

DECIDABILITY OF REACHABILITY  
IN VECTOR ADDITION SYSTEMS

Preliminary Version

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Abstract

A convincing proof of the decidability of reachability in vector addition systems is presented. No drastically new ideas beyond those in Sacerdote and Tenney, and Mayr are made use of. The complicated tree constructions in the earlier proofs are completely eliminated.

I. Introduction

There already exist two proofs for the decidability of reachability in vector addition systems [4,5,6]. The two approaches have many common features. For example, even though the central concept of 'cones' of [5,6] does not appear explicitly in [4], it seems to have played an equally important role in [4] too. (The surprisingly complicated and unconvincing cone construction of [5,6] can be reduced to a trivial construction.) I will discuss these similarities in the final version. In fact it turns out that no significantly new ideas beyond those of [5,6] are needed to solve this problem. The complicated tree constructions of [4,5,6] can be completely disposed of. I view the tree constructions as convenient tools to test some simple properties of the systems. However, in [4,5,6] certain complicated trees are first constructed and the proof of decidability is built on top of these.

The rest of this section is devoted to pointing out some elementary properties. The main results are presented in the next two sections. Let  $Z$  and  $N$  be the set of integers and the set of nonnegative integers, respectively. For every  $n$ -tuple  $x$ , let  $\pi_i(x)$  be the value of the  $i^{\text{th}}$  component of  $x$ . For every  $m \geq 0$ , let  $\bar{m}$  be the vector of all  $m$ 's. The usual componentwise addition,  $+$ , and comparison,  $\geq$ , are assumed for vectors. For any vector  $x$ ,  $-x = \bar{0} - x$ .

We assume familiarity with semilinear sets, Presburger formulas and their elementary properties [1]. The following somewhat specialized, but trivial, properties are also useful. A linear set with constant  $c$  and periods  $p_1, \dots, p_k$  is denoted by  $L(c; p_1, \dots, p_k)$ .

Lemma 1: Let  $L$  be a semilinear set, and let  $A$  be a subset of  $\{1, 2, \dots, n\}$ . If  $L$  does not satisfy the property that for every  $m \geq 1$  there exists an  $x \in L$  s.t. for every  $j \in A$ ,

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<sup>†</sup> Supported by the National Science Foundation under grant MCS-79-05163.

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$\pi_j(x) \geq m$ , then there exists a constant  $c$  such that when  $L$  is expressed as a union of linear sets  $\bigcup_{i=1}^k L_i$ , for every  $L_i$  there exists a  $j \in A$  s.t. the  $j^{\text{th}}$  component of every period of  $L_i$  has value 0 (the  $j^{\text{th}}$  component of the sum of the periods of  $L_i$  has zero value).

Let  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  and  $S = \{i_1, i_2, \dots, i_k\}$ . For any  $x \in \mathbb{Z}^n$ ,  $x(S) = (\pi_{i_1}(x), \pi_{i_2}(x), \dots, \pi_{i_k}(x))$ . For any  $L \subseteq \mathbb{N}^n$ , let  $L(S) = \{x(S) \mid x \in L\}$ .

Lemma 2:

- (a) For any semilinear set  $L$  and any set of coordinates  $S$ ,  $L(S)$  is a semilinear set.
- (b) If  $L$  is a linear set, then  $L(S)$  is also a linear set.
- (c) If  $L$  is a linear set with one period, then  $L(S)$  can be expressed with one period also. If  $L$  is  $L(c;p)$ , then  $L(S)$  can be given by  $L(c(s);p(S))$ .

Thus the projection of a semilinear set into a subset,  $\{i_1, \dots, i_k\}$ , of coordinates results in a semilinear set.

Lemma 3: Let  $L$  be a semilinear set, and  $S$  a subset of its coordinates. If  $L$  has the property that

- (a) for every  $x \in L$  and every  $j \notin S$ ,  $\pi_j(x)$  has a fixed value, and
- (b) for every  $m \geq 1$  there exists an  $x \in L$  s.t. for every  $j \in S$ ,  $\pi_j(x) \geq m$ , then  $L$  has a linear subset  $L(c;p)$  where the  $j^{\text{th}}$  component of  $p$  is nonzero iff  $j \in S$ .

We also need a few concepts from graph theory. Consider a graph (directed and possibly having parallel arcs) in which all the arcs are labeled distinctly. Let these labels be  $t_1, t_2, \dots, t_k$ . Throughout, we allow nodes without any in- and out-arcs. Such nodes are isolated nodes.

The folding of any (directed) path,  $p$ , of  $G$  is a  $k$ -tuple  $z \in \mathbb{N}^k$  such that  $\pi_i(z)$  equals the number of occurrences of  $t_i$  in the path. Let this folding be denoted by  $\pi(p)$ . Given a  $k$ -tuple  $z \in \mathbb{N}^k$ , a start node  $q_1$ , and an end node  $q_2$ , the unfolding of  $z$  is any path from  $q_1$  to  $q_2$  whose folding is  $z$ . For some combinations of  $z$ ,  $q_1$  and  $q_2$ , an unfolding might not exist. Note also that unfolding is not a functional map.

For every  $z \in \mathbb{N}^k$  and every node  $q$  of  $G$ , let the in-degree of  $q$  w.r.t.  $z$  be  $\sum \pi_i(z)$  if  $t_i$  is an in-arc of  $q$ . This is denoted by  $\text{in}(q, z)$ . The out-degree of  $q$  w.r.t.  $z$  is similarly defined as  $\sum \pi_i(z)$  if  $t_i$  is an out-arc of  $q$ , and is denoted by  $\text{out}(q, z)$ .

In Lemmas 4 to 7,  $G$  is any graph and  $q_1, q_2$  and  $q$  are any nodes of  $G$ .

Lemma 4: Let  $G$  be a strongly connected graph. For any  $k$ -tuple  $z \geq \bar{1}$ , an unfolding of  $z$  from  $q_1$  to  $q_1$  exists iff for every  $q$ ,  $\text{in}(q, z) = \text{out}(q, z)$ .

Proof: Obvious extension of the standard proofs for the existence of Euler trails (or walks) in directed graphs [2].

Lemma 5: In  $G$ , let  $z$  be the folding of any path from  $q_1$  to  $q_2$ . Then

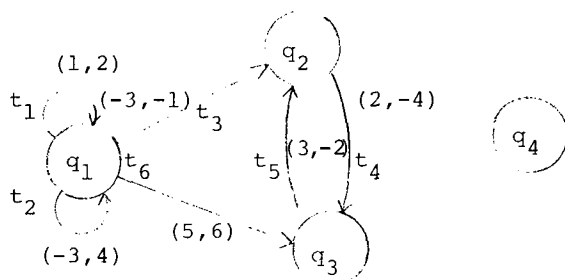
- (i) for every  $q \notin \{q_1, q_2\}$ ,  $\text{in}(q, z) = \text{out}(q, z)$ , and
- (ii) if  $q_1 = q_2$  then  $\text{in}(q_1, z) = \text{out}(q_1, z)$ , and  
if  $q_1 \neq q_2$  then  $\text{out}(q_1, z) = \text{in}(q_1, z) + 1$  and  
 $\text{in}(q_2, z) = \text{out}(q_2, z) + 1$ .

Lemma 6: In  $G$ , let  $z_1$  and  $z_2$  be the foldings of two paths from  $q_1$  to  $q_2$ . If  $z_1 - z_2 \geq \bar{1}$ , then for every non-isolated  $q$ , there exists an unfolding of  $z_1 - z_2$  from  $q$  to  $q$ .

Proof: We leave it to the reader to prove that existence of  $z_1$  and  $z_2$  with  $z_1 - z_2 \geq \bar{1}$  implies that all the non-isolated nodes are strongly connected. Now apply Lemma 5 to  $z_1$  and  $z_2$  and deduce that for every node  $q$ ,  $\text{in}(q, z_1 - z_2) = \text{out}(q, z_1 - z_2)$ . Finally apply Lemma 4.

Lemma 7: In  $G$ , let  $z_1$  be the folding of a path from  $q_1$  to  $q_1$ , and let  $z_2$  be the folding of a path from  $q_2$  to  $q_2$ , such that for some  $i_0$ ,  $i_0 z_1 - z_2 \geq \bar{1}$ . Then for any non-isolated  $q$ , there exists an unfolding of  $i_0 z_1 - z_2$  from  $q$  to  $q$ .

A vector addition system with states (VASS), as in [3], is an fsa or a directed graph in which the label of each arc is an  $n$ -tuple of integers. For uniformity of description, we allow the fsa to have states without any in- and out-arcs (isolated states). An example VASS is given below. Note that  $q_4$  is an isolated state. A configuration



of the VASS is given by  $(q, x)$  where  $q$  is a state and  $x$  is a point in  $\mathbb{Z}^n$ . Given an initial configuration  $(q_1, x)$ , a path from  $q_1$  in the fsa induces an obvious sequence of configurations. We informally denote this sequence of configurations or the corresponding sequence of points in  $\mathbb{Z}^n$  as a path. In the above example for the initial configuration  $(q_1, (7,7))$  and the path  $q_1 \xrightarrow{t_1} q_1 \xrightarrow{t_2} q_1 \xrightarrow{t_3} q_2 \xrightarrow{t_4} q_3$ , the corresponding sequences of configurations and points are  $(q_1, (7,7)) (q_1, (8,9)) (q_1, (5,13)) (q_2, (2,12)) (q_3, (4,8))$  and  $(7,7) (8,9) (5,13) (2,12) (4,8)$ , respectively. When the intent is clear, we sacrifice some precision for clarity. For example, we might say that point  $(8,9)$  is on the above path.

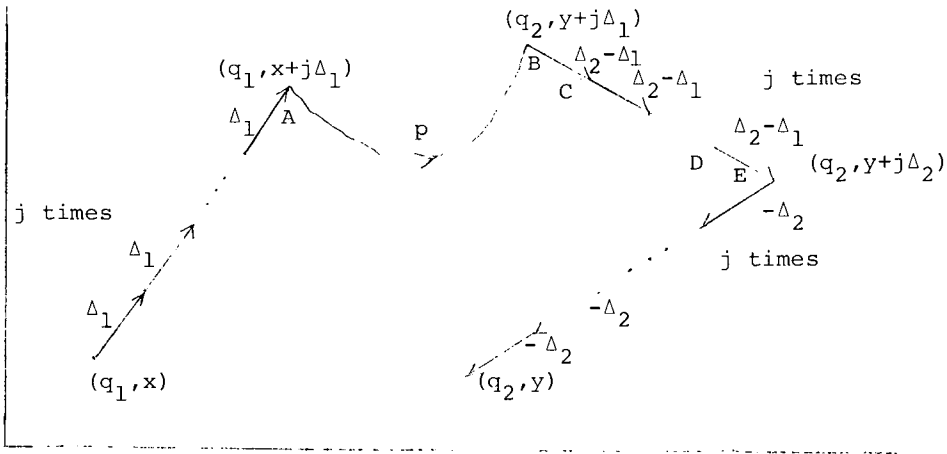
In a VASS,  $(q_2, y)$  is r-reachable from  $(q_1, x)$  iff there is a path,  $p$ , from  $(q_1, x)$  to  $(q_2, y)$ . We also write this as  $(q_2, y) \in r(q_1, x)$  or  $(q_2, y) \in r(q_1, x)$  by path  $p$ . The corresponding path is an r-path. Observe that an r-path can possibly pass through points in  $\mathbb{Z}^n$ . The configuration  $(q_2, y)$  is R-reachable from the configuration  $(q_1, x)$  iff there is a path from  $(q_1, x)$  to  $(q_2, y)$  s.t. for every configuration  $(q', v)$  on the path,  $v \in \mathbb{N}^n$ . We also denote this by  $(q_2, y) \in R(q_1, x)$ . The corresponding path is an R-path. Note that every R-path must lie completely in the positive orthant. Let  $A$  be a subset of the coordinates. The configuration  $(q_2, y)$  is semi R-reachable or SR-reachable w.r.t. A from the configuration  $(q_1, x)$  iff there is a path from  $(q_1, x)$  to  $(q_2, y)$  s.t. for every configuration  $(q', v)$  on the path and for every  $i \in A$ ,  $\pi_i(v) \geq 0$ . We also denote this by  $(q_2, y) \in SR(q_1, x)$  w.r.t.  $A$ . The corresponding path is an SR-path w.r.t. A. Note that when  $A$  is the empty set, the SR concept coincides with the  $r$  concept, and when  $A$  is the set of all  $n$  coordinates, the SR concept coincides with the  $R$  concept. Now we can state the reachability problem.

Reachability Problem: Design an algorithm to test for every  $q_1, x, q_2, y$  whether  $(q_2, y) \in R(q_1, x)$  or not.

- (b2)  $(q_2, y + \Delta_2) \in R_{\text{rev}}(q_2, y)$  , and  
(c)  $(q_2, \Delta_2 - \Delta_1) \in r(q_2, \bar{0})$  ,  
then  
 $(q_2, y) \in R(q_1, x)$  .

(Note that (a) states that  $(q_2, y)$  is reachable from  $(q_1, x)$  by some path in  $Z^n$  , (b) states that from  $(q_1, x)$  there exists a path in the positive orthant which increases all the coordinates of  $x$  , and similarly  $(q_2, y)$  can be reached from another point with all coordinates bigger, and (c) specifies certain spanning properties of  $G$  .

Proof: Let the r-path from  $(q_1, x)$  to  $(q_2, y)$  be  $p$  (condition (a) assures its existence). Then the R-reachability of  $(q_2, y)$  from  $(q_1, x)$  can be represented by the following schematic path. Note that the subpaths with  $\Delta_1$  and  $-\Delta_2$  shifts are R-paths, and the subpaths with  $\Delta_2 - \Delta_1$  shifts are r-paths. Select the minimum  $j$  such that paths



$p$  ,  $BC$  and  $DE$  lie in the positive orthant. (As  $j$  increases, every coordinate value of  $A, B$  and  $E$  increases. Consequently such a  $j$  exists). By Lemma 9, the complete path from  $B$  to  $E$  is in the positive orthant. Thus the complete path is an R-path.

The schematic path shown earlier is very similar to the schematic path in page 11 of [6] and also the proof of Theorem 2.6 in [5]. I want to emphasize this similarity to indicate that the basic idea in [5,6] is sound. In the rest of the paper we generalize this result and solve the reachability problem. The next result is our first variant of Theorem 1.

Theorem 2: In a VASS,  $G$  , for every  $q_1, x, q_2, y$  if there exist  $\Delta_1, \Delta_2 \geq \bar{1}$  s.t.

- (a)  $(q_2, y) \in r(q_1, x)$  ,  
(b1)  $(q_1, x + \Delta_1) \in R(q_1, x)$  ,  
(b2)  $(q_2, y + \Delta_2) \in R_{\text{rev}}(q_2, y)$  , and  
(c')  $(q_1, \bar{0})$  is r-reachable from  $(q_1, \bar{0})$  by a path  $p$  s.t.  $\pi(p) \geq \bar{1}$  ,  
then  $(q_2, y) \in R(q_1, x)$  .

Proof: It is sufficient to show that the above conditions imply condition (c) of Theorem 1.

Application of Lemma 8 to condition (b1) leads to  $(q_1, \Delta_1) \in r(q_1, \bar{0})$  . Similarly

For a VASS,  $G$ , let its reverse, denoted  $G_{\text{rev}}$ , be obtained by reversing the arcs in the fsa and then replacing every label  $x$  by  $-x$ . The  $r$ ,  $R$  and  $SR$  reachabilities of  $G_{\text{rev}}$  are denoted by  $r_{\text{rev}}$ ,  $R_{\text{rev}}$  and  $SR_{\text{rev}}$ , respectively.

Lemma 8: In any VASS,  $G$ , if  $(q_2, y) \in r(q_1, x)$ , then for every  $\Delta$ ,  $(q_2, y+\Delta) \in r(q_1, x+\Delta)$ . In particular,  $(q_2, y-x) \in r(q_1, \bar{0})$ .

Proof: Simply shift the starting point, keeping the old path.

If  $p$  is a path from  $\bar{0}$  to  $v$ , then  $p$  has the "effect" of shifting any point by  $v$ . We denote this effect by shift( $p$ ); i.e.  $\text{shift}(p) = v$ .

Lemma 9: In a VASS,  $G$ , let  $(q_1, \Delta_1) \in r(q_1, \bar{0})$  by a path  $p$ . Consider a sequence of configurations  $(q_1, x)$ ,  $(q_1, x+\Delta_1)$ ,  $(q_1, x+2\Delta_1)$ , ...,  $(q_1, x+k\Delta_1)$  where path  $p$  is applied from  $(q_1, x+i\Delta_1)$  to  $(q_1, x+(i+1)\Delta_1)$  for  $i=0, \dots, k-1$ . If the two paths  $(q_1, x) \xrightarrow{p} (q_1, x+\Delta_1)$  and  $(q_1, (x+(k-1)\Delta_1)) \xrightarrow{p} (q_1, x+k\Delta_1)$  are  $R$ -paths, then the complete path is an  $R$ -path.

Proof: Observe that for every  $1 \leq i \leq n$  and  $1 \leq j \leq k-1$ ,  $\pi_i(x+j\Delta_1)$  is in between  $\pi_i(x)$  and  $\pi_i(x+k\Delta_1)$ .

Lemma 10: In any VASS,  $G$ , and for any initial configuration  $(q_1, x)$ , the following hold:

- (a) it can be effectively decided whether there exists a  $\Delta \geq \bar{1}$  s.t.  $(q_1, x+\Delta) \in R(q_1, x)$ , and
- (b) if there does not exist any  $\Delta \geq \bar{1}$  satisfying  $(q_1, x+\Delta) \in R(q_1, x)$ , then a constant  $c$  s.t. every point  $R$ -reachable from  $(q_1, x)$  has some coordinate value  $\leq c$  can be effectively computed.

Proof: A trivial tree construction establishes this lemma.

The next lemma is a simple generalization of Lemma 10.

Lemma 11: In any VASS,  $G$ , for any set of coordinates  $A$  and any initial configuration  $(q_1, x)$ , the following hold:

- (a) it can be effectively decided whether there exists a  $\Delta \in \mathbb{Z}^n$ , s.t. for every  $j \in A$ ,  $\pi_j(\Delta) \geq 1$ , and  $(q_1, x+\Delta) \in SR(q_1, x)$  w.r.t.  $A$ , and
- (b) if there does not exist any  $\Delta$  satisfying (a), then a constant  $c$  s.t. every point  $SR$ -reachable w.r.t.  $A$  from  $(q_1, x)$  has  $i^{\text{th}}$  component value  $\leq c$  for some  $i$  in  $A$  can be effectively computed.

Proof: Suppress the coordinates which are not in  $A$  from the labels of all the arcs of  $G$  and also from  $x$  and  $y$ , and then apply Lemma 10.

## II. A Result on VASS

The following two theorems form the bridge between the approaches of [4,5,6] and the proof given here. They do not form a part of the main approach, and the anxious reader is advised to go directly to Theorem 3. Theorem 4 is the main result that will be applied in the next section. In all these theorems  $q_1$  and  $q_2$  are any states of  $G$ , and  $x$  and  $y$  are in  $\mathbb{N}^n$ .

Theorem 1: In a VASS,  $G$ , for every  $q_1, x, q_2, y$  if there exist  $\Delta_1, \Delta_2 \geq \bar{1}$  s.t.

- (a)  $(q_2, y) \in r(q_1, x)$ ,
- (b1)  $(q_1, x+\Delta_1) \in R(q_1, x)$

condition (b2) implies  $(q_2, \Delta_2) \in r_{\text{rev}}(q_2, \bar{0})$ , which further implies  $(q_2, -\Delta_2) \in r(q_2, \bar{0})$ . Let  $p_1$  be an  $r$ -path from  $(q_1, \bar{0})$  to  $(q_1, \Delta_1)$ , and let  $p_2$  be an  $r$ -path from  $(q_2, \bar{0})$  to  $(q_2, -\Delta_2)$ . Choose the minimum  $i_0$  s.t.

$i_0 \pi(p) \geq \pi(p_1) + \pi(p_2) + \bar{1}$  (this can be done since  $\pi(p) \geq \bar{1}$ ). Then application of Lemma 7 twice ( $i_0 \pi(p) - \pi(p_1) \geq \bar{1}$  and  $(i_0 \pi(p) - \pi(p_1)) - \pi(p_2) \geq \bar{1}$ ), results in the existence of a path from  $q_2$  to  $q_2$  ( $q_2$  is non-isolated due to b2) whose folding is  $i_0 \pi(p) - (\pi(p_1) + \pi(p_2))$ . Hence there exists an  $r$ -path from  $(q_2, \bar{0})$  to  $(q_2, \Delta_2 - \Delta_1)$ . (Note that  $\text{shift}(p) = \bar{0}$ ,  $\text{shift}(p_1) = \Delta_1$ , and  $\text{shift}(p_2) = -\Delta_2$ ). Thus condition (c) of Theorem 1 holds.

In this theorem, condition (c') gives a spanning property of the VASS. In the spirit of [5,6], it can be interpreted as that if any vector  $v$  is in the "positive span" then  $-v$  is also in the positive span.

The next theorem is an important generalization of Theorem 2.

**Theorem 3:** In a VASS, for every  $q_1, x, q_2, y$  and  $\Delta x, \Delta y \geq \bar{0}$ , if there exist  $\Delta_1, \Delta_2 \in \mathbb{Z}^n$  and  $m_1, m_2 \geq 0$ , s.t.

- (a) for every  $i \geq 1$ ,  $\pi_i(\Delta x) = 0 \Rightarrow \pi_i(\Delta_1) \geq 1$ , and  
 $\pi_i(\Delta y) = 0 \Rightarrow \pi_i(\Delta_2) \geq 1$ ,
- (b)  $(q_2, y) \in r(q_1, x)$ ,
- (c)  $(q_1, x + m_1 \Delta x + \Delta_1) \in R(q_1, x + m_1 \Delta x)$ , and  
 $(q_2, y + m_2 \Delta y + \Delta_2) \in R_{\text{rev}}(q_2, y + m_2 \Delta y)$ , and
- (d)  $(q_1, \Delta y)$  is  $r$ -reachable from  $(q_1, \Delta x)$  by a path  $p$  s.t.  $\pi(p) \geq \bar{1}$ ,  
then  
 $(\exists j_0) (\forall j \geq j_0) (q_2, y + j \Delta y) \in R(q_1, x + j \Delta x)$ .

Note that if  $\Delta x = \Delta y = \bar{0}$ , then this result degenerates into Theorem 2.

**Proof:** First we prove that the above conditions imply the condition (d') and (e) given below.

- (d')  $(q_2, \Delta y - \Delta x) \in r(q_2, \bar{0})$ , and
- (e) there exists an integer  $\alpha \geq 1$  s.t.  
 $\alpha \Delta x + \Delta_1 \geq \bar{1}$ ,  
 $\alpha \Delta y + \Delta_2 \geq \bar{1}$ , and  
 $(q_2, \alpha(\Delta y - \Delta x) + \Delta_2 - \Delta_1) \in r(q_2, \bar{0})$ .

By Lemma 8, (d) implies that  $(q_1, \Delta y - \Delta x) \in r(q_1, \bar{0})$  by path  $p$ . Since  $p$  makes use of every arc,  $p$  goes through every non-isolated state. Remove a prefix of  $p$  that corresponds to a subpath from  $q_1$  to  $q_2$  and append it as the suffix of  $p$  ( $q_2$  is non-isolated due to (c)). This establishes (d').

Since  $\Delta x, \Delta y \geq \bar{0}$ , (a) implies that there exists a  $\beta \in \mathbb{N}$  s.t. for every  $\beta' \geq \beta$ :

$$\beta' \Delta x + \Delta_1 \geq \bar{1}, \text{ and}$$

$$\beta' \Delta y + \Delta_2 \geq \bar{1}.$$

Application of Lemma 8 to (c) results in

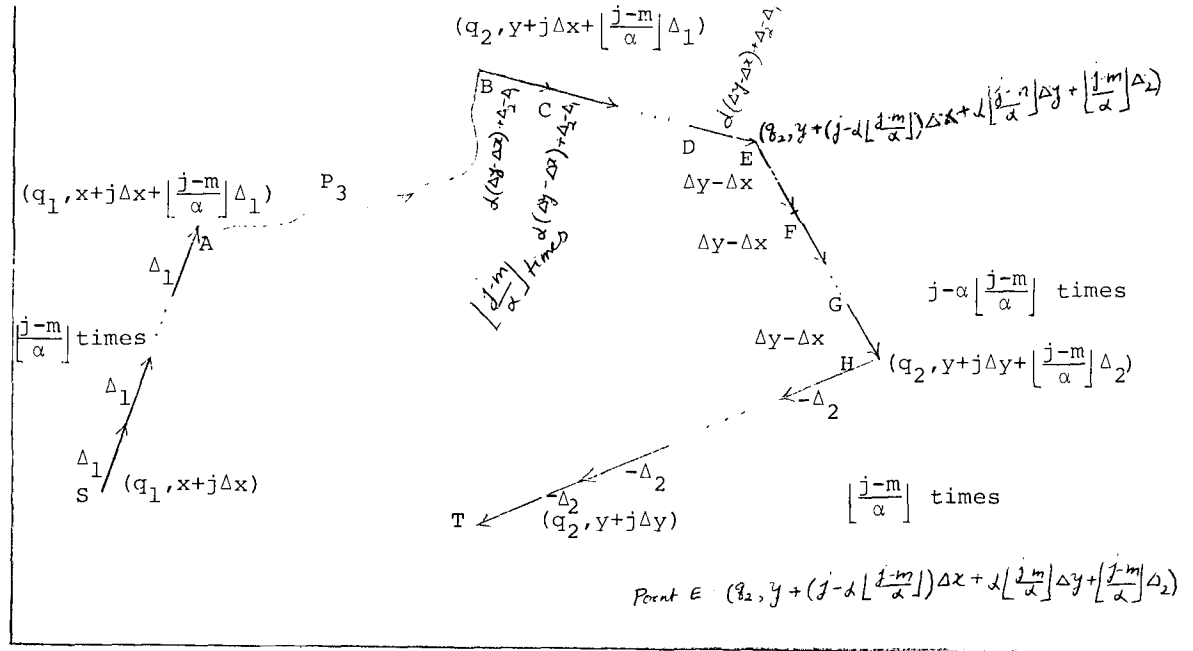
$$(q_1, \Delta_1) \in r(q_1, \bar{0}), \text{ and}$$

$$(q_2, -\Delta_2) \in r(q_2, \bar{0}).$$

Let  $p_1$  be an r-path from  $(q_1, \bar{0})$  to  $(q_1, \Delta_1)$ , and let  $p_2$  be an r-path from  $(q_2, \bar{0})$  to  $(q_2, -\Delta_2)$ . Let  $\gamma$  be the minimum value  $\geq 1$  s.t.  $\gamma\pi(p) \geq \pi(p_1) + \pi(p_2) + \bar{1}$  (this is feasible since  $\pi(p) \geq \bar{1}$ ). For any  $\gamma' \geq \gamma$ ,  $\gamma'\pi(p) \geq \pi(p_1) + \pi(p_2) + \bar{1}$ . Application of Lemma 7 twice ( $\gamma'\pi(p) - \pi(p_1) \geq \bar{1}$ , and  $(\gamma'\pi(p) - \pi(p_1)) - \pi(p_2) \geq \bar{1}$ ) results in the existence of a path from  $q_2$  to  $q_2$  whose folding is  $\gamma'\pi(p) - (\pi(p_1) + \pi(p_2))$ . Finally  $\gamma' \text{shift}(p) - (\text{shift}(p_1) + \text{shift}(p_2)) = \gamma'(\Delta y - \Delta x) + \Delta_2 - \Delta_1$ . Then choose  $\alpha = \max\{\beta, \gamma\}$ ,  $\beta' = \alpha$ , and  $\beta'' = \alpha$ , establishing (e).

Now we shall make use of conditions (b), (c), (d') and (e) to prove the theorem.

Let an r-path from  $(q_1, x)$  to  $(q_2, y)$  be  $p_3$ . The R-reachability of  $(q_2, y + j\Delta y)$  from  $(q_1, x + j\Delta x)$  can be represented by the following schematic path ( $m = \max\{m_1, m_2\}$ ).



Note that when  $j$  increases by a step of  $\alpha$ , all the components of A, B, E and H increase (This for E follows easily from the first two conditions of (e). For A, B and H a little more justification is needed. As an example, for A, observe that  $x + j\Delta x + \lfloor \frac{j-m}{\alpha} \rfloor \Delta_1 = x + \lfloor \frac{j}{\alpha} \rfloor (\alpha\Delta_2 + \Delta_1) + (j - \alpha \lfloor \frac{j}{\alpha} \rfloor) \Delta x$ . In addition, if  $j-m$  is a multiple of  $\alpha$  and if  $j$  increases by any step in between 1 and  $\alpha$ , then no component of A, B, E and H decreases, since  $\Delta x, \Delta y \geq \bar{0}$  (caution:  $\Delta x + \Delta_1$  and  $\Delta y + \Delta_2$  can possibly have negative components). Now keep  $j-m$  as a multiple of  $\alpha$  and increase it in steps of  $\alpha$  until the paths  $p_3$ , BC, DE, EF and GH are entirely in the positive orthant. Let the minimum value of  $j$  when this happens be  $j_0$ . For that choice of  $j$ , by Lemma 9, paths BE and EH are in the positive orthant. From the above observations, for every  $j \geq j_0$  every component of A, B, E and H is greater than or equal to the corresponding component of  $j_0$ . Hence for every  $j \geq j_0$ , the corresponding path is an R-path.

The next theorem is a minor generalization of Theorem 3, and forms the heart of the decision procedure.

**Theorem 4:** In a VASS,  $G$ , for every  $q_1, x, q_2, y$  and  $\Delta x, \Delta y \geq \bar{0}$ , if there exists  $\Delta_1, \Delta_2 \in \mathbb{Z}^n$ ,  $A \subseteq \{1, 2, \dots, n\}$  and  $m_1, m_2 \geq 0$  s.t.

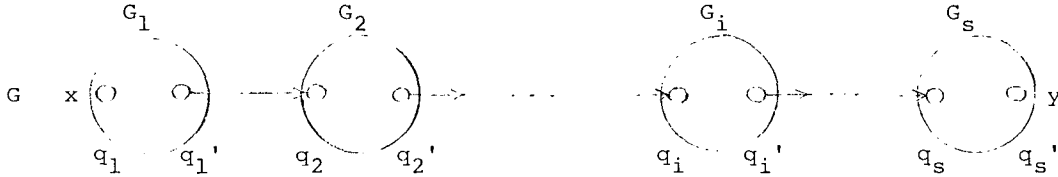
- (a) for every  $j \in A$ , the  $j^{\text{th}}$  component of the label of every arc of  $G$  has value 0, and  
for every  $j \notin A$ ,  $\pi_j(\Delta x) = 0 \Rightarrow \pi_j(\Delta_1) \geq 1$ , and  
 $\pi_j(\Delta y) = 0 \Rightarrow \pi_j(\Delta_2) \geq 1$ ,
- (b)  $(q_2, y) \in r(q_1, x)$
- (c)  $(q_1, x+m_1\Delta x+\Delta_1) \in R(q_1, x+m_1\Delta x)$ , and  
 $(q_2, y+m_2\Delta y+\Delta_2) \in R_{\text{rev}}(q_2, y+m_2\Delta y)$ , and
- (d)  $(q_1, \Delta y)$  is  $r$ -reachable from  $(q_1, \Delta x)$  by a path  $p$  s.t.  $\pi(p) \geq \bar{1}$ ,  
then  
 $(\exists j_0)(\forall j \geq j_0)((q_2, y+j\Delta y) \in R(q_1, x+j\Delta x))$ .

Proof: For every  $j \in A$ , the  $j^{\text{th}}$  component of the label of every arc has value 0, which implies that for any path in  $G$  the  $j^{\text{th}}$  component will not change. Thus from (b) we can infer that for every  $j \in A$ ,  $\pi_j(x) = \pi_j(y)$ . Condition (d) implies that for every  $j \in A$ ,  $\pi_j(\Delta x) = \pi_j(\Delta y)$ . Hence for every  $i \geq 0$  and every  $j \in A$ ,  $\pi_j(x+i\Delta x) = \pi_j(y+i\Delta y)$ . Now from the labels of all the arcs of  $G$  and also from all the vectors involved suppress the coordinates in  $A$  and apply Theorem 3. Any  $R$ -path so obtained from  $(q_1, x+i\Delta x)$  to  $(q_2, y+i\Delta y)$  is also an  $R$ -path for the original  $G$  (In this statement, reference to  $x+i\Delta x$  and  $y+i\Delta y$  is technically incorrect, since after the coordinates of  $A$  are suppressed the new vectors are  $n-|A|$  dimensional. However the intent is clear, and we will not indulge in unnecessary new notation).

In the next section, we will introduce a more general model of VASS's, known as GVASS. We will then give a decision procedure for solving its reachability problem.

### 3. Generalized VASS (GVASS)

Consider the following chain of  $n$ -dimensional VASS's. Every  $G_i$  is a VASS and



there is one arc from  $q_i'$  to  $q_{i+1}$ , for  $i=1, \dots, s-1$ . Every  $r$ -path from  $(q_1, x)$  to  $(q_s, y)$  goes through the arc  $(q_{i-1}', q_i)$ , for  $i=2, \dots, s$ , exactly once. When the  $r$ -path reaches  $q_i$  for the first time, the corresponding  $n$ -dimensional point is denoted as the input point of  $G_i$ ; similarly, when the path reaches  $q_i'$  for the last time, the corresponding  $n$ -dimensional point is the output point of  $G_i$ . For  $G_1$  the input point is  $x$ , and for  $G_s$  the output point is  $y$ . Note that an  $r$ -path from  $(q_1, x)$  to  $(q_s, y)$  need not have the input and output points of every  $G_i$  in the positive orthant. From now on we shall be interested only in  $r$ -paths which have all the intermediate input and output points in the positive orthant. With the help of Presburger formulation, it can be shown that whether there exists such an  $r$ -path from  $(q_1, x)$  to  $(q_s, y)$  can be effectively decided.

We further tighten the concept of an  $r$ -path as follows. We place the additional constraint that for every  $G_i$  certain specified coordinates of its input and output points must have fixed specified values. To make this precise, we introduce a new symbol,  $\omega$ , as in [4], which stands for "don't care" or simply " $\geq 0$ ". For each  $G_i$  we impose 2 constraints: an input constraint,  $V_i$ , and an output constraint,  $V_i'$ , where  $V_i, V_i' \subseteq$



$(Nu\{\omega\})^n$ . An r-path from  $(q_1, x)$  to  $(q'_s, y)$  is said to satisfy the input constraint of  $G_i$  if for every  $j$  with  $\Pi_j(V_i) \in N$  the  $j^{\text{th}}$  component of the input point of  $G_i$  has the value  $\Pi_j(V_i)$ , and for every  $j$  with  $\Pi_j(V_i) = \omega$  the  $j^{\text{th}}$  component of the input point of  $G_i$  has a value  $\geq 0$ . That is, non- $\omega$ 's specify exact values, and  $\omega$ 's are don't cares. The path satisfies the output constraint of  $G_i$  if for every  $j$  with  $\Pi_j(V_i) \in N$  the  $j^{\text{th}}$  component of the output point of  $G_i$  has the value  $\Pi_j(V_i)$ , and for every  $j$  with  $\Pi_j(V_i) = \omega$  the  $j^{\text{th}}$  component of the output point of  $G_i$  has a value  $\geq 0$ . Let us denote any r-path which satisfies the input and output constraints for all  $G_i$ 's as a cr-path. We express this by  $(q'_s, y) \in \text{cr}(q_1, x)$ . Note that when a cr-path is inside<sup>†</sup> a  $G_i$ , the corresponding points can have negative components.

A CR-path from  $(q_1, x)$  to  $(q'_s, y)$  is a cr-path from  $(q_1, x)$  to  $(q'_s, y)$ , and, in addition, no intermediate point can have a negative component. We express this by  $(q'_s, y) \in \text{CR}(q_1, x)$ . Note that a CR-path is a cr-path and also an R-path.\* Then in any cr-path from  $(q_1, x)$  to  $(q'_s, y)$ , the  $j^{\text{th}}$  component of the input and the output points of  $G_i$  must be the same. This holds since any path in  $G_i$  does not change the  $j^{\text{th}}$  component. In addition, when the path is inside  $G_i$  the  $j^{\text{th}}$  component of every point on this subpath is positive (in fact equal to the  $j^{\text{th}}$  component of the input point which is the same as the  $j^{\text{th}}$  component of the output point).

The above chain of VASS's with the input and output constraints and the rigid set specified for each  $G_i$  is a generalized VASS, denoted by GVASS.

The reachability problem for a GVASS is to test whether  $(q'_s, y) \in \text{CR}(q_1, x)$  for every given  $q_1, x, q_s$  and  $y$ .

Observe that when  $s=1$ , the GVASS becomes a VASS. Consequently decidability of reachability for GVASS's implies decidability of reachability for VASS's. In this section we give an effective procedure for the reachability problem for GVASS's. Throughout,  $G$  refers to the GVASS shown above.

Let the total number of arcs in  $G_1, \dots, G_s$  be  $k$ , where each  $G_i$  has  $k_i$  arcs ( $k = \sum k_i$ ). Any cr-path from  $(q_1, x)$  to  $(q'_s, y)$  can be mapped into a  $(2sn+k)$ -tuple, with the following interpretation.

$$\begin{array}{ccccccc} \boxed{\text{1st } n \text{ block}} & \boxed{\text{2nd } n \text{ block}} & \dots & \boxed{\text{2s}^{\text{th}} \text{ } n \text{ block}} & \boxed{\text{k block}} & & \\ \text{input point of } G_1 & \text{output point of } G_1 & & \text{output point of } G_s & \text{arcs} & & \\ "x" & & & "y" & & & \end{array}$$

For  $i=1, \dots, s$ , the  $(2i-1)^{\text{th}}$   $n$  block specifies the input point of  $G_i$ ; for  $i=1, \dots, s$ , the  $2i^{\text{th}}$   $n$  block specifies the output point of  $G_i$ ; the last  $k$  block specifies the number of times each arc in  $G_1, \dots, G_s$  is used in the path. Let us denote this "extended folding" by  $\Pi^e$ . Let

$$L_G = \{\Pi^e(p) \mid p \text{ is a cr-path from } (q_1, x) \text{ to } (q'_s, y)\}.$$

It can be easily shown, with the help of Presburger formulation, that  $L_G$  is an effectively computable semilinear set. If we project  $L_G$  into the  $2n+k_i$  dimensional subspace corresponding to the input ( $n$  components), and output ( $n$  components) points and the arcs ( $k_i$  components) of  $G_i$ , then the resulting set is semilinear (Lemma 2a), and this set gives the extended foldings of subpaths in  $G_i$ . Let this projection be denoted by  $L_G[i]$ .

<sup>†</sup> "path inside a  $G_i$ " refers to the subpath every one of whose states is in  $G_i$ .

\* Some definitions were accidentally omitted; they are given on the bottom of p. 281.

In  $V_i$ , replace every  $\omega$  component by 0, and let the new vector be denoted by  $v_i$ ; similarly  $v_i'$  is obtained from  $V_i'$  by replacing every  $\omega$  by 0. Now we define a very crucial property of GVASS's.

The GVASS,  $G$ , with initial configuration  $(q_1, x)$  and final configuration  $(q_s', y)$  satisfies property 0 iff the following conditions are satisfied:

- 01: for every  $m \geq 1$ , there exists a cr-path from  $(q_1, x)$  to  $(q_s', y)$  s.t.
- (a) every arc in every  $G_i$  is used at least  $m$  times, and
  - (b) for every  $i$  and  $j$ , if  $j \notin S_i$  then the  $j^{\text{th}}$  component of the input point of  $G_i$  has a value  $\geq m$ , and if  $j \notin S_i'$  then the  $j^{\text{th}}$  component of the output point of  $G_i$  has a value  $\geq m$ , and
- 02: for every  $i$ , there exist  $\Delta_i, \Delta_i' \subseteq \mathbb{Z}^n$  s.t. for every  $j \in S_i - R_i$ ,  $\pi_j(\Delta_i) \geq 1$  and for every  $j \in S_i' - R_i$ ,  $\pi_j(\Delta_i') \geq 1$ , and
- (a)  $(q_i, v_i + \Delta_i) \in \text{SR}(q_i, v_i)$  w.r.t.  $S_i - R_i$  in  $G_i$ , and
  - (b)  $(q_i', v_i' + \Delta_i') \in \text{SR}_{\text{rev}}(q_i', v_i')$  w.r.t.  $S_i' - R_i$  in  $G_i$ .

(Informally condition 01(b) states that every unconstrained input or output coordinate of every  $G_i$  must have a value  $\geq m$ . Condition 02 states that for every  $G_i$  in the subspace of its constrained, but nonrigid, input coordinates, all the components of its input constraint vector can be simultaneously increased by an R-path; and in the subspace of its constrained, but nonrigid, output coordinates, all the components of its output constraint vector can be simultaneously increased by an R-path in  $G_{i_{\text{rev}}}$ ).

**Theorem 5:** In the GVASS,  $G$ , there is a CR-path from  $(q_1, x)$  to  $(q_s', y)$  if  $G$  satisfies property 0.

**Proof:** Recall that  $L_G = \{\pi^e(p) \mid p \text{ is a cr-path from } (q_1, x) \text{ to } (q_s', y)\}$  is a semi-linear set. Since  $G$  satisfies property 0, for every  $m$ ,  $L_G$  contains an element whose components corresponding to every unconstrained coordinate and every arc have values  $\geq m$  (from 01). Thus by Lemma 3,  $L_G$  contains a linear subset of the form  $L(c; p)$ , denoted  $\hat{L}_G$ , where  $p$  has nonzero components corresponding to every unconstrained coordinate and every arc, and zero components corresponding to every constrained coordinate. Project  $\hat{L}_G$  into  $G_i$ ; i.e. consider  $\hat{L}_G[i]$ . Every element of  $\hat{L}_G[i]$  is in  $\mathbb{N}^{2n+k_i}$  and is the extended folding of a subpath of a cr-path from  $(q_1, x)$  to  $(q_s', y)$ . By Lemma (2c),  $\hat{L}_G[i]$  is a linear set and can be written as  $L((x^i, y^i, z^i); (\Delta x^i, \Delta y^i, \Delta z^i))$ . Where  $x^i, y^i, \Delta x^i$  and  $\Delta y^i$  are  $n$ -tuples and  $z^i$  and  $\Delta z^i$  are  $k_i$ -tuples. The  $x$ 's and  $y$ 's correspond to the input and the output points, respectively, of  $G_i$ . The  $z$ 's correspond to the foldings of subpaths in  $G_i$ . By Lemma (2c) (period of projection =  $p(s)$ ), we can infer that for every  $j$ ,

$$\pi_j(\Delta x^i) \geq 1 \text{ iff } j \notin S_i, \text{ and } \pi_j(\Delta y^i) \geq 1 \text{ iff } j \notin S_i', \text{ and}$$

$$\Delta z^i \geq \bar{1} \quad \dots (*)$$

In the following we establish that  $G_i$  satisfies conditions (a) to (d) of Theorem 4 (with  $q_1 = q_i, x = x^i, q_2 = q_i', y = y^i, \Delta x = \Delta x^i, \Delta y = \Delta y^i, \Delta_1 = \Delta_i, \Delta_2 = \Delta_i'$  and  $A = R_i$ ).

For every  $j \in R_i$  the  $j^{\text{th}}$  component of the label of every arc in  $G_i$  has value 0. For every  $j \notin R_i$ , if  $\pi_j(\Delta x^i) = 0$ , then  $j \in S_i - R_i$  (from \*), which, by condition 02, implies that  $\pi_j(\Delta_i) \geq 1$ . For every  $j \notin R_i$ , if  $\pi_j(\Delta y^i) = 0$ , then  $j \in S_i' - R_i$  (from \*), which, by condition 02, implies that  $\pi_j(\Delta_i') > 0$ . Hence for  $G_i$  condition (a) of Theorem 4 holds.

From the definition of  $\hat{L}_G[i]$ , it is easily seen that, for every  $j \geq 0$ , there is an  $r$ -path in  $G_i$  from  $(q_i, x^i + j\Delta x^i)$  to  $(q_i', y^i + j\Delta y^i)$ , the folding of the path being  $z^i + j\Delta z^i$ . For  $j=0$ , the above implies that  $(q_i', y^i) \in r(q_i, x^i)$ , which establishes condition (b) of Theorem 4.

Since there is a  $cr$ -path from  $(q_1, x)$  to  $(q_s', y)$  with the input and output points of  $G_i$  being  $x^i$  and  $y^i$ , respectively (from the fact that  $(x^i, y^i, z^i) \in \hat{L}_G[i]$ ), we can infer (from the definition of a  $cr$ -path) that for every  $j \in S_i$ ,  $\pi_j(V_i) = \pi_j(x^i)$ , and for every  $j \in S_i$ ,  $\pi_j(V_i) = \pi_j(y^i)$ .

Due to  $\theta 2(a)$ , there exists an  $SR$ -path, say  $p_1$ , w.r.t.  $S_i - R_i$  in  $G_i$  from  $(q_i, v_i)$  to  $(q_i, v_i + \Delta_i)$ . Since path  $p_1$  is entirely in  $G_i$ ,  $v_i \geq \bar{0}$ , and for every  $j \in R_i$  the  $j^{th}$  component of every arc in  $G_i$  has value 0, we can infer that path  $p_1$  is an  $SR$ -path w.r.t.  $S_i$  from  $(q_i, v_i)$  to  $(q_i, v_i + \Delta_i)$ . From the definition of  $v_i$ , it is easily seen that for every  $j \in S_i$ ,  $\pi_j(v_i) = \pi_j(V_i)$ . Hence for every  $j \in S_i$ ,  $\pi_j(v_i) = \pi_j(V_i) = \pi_j(x^i)$ . Consequently, path  $p_1$  is an  $SR$ -path w.r.t.  $S_i$  in  $G_i$  from  $(q_i, x^i)$  to  $(q_i, x^i + \Delta_i)$  (shift the starting point of  $p_1$  from  $v_i$  to  $x^i$ ). Since  $\Delta x^i \geq \bar{0}$ , and for every  $j \notin S_i$ ,  $\pi_j(\Delta x^i) \geq 1$  (from \*), there exists an  $m_1$ , s.t. the path  $p_1$  from  $(q_i, x^i + m_1 \Delta x^i)$  to  $(q_i, x^i + m_1 \Delta x^i + \Delta_i)$  is an  $R$ -path in  $G_i$  (shift the starting point from  $x^i$  to  $x^i + m_1 \Delta x^i$ ). Making use of  $\theta 2(b)$  and following a similar argument we can prove that there exists an  $m_2$  s.t. there is an  $R$ -path in  $G_i$  from  $(q_i, y^i + m_2 \Delta y^i)$  to  $(q_i, y^i + m_2 \Delta y^i + \Delta_i)$ . Hence condition (c) of Theorem 4 holds.<sup>rev</sup>

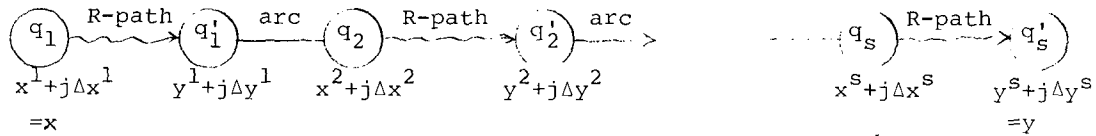
Recall that for every  $j \geq 0$ , there is an  $r$ -path in  $G_i$  from  $(q_i, x^i + j\Delta x^i)$  to  $(q_i', y^i + j\Delta y^i)$  the folding of the path being  $z^i + j\Delta z^i$ . Consider two such paths, one for  $j=0$  and the other for  $j=1$ , and apply Lemma 6. This yields that there is an unfolding of  $\Delta z^i$  from  $(q_i, \Delta x^i)$  to  $(q_i, \Delta y^i)$ . Since  $\Delta z^i \geq \bar{1}$  (condition \*),  $(q_i, \Delta y^i)$  is  $r$ -reachable from  $(q_i, \Delta x^i)$  by a path whose folding is  $\geq \bar{1}$ . This proves condition (d) of Theorem 4. Hence for  $G_i$ , by Theorem 4,

$$(\exists j_0)(\forall j \geq j_0)((q_i', y^i + j\Delta y^i) \in R(q_i, x^i + j\Delta x^i)).$$

For every  $i$  there is a constant  $j_0$  as given by the above statement. Now choose the maximum among such constants, as  $i$  varies from 1 to  $s$ . Let this be  $J$ . Then

$$(\forall i)(\forall j \geq J)(q_i', y^i + j\Delta y^i) \in R(q_i, x^i + j\Delta x^i).$$

For any such  $j$ , the above statement specifies a  $CR$ -path from  $(q_1, x)$  to  $(q_s', y)$ , which is schematically shown below. Note that  $x^1 = x$ ,  $\Delta x^1 = \bar{0}$ ,  $y^s = y$  and  $\Delta y^s = \bar{0}$



Recall that for every  $i$ , the linear set  $L((x^i, y^i, z^i); (\Delta x^i, \Delta y^i, \Delta z^i))$  was obtained by projecting  $\hat{L}_G$  into  $G_i$ . Hence for every  $j \geq 0$ ,  $(x^1, y^1, z^1, x^2, y^2, z^2, \dots, x^s, y^s, z^s) + j(\Delta x^1, \Delta y^1, \Delta x^2, \Delta y^2, \dots, \Delta x^s, \Delta y^s, \Delta z^1, \Delta z^2, \dots, \Delta z^s) \in \hat{L}_G$ . Consequently, for every  $i=1, \dots, s-1$  and  $j \geq 0$ , there is an arc from  $y^i + j\Delta y^i$  to  $x^{i+1} + j\Delta x^{i+1}$  (in fact this is the arc  $(q_i', q_{i+1})$ ). Thus the complete path shown above is an  $R$ -path.

We exhibited an infinite number of  $CR$ -paths from  $(q_1, x)$  to  $(q_s', y)$ , even though it is sufficient to exhibit just one path.

Now we want to show that the conditions of Theorem 5 can be effectively tested.

Theorem 6: It is effectively decidable whether a GVASS,  $G$ , satisfies property  $\theta$ .

Proof: To test for condition (1), compute the semilinear set  $L_G$  as a finite union of linear sets. At least one of these linear sets must have the property that the sum of its periods has a nonzero entry corresponding to every unconstrained coordinate and also every arc. (It trivially follows from the definition of  $L_G$  that for every linear subset of  $L_G$ , every one of its periods, and hence the sum of its periods, has a 0 value corresponding to every constrained coordinate). This can be easily tested. To test for property 2, apply Lemma 11 to each  $G_i$  twice (once for  $G_i$  and another time for  $G_{i,rev}$ ).

Now we shall establish that if  $G$  does not satisfy property  $\theta$ , then the "size" of the GVASS can be reduced. For every  $G_i$ , define its size by a triple  $(n_{i1}, n_{i2}, n_{i3})$  where:

$n_{i1}$  = number of rigid coordinates of  $G_i (=n - |R_i|)$ ,

$n_{i2}$  = number of arcs of  $G_i (=k_i)$ , and

$n_{i3}$  = number of unconstrained input and output coordinates of  $G_i (=2n - |S_i| - |S_i'|)$ .

The size of  $G$ , denoted  $SS(G)$ , is given by the multiset of sizes of its  $G_i$ 's. Let ' $<$ ' refer to the dictionary order among triples; i.e.  $(a_1, a_2, a_3) < (b_1, b_2, b_3)$  iff  $((a_1 < b_1) \text{ or } (a_1 = b_1 \text{ and } a_2 < b_2) \text{ or } (a_1 = b_1, a_2 = b_2 \text{ and } a_3 < b_3))$ . In Theorem 7, we shall establish that if  $G$  does not satisfy property  $\theta$ , then we can replace  $G$  by a finite number of GVASS's,  $G^1, G^2, \dots$ , such that for every  $i$ ,  $SS(G^i)$  can be obtained from  $SS(G)$  by replacing a triple by a finite number of smaller triples, and in addition, the CR-reachability of  $G$  has a 'yes' answer iff the corresponding problem for some  $G^i$  has a 'yes' answer. This establishes that only a finite number of modifications are possible. If the procedure terminates without satisfying property  $\theta$ , the size of every  $G_i$  will be  $(0, 0, 0)$ ; i.e. every coordinate of  $G_i$  is rigid and  $G_i$  is a single node or many isolated nodes without any arcs. If property  $\theta$  does not hold, at this stage, then it can be reported that there is no CR-path from  $(q_1, x)$  to  $(q_s, y)$ .

Theorem 7: If the GVASS,  $G$ , does not satisfy property  $\theta$  and if  $SS(G)$  contains an element different from  $(0, 0, 0)$ , then  $G$  can be replaced by a finite number of GVASS's,  $G^1, G^2, \dots$ , such that

- (1) for every  $i$ ,  $SS(G^i)$  can be obtained from  $SS(G)$  by replacing a triple by a finite number of triples, each of which is less than the triple being replaced, and
- (2) the CR-reachability of  $G$  has a 'yes' answer iff the CR-reachability of some  $G^i$  has a 'yes' answer.

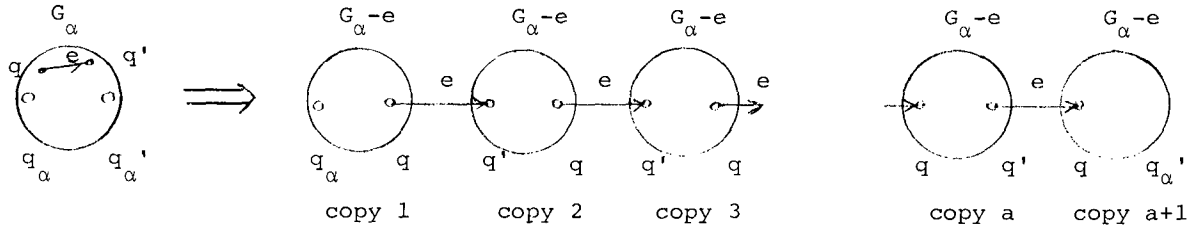
Proof:  $L_G$  is a semilinear set and can be expressed as the union of a finite number of linear sets:

$$L_G = L_1 \cup L_2 \cup \dots \cup L_\beta, \text{ where } L_i = L(c_i, p_{i1}, p_{i2}, \dots, p_{iy_i}).$$

If  $L_G = \text{empty set}$ , then remove all arcs of  $G$  and set all unconstrained coordinates to 0's.

Condition  $\theta 1(a)$

If condition  $\theta 1(a)$  fails, then we shall replace  $G$  by  $\beta$  GVASS's,  $G^1, \dots, G^\beta$ , the  $G^i$  being based on  $L_i$ . When  $\theta 1(a)$  fails, by Lemma 1, for every  $i$  there exists an  $i_0$  s.t. the  $i_0^{\text{th}}$  component of  $p_{i1} + p_{i2} + \dots + p_{iy_i}$  is 0, and this  $i_0^{\text{th}}$  component corresponds to some arc. Let this arc be  $e = (q, q')$ , and let it be in some  $G_\alpha$ . Note also that  $\pi_{i_0}(c_i)$  specifies the number of times arc  $e$  in  $G_\alpha$  gets used on each cr-path whose extended folding is in  $L_i$ .



$G_{\alpha} - e$  is obtained from  $G_{\alpha}$  by simply removing the arc  $e$ . The chain shown above has  $a+1$  copies of  $G_{\alpha} - e$ , and for each copy let the set of rigid components be  $R_{\alpha}$ . The input constraint of copy 1 is  $V_{\alpha}$  and the output constraint of copy  $a+1$  is  $V_{\alpha}'$ . The remaining input and output constraints are identical (say  $= W$ ) as given by:

$$\pi_j(W) = \begin{cases} \omega & \text{if } j \notin R_{\alpha} \\ \pi_{i0}(V_{\alpha}) & \text{if } j' \in R_{\alpha} \end{cases}.$$

The first two components of the size of each copy of  $G_{\alpha} - e$  are  $n_{\alpha 1}$  and  $n_{\alpha 2} - 1$ . Thus the size of each  $G_{\alpha} - e$  is less than the size of  $G_{\alpha}$ .

After the above transformation, let the new GVASS be  $G^i$ . It can be easily shown that:

in  $G$  there exists a CR-path, from  $(q_1, x)$   
to  $(q_s', y)$ , whose extended folding is in  $L_i$   
iff in  $G^i$  there exists a CR-path from  $(q_1, x)$   
to  $(q_s', y)$ .

Thus the theorem holds in this case.

#### Condition 01(b)

If condition 01(b) fails, then we shall replace  $G$  by  $\beta$  GVASS's,  $G^1, \dots, G^{\beta}$ , the  $G^i$  being based on  $L_i$ . By Lemma 1, for every  $i$  there exists an  $i0$  s.t. the  $i0^{\text{th}}$  component of  $p_{i1} + \dots + p_{iy_i}$  is 0, and the  $i0^{\text{th}}$  component corresponds to some unconstrained input or output coordinate in some  $G_{\alpha}$ . Note that  $\pi_{i0}(c_i)$  specifies the required fixed value of that coordinate which had previously the value  $\omega$ . Replace that  $\omega$  in the corresponding constraint vector (input or output) of  $G_{\alpha}$  by  $\pi_{i0}(c_i)$ , and let the resulting GVASS be  $G^i$ . Note that the size of  $G_{\alpha}$  changes from  $(c_{\alpha 1}, c_{\alpha 2}, c_{\alpha 3})$  to  $(c_{\alpha 1}, c_{\alpha 2}, c_{\alpha 3-1})$ . Now we can complete the argument as in 01(a).

#### Condition 02

Conditions 02(a) and 02(b) can be handled similarly, and we show the transformation when 02(a) fails. Let 02(a) fail for some  $G_{\alpha}$ . Then there does not exist a  $\Delta_{\alpha}$  s.t. for every  $i \in S_{\alpha} - R_{\alpha}$   $\pi_i(\Delta_{\alpha}) \geq 1$  and  $(q_{\alpha}, v_{\alpha} + \Delta_{\alpha}) \in \text{SR}(q_{\alpha}, v_{\alpha})$  w.r.t.  $S_{\alpha} - R_{\alpha}$  in  $G_{\alpha}$ . Thus by Lemma 11, we can effectively compute a constant  $c$  s.t. every point SR-reachable w.r.t.  $S_{\alpha} - R_{\alpha}$  from  $(q_{\alpha}, v_{\alpha})$  has  $i^{\text{th}}$  component value  $\leq c$  for some  $i$  in  $S_{\alpha} - R_{\alpha}$ . Since every CR-path from  $(q_1, x)$  to  $(q_s', y)$  must satisfy the input constraint of  $G_{\alpha}$ , when such a CR-path is inside  $G_{\alpha}$ , any point on it must have a component value  $\leq c$  for some coordinate in  $S_{\alpha} - R_{\alpha}$ . We shall treat the reduction as  $|S_{\alpha} - R_{\alpha}|$  cases, one corresponding to each element in  $S_{\alpha} - R_{\alpha}$ . In each case we shall modify  $G_{\alpha}$  into at most  $c+1$  new  $G_{\alpha}$ 's. In total we generate at most  $|S_{\alpha} - R_{\alpha}|(c+1)$  GVASS's.

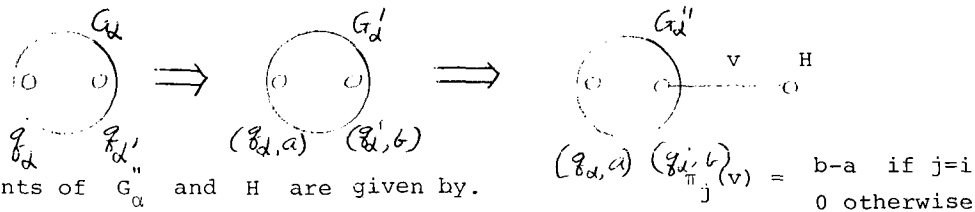
Select any  $i$  in  $S_\alpha - R_\alpha$ .

(i) If  $\pi_i(V'_\alpha) = \omega$ , then replace that  $\omega$  by  $0, 1, 2, \dots, c$  each giving a new  $G_\alpha$ , which results in  $c+1$  new GVASS's. Note that this additional restriction will not remove any old CR-path, from  $(q_1, x)$  to  $(q_s, y)$ , which satisfies the property that when the path is inside  $G_\alpha$  (including its outputs point) its  $i^{\text{th}}$  coordinate is  $\leq c$ .

(ii) Let  $\pi_i(V'_\alpha) \in N$ .

Let  $\pi_i(V'_\alpha) = a$  and  $\pi_i(V'_\alpha) = b$ . Note that for any CR-path from  $(q_1, x)$  to  $(q_s, y)$ , the input and the output points of  $G_\alpha$  have  $i^{\text{th}}$  component values of  $a$  and  $b$ , respectively. In addition when the path is inside  $G_\alpha$ , the  $i^{\text{th}}$  component of the corresponding points must lie in between  $0$  and  $c$ . By making use of the  $i^{\text{th}}$  components of the labels of the arcs of  $G_\alpha$  we can construct an fsa which captures all paths from  $a$  to  $b$  with intermediate values in between  $0$  and  $c$ . If we take the cross-product of  $G_\alpha$  and this fsa, the new VASS,  $G'_\alpha$ , has the additional constraint that the  $i^{\text{th}}$  component of each of its intermediate point  $p$  is in between  $0$  and  $c$ . At this stage, if the  $i^{\text{th}}$  components of the labels of all the arcs of  $G'_\alpha$  are replaced by  $0$ 's the paths inside  $G'_\alpha$  will not be affected but at the output point the  $i^{\text{th}}$  component will have a value  $a$  instead of  $b$ . This can be corrected by an arc, whose label is a vector of all  $0$ 's, except that its  $i^{\text{th}}$  component is  $b-a$ . Even though this is a sufficient description, we precisely describe the procedure below and show that the size decreases in the transformation.

Let  $G_\alpha$  have  $g$  states. Then create  $g(c+1)$  states, the label of each state being a pair  $(q, \alpha)$  where  $q$  is a state of  $G_\alpha$  and  $0 \leq \alpha \leq c$ . There exists an arc labeled  $u$  from  $(q, \alpha)$  to  $(q', \alpha')$  iff in  $G_\alpha$  there is an arc labeled  $u$  from  $q$  to  $q'$  and  $\alpha' = \alpha + \pi_i(u)$ . Let the input and the output states of  $G'_\alpha$  be  $(q_\alpha, a)$  and  $(q_\alpha, b)$ , respectively. Now in  $G'_\alpha$  replace the  $i^{\text{th}}$  component of the label of every arc by  $0$  and add an arc,  $v$ , to a new state from  $(q_\alpha, b)$  as shown below,



The constraints of  $G''_\alpha$  and  $H$  are given by.

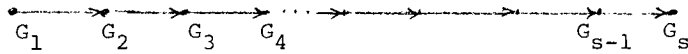
Input constraint of  $G''_\alpha = V_\alpha$ , output constraint of  $G''_\alpha = V$ , set of rigid coordinates of  $G''_\alpha = R_\alpha \cup \{i\}$ , input constraint of  $H = V'_\alpha$ , output constraint of  $H = V'_\alpha$ , and set of rigid coordinates of  $H = R_\alpha \cup \{i\}$ , where  $V$  is given by  $\pi_j(V) = \begin{cases} a & \text{if } j=i \\ \pi_j(V'_\alpha) & \text{otherwise.} \end{cases}$

Note that the first component of the sizes of  $G''_\alpha$  and  $H$  is  $n_{i1}-1$ , which is less than  $n_{i1}$ . Hence the size of each of  $G''_\alpha$  and  $H$  is less than the size of  $G_\alpha$ . After the replacement of  $G_\alpha$  by  $G''_\alpha$  and  $H$  let the GVASS be  $G^i$ . It is easily seen that there exists a CR-path in  $G$  s.t. when the path is inside  $G_\alpha$  the  $i^{\text{th}}$  coordinate of every point is  $\leq c$  iff there exists a CR-path in  $G^i$ . Thus the theorem holds.

The next theorem gives the termination condition.

**Theorem 8:** For the GVASS,  $G$ , if every member of its size set is  $(0, 0, 0)$  and if  $G$  does not satisfy property  $\Phi$ , then there is no CR-path from  $(q_1, x)$  to  $(q_s, y)$ .

**Proof:** Note that there are no arcs within any  $G_i$ . If for some  $G_i, q_1 \neq q_s$  then there is no directed path from  $q_1$  to  $q_s$ , hence there is no CR-path from  $(q_1, x)$  to  $(q_s, y)$ . Otherwise, excluding the isolated nodes,  $G$  is as shown below.



For such a GVASS, property  $\theta$  degenerates into existence of a cr-path from  $(q_1, x)$  to  $(q_s, y)$ . Also note that in such a GVASS every cr-path is a CR-path.

Now we can outline the decision procedure for the reachability problem of GVASS's.

```

L:  Test whether the GVASS satisfies property  $\theta$ 
    if property  $\theta$  holds then report 'yes' and halt
    else
      if the size set has a member  $\neq (0,0,0)$  then reduce
        the problem size, and goto L
      else report 'no' and halt.
  
```

#### IV. Conclusions:

We are able to establish the decidability of reachability by making use of known simple observations. It is particularly gratifying that the simple idea embodied in Theorem 1 can be used to solve the general problem. At about the time [5] was announced, I was able to establish the decidability of reachability in 4 dimensions. The result became outdated even before I could work out the details, due to [3] and [5]. However what intrigued me most was that my approach was based on a special form of Theorem 1. By such a theorem, I was able to reduce the dimensionality by 1, but I was forced to fall back on a technique analogous to that of [7]. What was lacking was the inductive step, which [5] claimed to have developed. Even though [5] does not explicitly consider the extension of VAS's to VASS's as in here and also in [3,4], it does make use of chains of VAS's in a manner not too different from the technique here. In fact, after going through our proof one should not have any difficulty in reformulating this technique for chains of VAS's.

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For  $G_i$ , let  $S_i = \{j \mid \pi_j(V_i) \neq \omega\}$ , and  $S'_i = \{j \mid \pi_j(V'_i) \neq \omega\}$ . The coordinates in  $S_i$  and  $S'_i$  are the constrained input and output coordinates respectively of  $G_i$ . A subset,  $R_i$ , of  $S_i \cap S'_i$  is denoted as the set of rigid coordinates, and has the following significance. For every  $j \in R_i$ , the  $j$ 'th component of the label of every arc in  $G_i$  must have value 0.