core XPath [17]. Also, in predicate expressions, we allow comparisons of attribute values with $=$ and $\neq$ as long as one of the following holds.

1. At least one side of the (in-)equality is an *absolute* location path (i.e., one starting at the root, or document node); or
2. The comparison is relative, but *safe* (as defined in [6]); or
3. Both sides have location expressions that start with `child` or `descendant` and do not use `parent` or `ancestor`.

Using the same proof, but aided by our more powerful two-variable logic, we can upgrade the main XPath result from [6]:

Theorem 4. *Over trees of bounded depth, Satisfiability and Containment for (unary or binary) BDXPath is decidable. This holds even relative to a schema consisting of a regular tree language and unary key and inclusion constraints.*

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